Problems - 2013.04.02

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1. For an integer $n \geq 3$ consider the sets

$$S_n = \{(x_1, x_2, \ldots, x_n) : \forall i \; x_i \in \{0, 1, 2\}\}$$

$$A_n = \{(x_1, x_2, \ldots, x_n) \in S_n : \forall i \leq n-2 \; |\{x_i, x_{i+1}, x_{i+2}\}| \neq 1\}$$

and

$$B_n = \{(x_1, x_2, \ldots, x_n) \in S_n : \forall i \leq n-1 \; |\{x_i = x_{i+1} \implies x_i \neq 0\}.$$

Prove that $|A_{n+1}| = 3|B_n|$.

2. Let $A_1, A_2, \ldots, A_n$ be finite nonempty sets. Define the function

$$f(t) = \sum_{k=1}^{n} \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} (-1)^{k-1} t^{|A_{i_1} \cup A_{i_2} \cup \cdots \cup A_{i_k}|}.$$

Prove that f is nondecreasing on $[0, 1]$.

3. A triangulation of a finite point set A in the plane is defined as a partition of its convex hull into triangles such that the vertices of each triangle are elements of A, and no triangle contains any element of A apart from its vertices. Prove that all triangulations of a set A consist of the same number of triangles.

4. For a finite point set A in the plane, let $\nu(A)$ denote the number of triangles in a triangulation of set A, and let $A + A = \{x + y \mid x, y \in A\}$, where the sum of two points is defined as the point whose position vector is the sum of the position vectors of the terms. Prove that $\nu(A + A) \geq 4\nu(A)$.

5. Someone selected n not necessarily different non-negative integers. He wrote down on a sheet of paper all of the $(2^n - 1)$ sums that can be formed out of the n numbers. Is it possible to determine the original numbers from this information?

6. Someone selected n not necessarily different integers. He wrote down on a sheet of paper all of the $(2^n - 1)$ sums that can be formed out of the n numbers. Given that 0 was not obtained, is it possible to determine the original numbers from this information?

Hard nuts

7. For which values of n can you construct two different sets $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$ such that the (multi)sets $\{a_i + a_j | 1 \leq i < j \leq n\}$ and $\{b_i + b_j | 1 \leq i < j \leq n\}$ are the same?

8. For a permutation $\sigma = (i_1, i_2, \ldots, i_n)$ of $(1, 2, \ldots, n)$ define $D(\sigma) = \sum\limits_{k=1}^{n} |i_k - k|$. Let $Q(n, d)$ be the number of permutations σ of $(1, 2, \ldots, n)$ with $d = D(\sigma)$. Prove that $Q(n, d)$ is even for $d \geq 2n$.