Problems - 2013.03.12

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1. Let a, b, c be positive reals. Prove that

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{3}{2}.$$

2. Let S_n be the set of all sums $\sum_{k=1}^{n} x_k$, where $n \geq 2$, $0 \leq x_1, x_2, \ldots, x_n \leq \pi/2$ and

$$\sum_{k=1}^{n} x_k = 1.$$

a) Show that S_n is an interval.
b) Let l_n be the length of S_n. Find $\lim\limits_{n\to\infty} l_n$.

3. Compare $\tan\sin(x)$ and $\sin\tan(x)$ for all $x \in (0, \pi/2)$.

4.* Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{1}{a^3(b+c)}+\frac{1}{b^3(c+a)}+\frac{1}{c^3(a+b)}\geq\frac{3}{2}.$$

Hard nuts

5. Let n be a positive integer and let $x_1 \leq x_2 \leq \cdots \leq x_n$ be real numbers. Prove that

$$\left(\sum_{i,j=1}^{n} |x_i - x_j|\right)^2 \leq \frac{2(n^2-1)}{3}\sum_{i,j=1}^{n}(x_i - x_j)^2.$$

Show that the equality holds if and only if $x_1, \ldots, x_n$ is an arithmetic sequence.

6. Suppose that a, b, c are real numbers in the interval $[-1, 1]$ such that

$$1 + 2abc \geq a^2 + b^2 + c^2.$$

Prove that

$$1 + 2(abc)^n \geq a^{2n} + b^{2n} + c^{2n}.$$

for all positive integers n.