Problems - 2013.03.05

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1. Prove that every group of exponent 2 is commutative. (A group G has exponent 2, if $x^2 = 1_G$ for all $x \in G$, where 1_G stands for the unit element of G.)

2. Suppose that in a not necessarily commutative ring R the square of any element is 0. Prove that $abc + abc = 0$ for any three elements a, b, c.

3. Let $a_1, a_2, \ldots, a_{51}$ be nonzero elements of a field. We simultaneously replace each element with the sum of the 50 remaining ones. In this way we get a sequence $b_1, \ldots, b_{51}$. If this new sequence is a permutation of the original one, what can be the characteristic of the field? (The characteristic of a field is p, if p is the smallest positive integer such that $\underbrace{x + x + \ldots + x}_{p} = 0$ for any element x of the field. If there exists no such p, the characteristic is 0.)

4. Given an integer $n > 1$, let S_n be the group of permutations of the numbers $1, 2, \ldots, n$. Two players, A and B, play the following game. Taking turns, they select elements (one element at a time) from the group S_n. It is forbidden to select an element that has already been selected. The game ends when the selected elements generate the whole group S_n. The player who made the last move loses the game. The first move is made by A. Which player has a winning strategy?

Hard nuts

5. Given a finite group G, denote by $c(G)$ the number of conjugacy classes in G. Prove that $c(G) \to \infty$ as $|G| \to \infty$.

6. Let p be a prime number and $\mathbb{F}_p$ be the field of residues modulo p. Let W be the smallest set of polynomials with coefficients in $\mathbb{F}_p$ such that (1) the polynomials $x + 1$ and $x^{p-2} + x^{p-3} + \ldots + x^2 + 2x + 1$ are in W; and (2) for any polynomials $h_1(x)$ and $h_2(x)$ in W, the polyinomial $r(x)$, which is the remainder of $h_1(h_2(x))$ modulo $x^p - x$ is also in W. How many polynomials are there in W?