Problems - 2013.02.12

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1. (a) Let $a_1, a_2, \dots$ be a sequence of real numbers such that $a_1 = 1$ and $a_{n+1} > \frac{3}{2}a_n$ for all n. Prove that the sequence $\frac{a_n}{\left(\frac{3}{2}\right)^{n-1}}$ has a finite limit or tends to infinity. (b) Prove that for all $\alpha > 1$ there exists a sequence $a_1, a_2, \dots$ with the same properties such that $\lim \frac{a_n}{\left(\frac{3}{2}\right)^{n-1}} = \alpha$.

2. Does there exist a bijective map $\pi : \mathbb{N} \to \mathbb{N}$ such that $\sum\limits_{k=1}^{\infty} \frac{\pi(n)}{n^2} < \infty$?

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $(f(x))^n$ is a polynomial for every $n = 2, 3, \dots$. Does it follow that f is a polynomial?

4. Consider a polynomial $f(x) = x^{2012} + a_{2011}x^{2011} + \dots + a_1x + a_0$. Albert Einstein and Homer Simpson are playing the following game. In turn, they choose one of the coefficients $a_0, \dots, a_{2011}$ and assign a real value to it. Albert has the first move. Once a value is assigned to a coefficient, it cannot be changed any more. The game ends after all the coefficients have been assigned values. Homer's goal is to make $f(x)$ divisible by a fixed polynomial $m(x)$ and Albert's goal is to prevent this.
(a) Which of the players has a winning strategy if $m(x) = x - 2012$?
(b) Which of the players has a winning strategy if $m(x) = x^2 + 1$?

5. (a) A sequence $x_1, x_2, \dots$ of real numbers satisfies $x_{n+1} = x_n \cos x_n$ for all $n \geq 1$. Does it follow that this sequence converges for all initial values x_1? (b) A sequence $y_1, y_2, \dots$ of real numbers satisfies $y_{n+1} = y_n \sin y_n$ for all $n \geq 1$. Does it follow that this sequence converges for all initial values y_1?

6. Let $0 < a < b$. Prove that

$$\int_a^b (x^2 + 1)e^{-x^2} dx \geq e^{-a^2} - e^{-b^2}.$$

Hard nuts

7. Prove or disprove the following statements:
(a) There exists a monotone function $f : [0,1] \to [0,1]$ such that for each $y \in [0,1]$ the equation $f(x) = y$ has uncountably many solutions x.
(b) There exists a continuously differentiable function $f : [0,1] \to [0,1]$ such that for each $y \in [0,1]$ the equation $f(x) = y$ has uncountably many solutions x.

8.* (a) Show that for each function $f : \mathbb{Q} \times \mathbb{Q} \to \mathbb{R}$ there exists a function $g : \mathbb{Q} \to \mathbb{R}$ such that $f(x, y) \leq g(x) + g(y)$ for all $x, y \in \mathbb{Q}$. (b) Find a function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ for which there is no function $g : \mathbb{R} \to \mathbb{R}$ such that $f(x, y) \leq g(x) + g(y)$ for all $x, y \in \mathbb{R}$.