Problems - 2013.02.05

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1. At least how many open half-planes are needed to obtain the (a) open, (b) closed unit circle of $\mathbb{R}^2$ as their intersection? What about closed half-planes?

2. Find the minimum of

$$x^4 - 15x^2 - 18x$$

for $x \in \mathbb{R}$.

3. Given 2013 points on the $\mathbb{R}^2$ plane, no two of them are collinear with the origin. Show that there is a line passing through the origin such that it divides the given points into two sets of sizes 1006 and 1007. Does there necessarily exists a line dividing into 1005 – 1008?

4. Some Hungarian, English, French, German, Italian and Russian scientists attended a conference. They observed that each of them has exactly one acquaintance from each nation (inculding his/her own one). Prove that the number of participants is divisible by 12. Prove that for any natural number divisible by 12, we can "organize" (at least, theoretically) such a conference.

5. Find all functions $f : \mathbb{Q} \to \mathbb{C}$ satisfying

$$f(x + y) = f(x) + f(y) + xy.$$

for all $x, y \in \mathbb{Q}$.

6. Prove that for any positive real numbers x, y, z,

$$\sqrt{x^2 - xy + y^2} + \sqrt{y^2 - yz + z^2} \geq \sqrt{x^2 + xz + z^2}.$$

7. Given a graph on $2n$ vertices. Prove that we may erase less than half of the edges such that the remaining graph is bipartite with equally large classes.

Hard nuts

8. For an integer $k \geq 2$, a positive integer n is said to be k-splitting, if there exists $a \in \mathbb{N}$ and $p > 0$ prime such that

$$n = p + a^k.$$

Prove that there are infinitely many natural numbers that are **not** k-splitting for any $2 \leq k \leq 2013$.

9.* For $n \geq 2$, let $A_1, \ldots, A_n$ be events such that for any $1 \leq i \leq n$, $p(A_i) = 1/2$ and for any $1 \leq i < j \leq n$, $p(A_i \cap A_j) = 1/4$. Prove that the probabilty of that none of them happens is at most $\frac{1}{n+1}$. Prove that this can be sharp for infinitely many n.