Problems - 2012.10.31

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1. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Suppose that for any $c > 0$, the graph of f can be moved to the graph of cf using only a translation or a rotation. Does this imply that $f(x) = ax + b$ for some real numbers a and b?

2. Suppose that $f : [0, 1] \to \mathbb{R}$ has a continuous derivative and that $\int_0^1 f(x)\,dx = 0$. Prove that for every $\alpha \in (0, 1)$,
$$\left| \int_0^\alpha f(x)\,dx \right| \leq \frac{1}{8} \max_{0 \leq x \leq 1} |f'(x)|.$$

3. Let f be a real function on the real line with continuous third derivative. Prove that there exists a point a such that
$$f(a) \cdot f'(a) \cdot f''(a) \cdot f'''(a) \geq 0.$$

4. Find all differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that
$$f'(x) = \frac{f(x+n) - f(x)}{n}$$
for all real numbers x and all positive integers n.

5. Let $F_0(x) = \ln x$. For $n \geq 0$ and $x > 0$, let $F_{n+1}(x) = \int_0^x F_n(t)\,dt$. Evaluate
$$\lim_{n\to\infty} \frac{n! F_n(1)}{\ln n}.$$

6. Show that for every positive integer n,
$$\left(\frac{2n-1}{e}\right)^{\frac{2n-1}{2}} < 1 \cdot 3 \cdot 5 \cdots (2n-1) < \left(\frac{2n+1}{e}\right)^{\frac{2n+1}{2}}.$$