Problems - 2012.10.10

P. Maga & P. P. Pach

1. How many square numbers are in the sequence $1, 11, 111, \ldots$? (The base is 10.)
2. Let $n \geq 3$, and set $A = n - 2$. Prove that $\overline{A1}$ (in base n) is a square number.
3. Let $1 \leq a, b, c \leq 9$ be integers. Prove that the number $\overline{abcabc}$ (in base 10) is divisible by 77.
4. For any $n \geq 1$, let
$$A_n = \overbrace{11\ldots11}^{n}\overbrace{22\ldots22}^{n}.$$
Prove that A_n can be written as the product of two consecutive integers. (The base is 10.)
5. For any $n \geq 1$, let
$$B_n = \overbrace{44\ldots44}^{n}\overbrace{88\ldots8}^{n-1}9.$$
Prove that B_n is a square number. (The base is 10.)

   Prove that this problem is "equivalent" to the above one in the sense that each of them implies the other in a simple way.
6. How many prime numbers among the positive integers, written as usual in base 10, are alternating 1's and 0's, beginning and ending with 1?
7. A positive integer k is said to be *nice*, if there exists a multiple K of k such that all digits of K are equal. Determine the nice numbers.

Hard nuts

8. Can a countably infinite set have an uncountable collection of non-empty subsets such that the intersection of any two of them is finite?
9. Can a countably infinite set have an uncountable collection $(A_i)_{i \in \mathcal{I}}$ of subsets such that for any $i, j \in \mathcal{I}$ $(i \neq j)$, $A_i \subset A_j$ or $A_j \subset A_i$?