Problems - 2012.09.12

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1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function such that $f(x, y) + f(y, z) + f(z, x) = 0$ for all real numbers x, y and z. Prove that there is a function $g : \mathbb{R} \to \mathbb{R}$ such that $f(x, y) = g(x) - g(y)$ for all real numbers x and y.

2. Alan and Barbara play a game in which they take turns filling entries of an initially empty 2008×2008 array. Alan plays first. At each turn, a player chooses a real number and places it in a vacant entry. The game ends when all the entries are filled. Alan wins if the determinant of the resulting matrix is nonzero; Barbara wins if it is zero. Which player has a winning strategy?

3. Let n be an even positive integer. Write the positive numbers $1, \ldots, n^2$ in an $n \times n$ grid $(a_{i,j})_{1 \le i,j \le n}$ such that $a_{i,j} = (i - 1)n + j$. Color the grid such that in each row and in each column, half of the squares are red, and the other half are black. Prove that in any such coloring, the sum of the red numbers equals the sum of the black numbers.

4. Given a positive integer n, what is the largest k such that the numbers $1, \ldots, n$ can be put into k boxes such that the sum in each box is the same?

5. Is it true that if $f : [0, 1] \to [0, 1]$ is

   (a) monotone increasing

   (b) monotone decreasing

   then there exists an $x \in [0, 1]$ for which $f(x) = x$?

6. Let f be continuous and nowhere monotone on $[0, 1]$. Show that the set of points on which f attains local minima is dense in $[0, 1]$.

7. Let a_j, b_j, c_j be integers for $1 \le j \le N$. Assume that for all j, at least one of them is odd. Show that there exist integers r, s, t such that $ra_j + sb_j + tc_j$ is odd for at least $4N/7$ values of j.

8. Let A be a positive real number. What are the possible values of $\sum_{j=0}^{\infty} x_j^2$ given that $x_0, x_1, \ldots$ are positive real numbers for which $\sum_{j=0}^{\infty} x_j = A$?

Hard nuts

9. Given a finite simple graph. At each vertex, there is a lamp (of two states, on and off) and a switch. Each change in a switch changes the state of the corresponding and the neighboring lamps. Initially all lamps are off. Show that there is a sequence of changes that makes all lamps on.

10. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f(x)f(y) = 2f(x + yf(x))$$

for all $x, y \in \mathbb{R}^+$.