MONOTONE ARRAYS AND A MULTIDIMENSIONAL RAMSEY THEOREM

A foundational result in Ramsey theory appears in a paper Erdős and Szekeres from 1935: any sequence of n^2+1 distinct real numbers contains either an increasing or decreasing subsequence of length $n+1$. This simple result was one of the starting seeds for the development of Ramsey theory. There are a number of different ways to generalise the Erdős-Szekeres Theorem to higher dimensions but perhaps the most natural approach was developed thirty years ago by Fishburn and Graham. A *d-dimensional array* is an injective function f from $A_1 \times \ldots \times A_d$ to $\mathbb{R}$ where $A_1, \ldots A_d$ are non-empty subsets of $\mathbb{Z}$; we say f has *size* $|A_1| \times \cdots \times |A_d|$; if $|A_i| = n$ for each i, it will be convenient to say that f has *size* $[n]^d$. A multidimensional array is said to be *monotone* if for each direction all the 1-dimensional subarrays in that direction are increasing or decreasing. In other words, for every i, one of the following holds:

- For every choice of a_j, $j \neq i$, the function $f(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_d)$ is increasing in x.
- For every choice of a_j, $j \neq i$, the function $f(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_d)$ is decreasing in x.

Let $M_d(n)$ be the smallest N such that a d-dimensional array on $[N]^d$ contains a monotone d-dimensional subarray of size $[n]^d$. We will show how to improve the bounds on $M_d(n)$ recently proved by Bucić, Sudakov and Tran. More precisely, we will show that a doubly exponential upper bound holds in all dimensions.

Theorem 0.1. *For every $d \geq 2$, there is $C_d > 0$, such that for every positive n, $M_d(n) \leq 2^{n^{C_d n^{d-1}}}$.*

Finally, we will see how this is intimately connected to a generalisation of Ramsey Theorem on the cartesian product of cliques.

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