TRANSIENT NEAREST NEIGHBOR RANDOM WALK ON THE LINE

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Abstract: We prove strong theorems for the local time at infinity of a nearest neighbor transient random walk. First, laws of the iterated logarithm are given for the large values of the local time. Then we investigate the length of intervals over which the walk runs through (always from left to right) without ever returning.

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1. Introduction

Let $X_0 = 0,\ X_1, X_2, \ldots$ be a Markov chain with

$$E_i := \mathbf{P}(X_{n+1} = i+1 \mid X_n = i) = 1 - \mathbf{P}(X_{n+1} = i-1 \mid X_n = i) \tag{1.1}$$
$$= \begin{cases} 1 & if \quad i = 0 \\ 1/2 + p_i & if \quad i = 1, 2, \ldots, \end{cases}$$

where $0 \le p_i < 1/2,\ i = 1, 2, \ldots$. This sequence $\{X_i\}$ describes the motion of a particle which starts at zero, moves over the nonnegative integers and going away from 0 with a larger probability than to the direction of 0. We will be interested in the case when $\{p_i,\, i = 1, 2...\}$ goes to zero. That is to say 0 has a repelling power which becomes small if the particle is far away from 0. We intend to characterize the local time of this motion.

A slightly different but symmetric variation of the same motion can be defined as follows. Let $X_0^* = 0,\ X_1^*, X_2^*, \ldots$ be a Markov chain with

$$E_i^* := \mathbf{P}(X_{n+1}^* = i+1 \mid X_n^* = i) = 1 - \mathbf{P}(X_{n+1}^* = i-1 \mid X_n^* = i) =$$
$$= \begin{cases} 1/2 & if \quad i = 0, \\ 1/2 + p_i & if \quad i = 1, 2, \ldots, \\ 1/2 - p_{-i} & if \quad i = -1, -2, \ldots \end{cases}$$

Our results can be rephrased with the obvious modification for this walk as well. However to be in line with the existing literature we will use the definition in (1.1).

The properties of this model, often called birth and death chain, connections with orthogonal polynomials in particular, has been treated extensively in the literature. See e.g. the classical paper by Karlin and McGregor [11], or more recent papers by Coolen-Schrijner and Van Doorn [3] and Dette [5].

As it will turn out in this paper, the properties of the walk and its local time is very sensitive even for small changes in $\{p_i\}$-s. There is a well-known result in the literature (cf. e.g. Chung [2]) characterizing those sequences $\{p_i\}$ for which $\{X_i\}$ is transient (resp. recurrent).

Theorem A: ([2], page 74) *Let X_n be a Markov chain with transition probabilities given in* (1.1). *Define*

$$U_i := \frac{1 - E_i}{E_i} = \frac{1/2 - p_i}{1/2 + p_i} \qquad i = 1, 2... \tag{1.2}$$

Then X_n is transient if and only if

$$\sum_{k=1}^{\infty} \prod_{i=1}^{k} U_i < \infty.$$

This criteria however does not reveal explicitly what are the transient/reccurent type of $\{p_i\}$ sequences. Lamperti [13], [15] proved a more general theorem about recurrence and transience of real nonnegative processes (not necessarily Markov chains). Here we spell out his result in our setup only, which easily follows from Theorem A as well.

Corollary: *If for all i large enough,*

$$p_i \le \frac{1}{4i} + O\left(\frac{1}{i^{1+\delta}}\right) \qquad \delta > 0, \tag{1.3}$$

then $\{X_i\}$ is recurrent. If instead, for some $\theta > 1$

$$p_i \ge \frac{\theta}{4i} \tag{1.4}$$

for i large enough, then $\{X_i\}$ is transient.

As we proceed to find the necessary tools for getting results about the local time, as a byproduct, we will get a much sharper version of this Corollary.

In this paper we concentrate only on the transient case.

There are many results in the literature about the limiting behavior of $\{X_n\}$, depending on the sequence $\{p_i\}$. Lamperti [14] determined the limiting distribution of X_n.

Theorem B: ([14]) *If* $\lim_{i\to\infty} ip_i = B/4 > 0$, *then*

$$\lim_{n\to\infty} \mathbf{P}\left(\frac{X_n}{\sqrt{n}} < x\right) = \frac{1}{2^{B/2-1/2}\Gamma(B/2+1/2)} \int_0^x u^B e^{-u^2/2}\, du.$$

In fact, Lamperti [14] (see also Rosenkrantz [17]) proved weak convergence of $X_n/\sqrt{n}$ to a Bessel process as well. We intend to give further connections (strong invariance, etc.) between X_n and Bessel process in a subsequent paper.

The law of the iterated logarithm for X_n was given by Brézis et al. [1], Székely [19], Gallardo [8], Voit [20]. Their somewhat more general results specialized in our setup, reads as follows.

Theorem C: ([1], [19], [8], [20])

If $\lim_{i\to\infty} ip_i = c > 0$, *then*

$$\limsup_{n\to\infty} \frac{X_n}{\sqrt{2n\log\log n}} = 1 \qquad \text{a.s.}$$

Voit [21] has proved a law of large numbers for certain Markov chains, which we quote in our setup only.

Theorem D: ([21]) *If* $\lim_{i\to\infty} i^\alpha p_i = c > 0$ *for some* $0<\alpha<1$, *then*

$$\lim_{n\to\infty} \frac{X_n}{n^{1/(1+\alpha)}} = 2c(1+\alpha) \qquad \text{a.s.}$$

Our main concern in this paper is to study the local time of $\{X_n\}$, defined by

$$\xi(x,n) := \#\{k : 0 \le k \le n,\ X_k = x\}, \quad x = 0,1,2,\ldots, \tag{1.5}$$

and

$$\xi(x,\infty) := \lim_{n\to\infty} \xi(x,n). \tag{1.6}$$

2. Lemmas and Notations

For U_i defined in (1.2) we get by elementary calculation that

Fact 1.

$$\begin{aligned} U_i &= \frac{1-E_i}{E_i} = \frac{1/2 - p_i}{1/2 + p_i} = 1 - 4p_i + O(p_i^2) \\ &= \exp(-4p_i + O(p_i^2)) \quad (i = 0, \pm1, \pm2, \ldots). \end{aligned} \tag{2.1}$$

Introduce the notation

$$D(m,n) := \begin{cases} 0 & \text{if } n = m, \\ 1 & \text{if } n = m+1, \\ 1 + \sum_{j=1}^{n-m-1} \prod_{i=1}^{j} U_{m+i} = & \\ 1 + \sum_{j=1}^{n-m-1} \exp\left(-(1+o_m(1))4 \sum_{i=m+1}^{m+j} p_i\right) & \text{if } n \ge m+2. \end{cases}$$

Denote

$$\lim_{n\to\infty} D(m,n) =: D(m,\infty).$$

Lemma 2.1. *If* $p_i \downarrow 0$ *when* $i \to \infty$, *then for* m *large enough*

$$D(m,\infty) \ge \frac{C}{p_m},$$

where C *is an absolute constant. Consequently,*

$$\lim_{m\to\infty} D(m,\infty) = +\infty.$$

Proof: Let $\frac{1}{p_m} \le j \le \frac{2}{p_m}$, then from (2.1) for m big enough we have

$$\sum_{i=m}^{m+j} (p_i + Cp_i^2) \le \frac{2}{p_m}(p_m + Cp_m^2) \le 2(1 + Cp_m) \le 2(1 + C).$$

Consequently

$$\exp\left(-\sum_{i=m}^{m+j}(p_i + Cp_i^2)\right) \ge \exp(-2(1+C))$$

and

$$\begin{aligned} D(m,\infty) &\ge 1 + \sum_{j=0}^{\infty} \exp\left(-\sum_{i=m}^{m+j}(p_i + Cp_i^2)\right) \\ &\ge 1 + \sum_{j=[\frac{1}{p_m}]}^{[\frac{2}{p_m}]} \exp(-2(1+C)) \ge 1 + \frac{1}{p_m}\exp(-2(1+C)). \end{aligned}$$

□

For $0 \le a \le b \le c$ define

$$\begin{aligned} &p(a,b,c) := \\ &= \mathbf{P}(\min\{j: \ j > m, \ X_j = a\} < \min\{j: \ j > m, \ X_j = c\} \mid X_m = b), \end{aligned}$$

i.e. $p(a,b,c)$ is the probability that a particle starting from b hits a before c.

Lemma 2.2. *For* $0 \le a \le b \le c$

$$p(a,b,c) = 1 - \frac{D(a,b)}{D(a,c)}.$$

Especially

$$p(0,1,n) = 1 - \frac{1}{D(0,n)}, \quad p(n,n+1,\infty) = 1 - \frac{1}{D(n,\infty)}. \tag{2.2}$$

Proof: The proof of this lemma is fairly standard, we give it for completeness. Clearly we have

$$\begin{aligned}
&p(a,a,c) = 1,\\
&p(a,c,c) = 0,\\
&p(a,b,c) = E_b p(a,b+1,c) + (1-E_b)p(a,b-1,c).
\end{aligned}$$

Consequently

$$p(a,b+1,c) = \frac{1}{E_b}p(a,b,c) - \frac{1-E_b}{E_b}p(a,b-1,c)$$

and

$$\begin{aligned}
p(a,b+1,c) - p(a,b,c) &= \frac{1-E_b}{E_b}(p(a,b,c) - p(a,b-1,c)) =\\
&= U_b(p(a,b,c) - p(a,b-1,c)).
\end{aligned}$$

By iteration we get

$$\begin{aligned}
&p(a,b+1,c) - p(a,b,c) =\\
&= U_b U_{b-1}(p(a,b-1,c) - p(a,b-2,c))\\
&= \ldots = U_b U_{b-1} \cdots U_{a+1}(p(a,a+1,c) - p(a,a,c)) =\\
&= U_b U_{b-1} \cdots U_{a+1}(p(a,a+1,c) - 1).
\end{aligned} \tag{2.3}$$

Starting with the trivial identity

$$p(a,a+1,c) - p(a,a,c) = p(a,a+1,c) - 1$$

and adding to it the above equations for $b = a+1, \ldots, c-1$ we get

$$-1 = p(a,c,c) - p(a,a,c) = D(a,c)(p(a,a+1,c) - 1),$$

i.e.

$$p(a,a+1,c) = 1 - \frac{1}{D(a,c)}. \tag{2.4}$$

Hence (2.3) and (2.4) imply

$$p(a,b+1,c) - p(a,b,c) = -\frac{1}{D(a,c)} U_b U_{b-1} \cdots U_{a+1}.$$

Adding these equations we obtain

$$\begin{aligned} p(a,b+1,c) - 1 = p(a,b+1,c) - p(a,a,c) &= \\ &= -\frac{1}{D(a,c)}(1 + U_{a+1} + U_{a+1}U_{a+2} + \cdots + U_{a+1}U_{a+2}\cdots U_b) = \\ &= -\frac{D(a,b+1)}{D(a,c)}. \end{aligned}$$

Hence we have the lemma. □

Introduce the following notations:

$$\begin{aligned} &\lambda(0,i) = 1, \\ &\lambda(1,i) = i, \\ &\lambda(2,i) = \lambda(1,i)\log i, \ldots, \\ &\lambda(k,i) = \lambda(k-1,i)\log_{k-1} i \quad (k = 3,4,\ldots), \end{aligned}$$

where

$$\begin{aligned} &\log_0 i = i, \\ &\log_1 i = \log i, \ldots, \\ &\log_k i = \log\log_{k-1} i, \end{aligned}$$

and

$$\begin{aligned} \Lambda(0,i) &= 0, \\ \Lambda(K,i) &= \sum_{k=1}^{K} \frac{1}{\lambda(k,i)}, \quad (K = 1,2,\ldots) \\ \Lambda(K,i,B) &= \Lambda(K-1,i) + \frac{B}{\lambda(K,i)} \quad (B > 0). \end{aligned}$$

Note that

$$
\begin{aligned}
\Lambda(1,i,B) &= \frac{B}{i},\\
\Lambda(2,i,B) &= \frac{1}{i} + \frac{B}{i\log i},\\
\Lambda(3,i,B) &= \frac{1}{i} + \frac{1}{i\log i} + \frac{B}{i\log i\log\log i}.
\end{aligned}
$$

For some $K = 1, 2, \ldots,\ B > 0$ fixed, define

$$
i_0 = \min\left\{ i : \frac{1}{4}\Lambda(K,i,B) < \frac{1}{2} \right\}
$$

and let

$$
p_i = \begin{cases} p_{i_0}, & \text{if} \quad 1 \le i \le i_0 \\ \frac{1}{4}\Lambda(K,i,B) & \text{if} \quad i > i_0. \end{cases} \tag{2.5}
$$

Now we are interested in the case $\{p_i\}$ above. In fact, in the future for convenience, when we say that

$$
p_i = \frac{1}{4}\Lambda(K,i,B)
$$

we actually mean that p_i is defined by (2.5).

Lemma 2.3. *Let* $p_i = \frac{1}{4}\Lambda(K,i,B)$. *Then*

$$
D(0,\infty) \begin{cases} = \infty & \text{if} \quad B \le 1, \\ < \infty & \text{if} \quad B > 1, \end{cases} \tag{2.6}
$$

$$
p(0,1,\infty) \begin{cases} = 1 & \text{if} \quad B \le 1, \\ < 1 & \text{if} \quad B > 1. \end{cases} \tag{2.7}
$$

For $n \ge m+2$, $B \ne 1$ *and* m *big enough*

$$
\begin{aligned}
D(m,n) = (1+o_m(1))\lambda(K-1,m)(\log_{K-1} m)^B \times \\
\times \frac{1}{B-1}\left(\frac{1}{(\log_{K-1} m)^{B-1}} - \frac{1}{(\log_{K-1} n)^{B-1}} \right).
\end{aligned} \tag{2.8}
$$

If $B > 1$,

$$
D(m,\infty) = (1+o_m(1))\frac{\lambda(K,m)}{B-1}, \tag{2.9}
$$

$$p(m,m+1,\infty) = 1-(1+o_m(1))\frac{(B-1)}{\lambda(K,m)}. \tag{2.10}$$

Proof: To prove (2.8), observe that from (2.1) we have for $n \geq m+2$

$$\begin{aligned} D(m,n) &= 1+\sum_{j=1}^{n-m-1}\prod_{i=1}^{j} U_{m+i} \qquad (2.11)\\ &= 1+\sum_{j=1}^{n-m-1}\exp\left(-\sum_{i=m+1}^{m+j}(\Lambda(K,i,B))\right)\exp\left(O(1)\sum_{i=m+1}^{m+j}\Lambda^2(K,i,B)\right)\\ &= 1+(1+o_m(1))\sum_{j=1}^{n-m-1}\exp\left(-\sum_{i=m+1}^{m+j}\Lambda(K,i,B)\right)\\ &=: 1+(1+o_m(1))A(m,n,K). \end{aligned}$$

Now we give a lower bound for $A(m,n,K)$.

$$\begin{aligned} A(m,n,K) &\geq \sum_{j=1}^{n-m-1}\exp\left(-\int_m^{m+j}\Lambda(K,x,B)\,dx\right) \qquad (2.12)\\ &= \sum_{j=1}^{n-m-1}\frac{\lambda(K-1,m)(\log_{K-1} m)^B}{\lambda(K-1,m+j)(\log_{K-1}(m+j))^B}\\ &= \lambda(K-1,m)(\log_{K-1} m)^B\sum_{\ell=m+1}^{n-1}\frac{1}{\lambda(K-1,\ell)(\log_{K-1}\ell)^B}\\ &\geq \lambda(K-1,m)(\log_{K-1} m)^B\int_{m+1}^{n}\frac{1}{\lambda(K-1,x)(\log_{K-1}x)^B}\,dx\\ &= \lambda(K-1,m)(\log_{K-1} m)^B\left(\frac{(\log_{K-1} m)^{1-B}-(\log_{K-1} n)^{1-B}}{B-1}\right). \end{aligned}$$

It is easy to see that the proof of the upper bound goes the same way, resulting the same expression as in (2.12) with m replaced by $m+1$ which combined with (2.11) proves (2.8). The proof of (2.6) is similar, and the rest of the lemma follows from these two. □

Consequence: If for any $K = 1, 2...$

$$p_i = \frac{\Lambda(K, i, B)}{4},$$

then the Markov chain is recurrent if $B \le 1$ and transient if $B > 1$.

Now we would like to consider the case when p_i is essentially $\frac{B}{4i^\alpha}$, which should be understood in the same way as it was defined in (2.5). Namely, let

$$i_0 = \min\left\{i : \frac{B}{4i^\alpha} < \frac{1}{2}\right\}$$

and let

$$p_i = \begin{cases} p_{i_0}, & if \quad 1 \le i \le i_0 \\ \frac{B}{4i^\alpha} & if \quad i > i_0. \end{cases} \tag{2.13}$$

Lemma 2.4. *In case* $p_i = \frac{B}{4i^\alpha}$ $(0 < \alpha < 1)$ *we have*

$$D(m, \infty) = (1 + o_m(1))\frac{m^\alpha}{B}, \tag{2.14}$$

$$1 - p(m, m+1, \infty) = (1 + o_m(1))\frac{B}{m^\alpha}. \tag{2.15}$$

Proof: Consider the case $0 < \alpha < 1/2$ first. By (2.1)

$$\prod_{i=1}^{j} U_{m+i} \le \exp\left(-B\sum_{\nu=m+1}^{m+j} \nu^{-\alpha} + \sum_{\nu=m+1}^{m+j} C\nu^{-2\alpha}\right) =$$

$$\le (1 + o_m(1))\exp\left(\frac{-B}{1-\alpha}\left[(m+j)^{1-\alpha} - (m)^{1-\alpha}\right] + \frac{C}{1-2\alpha}\left[(m+j)^{1-2\alpha} - (m)^{1-2\alpha}\right]\right).$$

Consequently,

$$\begin{aligned} &D(m, n) \\ &\le (1 + o_m(1))\exp\left(\frac{Bm^{1-\alpha}}{1-\alpha} - \frac{Cm^{1-2\alpha}}{1-2\alpha}\right)\sum_{k=m+1}^{n} \exp(\left(-\frac{Bk^{1-\alpha}}{1-\alpha} + \frac{Ck^{1-2\alpha}}{1-2\alpha}\right) \\ &\le (1 + o_m(1))\exp\left(\frac{Bm^{1-\alpha}}{1-\alpha} - \frac{Cm^{1-2\alpha}}{1-2\alpha}\right)\int_{m+1}^{n} \exp\left(-\frac{Bx^{1-\alpha}}{1-\alpha}(1 - C\frac{1-\alpha}{1-2\alpha}x^{-\alpha})\right)dx \\ &\le (1 + o_m(1))\exp\left(\frac{Bm^{1-\alpha}}{1-\alpha} - \frac{Cm^{1-2\alpha}}{1-2\alpha}\right)\int_{m+1}^{n} \exp\left(-\frac{Bx^{1-\alpha}}{1-\alpha}h_m\right)dx, \end{aligned}$$

where
$$h_m = 1 - C\frac{1-\alpha}{1-2\alpha}(m+1)^{-\alpha}.$$
In the calculation above C is a positive constant the value of which is not important. In the future we will use C, C^* or $C_1, C_2 \ldots$ for which this remark applies, and their values might change from line to line. Using substitution and the asymptotic representation of the incomplete Gamma function (see e.g. Gradsteyn and Ryzhik [9] page 942, formula (8.357))

$$\Gamma(\beta, x) = \int_x^\infty t^{\beta-1} e^{-t}\,dt = x^{\beta-1}e^{-x}\left(1 + \frac{O(1)}{x}\right) \qquad \text{as } x \to \infty,$$

we conclude that as $m \to \infty$

$$\begin{aligned} D(m,\infty) &\le \left(1 + O(\frac{1}{m^{1-\alpha}})\right)\frac{m^\alpha}{Bh_m}\exp\left(\frac{Bm^{1-\alpha}}{1-\alpha} - \frac{Cm^{1-2\alpha}}{1-2\alpha}\right)\exp\left(-\frac{h_m B m^{1-\alpha}}{1-\alpha}\right) \\ &= \left(1 + O(\frac{1}{m^{1-\alpha}})\right)\frac{m^\alpha}{B}. \end{aligned}$$

A similar calculation (which we omit) gives the same lower bound. The case of $\alpha = 1/2$ goes along the same lines with obvious modifications. On the other hand, the case $1/2 < \alpha < 1$ can be worked out similarly, but it is obvious with less precise calculations as well. □

Lemma 2.5. *In case* $p_i = \dfrac{B}{4(\log i)^\alpha}$ *with* $\alpha > 0$, *there exist* $0 < K_1 < K_2$ *such that*

$$K_1(\log m)^\alpha \le D(m,\infty) \le K_2(\log m)^\alpha, \tag{2.16}$$

$$1 - p(m, m+1, \infty) = \frac{O(1)}{(\log m)^\alpha}. \tag{2.17}$$

Proof: First we give the upper bound. For $m \ge m_0$

$$\begin{aligned} \sum_{i=m}^{m+j}\left(\frac{B}{(\log i)^\alpha} - \frac{C}{(\log i)^{2\alpha}}\right) &= \sum_{i=m}^{m+j}\frac{B}{(\log i)^\alpha}\left(1 - \frac{C^*}{(\log i)^\alpha}\right) \\ &\ge \sum_{i=m}^{m+j}\frac{B(1-\varepsilon)}{(\log i)^\alpha} =: A(m,j,\varepsilon). \end{aligned}$$

Then for
$$\ell(\log m)^\alpha \le j < (\ell+1)(\log m)^\alpha \quad (\ell = 0, 1, 2, \ldots)$$

we have

$$A(m,j,\varepsilon) \geq \frac{B(1-\varepsilon)\ell(\log m)^{\alpha}}{(\log[m+(\ell+1)(\log m)^{\alpha}])^{\alpha}} =: H(m,\ell,\alpha).$$

It is easy to see now, that if $(\ell+1)(\log m)^{\alpha} \leq m$ then for an appropriate C_1

$$H(m,\ell,\alpha) \geq \frac{B(1-\varepsilon)\ell(\log m)^{\alpha}}{(\log(2m))^{\alpha}} \geq C_1\ell.$$

On the other hand, if $(\ell+1)(\log m)^{\alpha} \geq m$, then for an appropriate C_2

$$H(m,\ell,\alpha) \geq \frac{B(1-\varepsilon)\ell(\log m)^{\alpha}}{(\log(2(\ell+1)(\log m)^{\alpha})^{\alpha}} \geq C_2\ell^{1/(2\alpha)}.$$

Then with $N = N(\alpha) := [\frac{m}{(\log m)^{\alpha}}] - 1$.

$$D(m,\infty) \leq \sum_{\ell=0}^{N} e^{-C_1\ell}(\log m)^{\alpha} + \sum_{\ell=N}^{\infty} e^{-C_2\ell^{\frac{1}{2\alpha}}}(\log m)^{\alpha} = O(1)(\log m)^{\alpha}.$$

The lower bound follows from Lemma 2.1. $\square$

3 Local time

We intend to study the limit properties of the local time $\xi(R,\infty)$ in case of transient random walks. To this end we also define the number of upcrossings by

$$\xi(R,n,\uparrow) := \#\{k:\ 0 \leq k \leq n,\ X_k = R,\ X_{k+1} = R+1\}. \tag{3.1}$$

$$\xi(R,\infty,\uparrow) := \lim_{n\to\infty} \xi(R,n,\uparrow). \tag{3.2}$$

It is a well known fact that the local time of a transient Markov chain follows geometric distribution. It was first formulated in the seminal paper of Erdős and Taylor ([7]) though they just stated it for random walk in dimension $d \geq 3$. So we could only calculate the parameter of this geometric distribution. However we still provide a proof for completeness.

Lemma 3.1. *For $R = 0,1,2,\ldots$*

$$\mathbf{P}(\xi(R,\infty) = L) = \frac{1+2p_R}{2D(R,\infty)}\left(1 - \frac{1+2p_R}{2D(R,\infty)}\right)^{L-1}, \quad L = 1,2,\ldots \tag{3.3}$$

Moreover, the sequence

$$\xi(R,\infty,\uparrow), \qquad R=0,1,2,\ldots$$

is a Markov chain and

$$\mathbf{P}(\xi(R,\infty,\uparrow)=L)=\frac{1}{D(R,\infty)}\left(1-\frac{1}{D(R,\infty)}\right)^{L-1}, \quad L=1,2,\ldots \tag{3.4}$$

Remark: Defining $p_0=1/2$ which is in agreement with our definition $E_0=1$ we see that the local time and the upcrossing of the site 0 have the same distribution.

Proof: Clearly we have for $L=1,2,\ldots$

$$\mathbf{P}(\xi(R,\infty)=L)=\left(\frac{1}{2}+p_R\right)(1-p(R,R+1,\infty))\times$$
$$\times\sum_{j=0}^{L-1}\binom{L-1}{j}\left(\frac{1}{2}-p_R\right)^j\left(\left(\frac{1}{2}+p_R\right)p(R,R+1,\infty)\right)^{L-j-1}=$$
$$=\left(\frac{1}{2}+p_R\right)(1-p(R,R+1,\infty))\times$$
$$\times\left(1-\left(\frac{1}{2}+p_R\right)(1-p(R,R+1,\infty))\right)^{L-1},$$

implying (3.3) by (2.2).

The other statements of the Lemma are obvious. □

Theorem 3.1. *If $p_R\to 0$, as $R\to\infty$, then*

$$\lim_{R\to\infty}\mathbf{P}\left(\frac{\xi(R,\infty)}{2D(R,\infty)}>x\right)=\lim_{R\to\infty}\mathbf{P}\left(\frac{\xi(R,\infty,\uparrow)}{D(R,\infty)}>x\right)=e^{-x},$$

that is to say, $\dfrac{\xi(R,\infty)}{2D(R,\infty)}$ *and* $\dfrac{\xi(R,\infty,\uparrow)}{D(R,\infty)}$ *have exponential limiting distributions.*

The proof is a trivial consequence of Lemma 3.1.

Theorem 3.2. *Assume that $p_R\to 0$ as $R\to\infty$. Then with probability 1 we have*

$$\xi(R,\infty)\le 2(1+\varepsilon)D(R,\infty)\log R \tag{3.5}$$

for any $\varepsilon>0$ if R is large enough.

Moreover,

$$\xi(R,\infty)\ge MD(R,\infty) \quad \text{i.o. a.s.} \tag{3.6}$$

for any $M>0$.

In case $p_R = \dfrac{\Lambda(K,R,B)}{4}$ with $B>1$, instead of (3.5) and (3.6) we have the much sharper

Theorem 3.3. *For* $p_R = \dfrac{\Lambda(K,R,B)}{4}$, $B>1$, *we have*

$$\limsup_{R\to\infty} \frac{\xi(R,\infty)}{2D(R,\infty)\log\log R} \leq 1. \tag{3.7}$$

and

$$\limsup_{R\to\infty} \frac{\xi(R,\infty)}{2D(R,\infty)\log_{K+1} R} \geq 1. \tag{3.8}$$

Especially in case $p_R = \dfrac{\Lambda(1,R,B)}{4} = \dfrac{B}{4R}$, $B>1$, *being* $D(R,\infty) = \dfrac{R}{B-1}$, *we have*

$$\limsup_{R\to\infty} \frac{(B-1)\xi(R,\infty)}{2R\log\log R} = \limsup_{R\to\infty} \frac{(B-1)\xi(R,\infty,\uparrow)}{R\log\log R} = 1. \tag{3.9}$$

Consequences:

- If $p_R = \dfrac{1}{4}\Lambda(K,R,B)$, $(B>1)$ then for any $\varepsilon>0$

  $$\xi(R,\infty) \leq \frac{2(1+\varepsilon)}{B-1}\lambda(K,R)\log\log R \qquad \text{a.s.} \tag{3.10}$$

  if R is large enough,

  $$\xi(R,\infty) \geq \frac{2(1-\varepsilon)}{B-1}\lambda(K,R)\log_{K+1} R \qquad \text{i.o. a.s.} \tag{3.11}$$

  and

  $$\lim_{R\to\infty} \mathbf{P}\left(\frac{B-1}{2\lambda(K,R)}\xi(R,\infty) > x\right) = e^{-x}. \tag{3.12}$$

- If $p_R = \dfrac{B}{4R^\alpha}$ $(0<\alpha<1)$, then

  $$\xi(R,\infty) \leq \frac{2}{B}(1+\varepsilon)R^\alpha \log R \quad \text{a.s.} \tag{3.13}$$

  if R is large enough,

$$\xi(R,\infty) \geq MR^{\alpha} \quad \text{i.o. a.s.} \tag{3.14}$$

for any $M > 0$ and

$$\lim_{R\to\infty} \mathbf{P}\left(\frac{B\,\xi(R,\infty)}{2R^{\alpha}} > x\right) = e^{-x}. \tag{3.15}$$

- If $p_R = \dfrac{B}{4(\log R)^{\alpha}} \quad (\alpha > 0)$, then

$$\xi(R,\infty) \leq O(1)(\log R)^{1+\alpha} \quad \text{a.s.} \tag{3.16}$$

if R is large enough and

$$\xi(R,\infty) \geq M(\log R)^{\alpha} \quad \text{i.o. a.s.} \tag{3.17}$$

for any $M > 0$.

Proof of Theorem 3.2: (3.5) follows from Lemma 3.1. On the other hand, (3.3) also implies that for any $M > 0$

$$\liminf_{R\to\infty} \mathbf{P}\left(\xi(R,\infty) \geq MD(R,\infty)\right) > 0$$

from which it follows that

$$\mathbf{P}\left(\xi(R,\infty) \geq MD(R,\infty) \text{ i.o.}\right) > 0.$$

Now to finish our proof we need to apply the zero-one law (in a non-independent setup) exactly in the same way as in Doob [6] page 103, observing that the conditional probability of our tail event given the first n steps of our walk is the same as its unconditional probability, that is for any $n = 1, 2, \ldots$

$$\mathbf{P}\left(\xi(R,\infty) \geq MD(R,\infty) \text{ i.o.} \mid X_1, X_2, \ldots X_n\right) = \mathbf{P}\left(\xi(R,\infty) \geq MD(R,\infty) \text{ i.o.}\right).$$

According to the zero-one law mentioned above, the conditional probability on the left-hand side is either zero or one, and since the unconditional probability on the right-hand side is strictly positive, it should be equal to one, which, in turn, implies (3.6).

Proof of Theorem 3.3: To prove (3.7), we need a few lemmas. Recall the definition of the upcrossing in (3.1) and (3.2). For large values of the local time and upcrossing we have the following invariance principle.

Lemma 3.2. *As $R \to \infty$*

$$\xi(R,\infty) - 2\xi(R,\infty,\uparrow) = O((D(R,\infty)\log R)^{1/2+\varepsilon} + p_R D(R,\infty)\log R) \quad \text{a.s.} \tag{3.18}$$

Proof: Under the condition $\xi(R,\infty) = L$, $\xi(R,\infty,\uparrow) - 1$ has binomial distribution with parameters $(L-1, 1/2 + p_R)$. According to Hoeffding inequality,

$$\mathbf{P}\left(\left|\xi(R,\infty,\uparrow) - 1 - \left(\frac{1}{2} + p_R\right)(L-1)\right| \geq u(L-1)^{1/2}\right) \leq e^{-Cu^2}$$

with some $C > 0$, from which as $L \to \infty$,

$$\xi(R,\infty,\uparrow) - \frac{L}{2} = O(L^{1/2+\varepsilon} + Lp_R) \quad \text{a.s.}$$

Putting $L = \xi(R,\infty)$, we get (3.18) from (3.5).

Lemma 3.3. *Let*

$$\gamma_R = \left(\frac{1}{2} + p_R\right) p(R, R+1, \infty),$$

and

$$c_R = \frac{\gamma_R}{1 - \gamma_R}.$$

Then

$$\zeta(R) := \frac{\xi(R,\infty,\uparrow)}{c_1 \cdots c_R}, \quad R = 1, 2, \ldots$$

is a submartingale.

Proof: Let T_R be the first hitting time of R by $\{X_n\}$, e.g. $T_R = \min\{n : X_n = R\}$. Then we have

$$\mathbf{P}_R(\xi(R, T_{R-1}, \uparrow) = j, T_{R-1} < \infty) = \left(\frac{1}{2} - p_R\right)\gamma_R^j, \quad j = 0, 1, \ldots, \tag{3.19}$$

$$\mathbf{P}_R(\xi(R,\infty,\uparrow) = j, T_{R-1} = \infty) = \left(\frac{1}{2} + p_R - \gamma_R\right)\gamma_R^{j-1}, \quad j = 1, 2, \ldots \tag{3.20}$$

Observe that

$$\xi(R,\infty,\uparrow) = \sum_{m=1}^{\xi(R-1,\infty,\uparrow)-1} \xi_m + \widetilde{\xi},$$

where $\xi_m,\ m=1,2...$ has distribution (3.19) and $\widetilde{\xi}$ has distribution (3.20). Then

$$\mathbf{E}(e^{\lambda\xi(R,\infty,\uparrow)},\, \xi(R-1,\infty,\uparrow)=i) = (\mathbf{E}(e^{\lambda\xi_1}))^{i-1}\mathbf{E}(e^{\lambda\widetilde{\xi}}) \tag{3.21}$$

$$= \frac{\left(\frac{1}{2}+p_R-\gamma_R\right)e^{\lambda}\left(\frac{1}{2}-p_R\right)^{i-1}}{(1-\gamma_R e^{\lambda})^i},$$

hence

$$\mathbf{E}(e^{\lambda\xi(R,\infty,\uparrow)} \mid \xi(R-1,\infty,\uparrow)=i) = e^{\lambda}\left(\frac{1-\gamma_R}{1-\gamma_R e^{\lambda}}\right)^i,$$

from which

$$\mathbf{E}(\xi(R,\infty,\uparrow) \mid \xi(R-1,\infty,\uparrow)) = c_R\xi(R-1,\infty,\uparrow)+1, \tag{3.22}$$

which easily implies the lemma. □

Now we prove the upper bound, i.e.

$$\limsup_{R\to\infty}\frac{\xi(R,\infty,\uparrow)}{D(R,\infty)\log\log R} \le 1 \qquad \text{a.s.}, \tag{3.23}$$

which also implies (3.7) by Lemma 3.2.

With an easy calculation we get from (3.21) that

$$\mathbf{E}(e^{\lambda\xi(R,\infty,\uparrow)}) = \frac{e^{\lambda}}{D(R,\infty)-e^{\lambda}(D(R,\infty)-1)}. \tag{3.24}$$

Using that $\zeta(R)$ is submartingale, from (3.24) we have with $R_k=[\exp(k/\log k)]$, $C_k = c_1c_2\dots c_{R_k}$,

$$u_k = (1+\varepsilon)D(R_k,\infty)\log\log R_k,$$

$$\mathbf{P}\left(\max_{R_k\le R<R_{k+1}}\zeta(R)\ge\frac{u_{k+1}}{C_{k+1}}\right)$$
$$\le \exp(-\lambda u_{k+1}/C_{k+1})\mathbf{E}(\exp(\lambda\zeta(R_{k+1})))$$
$$= \frac{\exp(\lambda/C_{k+1})(1-u_{k+1})}{D(R_{k+1},\infty)-\exp(\lambda/C_{k+1})(D(R_{k+1},\infty)-1)}.$$

It can be seen that the optimal choice for λ is given by

$$\exp(\lambda/C_{k+1}) = \frac{(u_{k+1}-1)D(R_{k+1},\infty)}{u_{k+1}(D(R_{k+1},\infty)-1)},$$

and we get finally

$$\mathbf{P}\left(\max_{R_k\le R<R_{k+1}}\zeta(R)\ge \frac{u_{k+1}}{C_{k+1}}\right)=\frac{O(1)\log\log R_{k+1}}{(\log R_{k+1})^{1+\varepsilon}}.$$

Hence by Borel-Cantelli lemma for large k and $R_k\le R<R_{k+1}$ we have

$$\zeta(R)\le \frac{(1+\varepsilon)D(R,\infty)\log\log R}{c_1\cdots c_R c_{R+1}\cdots c_{R_{k+1}}},$$

i.e.

$$\xi(R,\infty,\uparrow)\le \frac{(1+\varepsilon)D(R,\infty)\log\log R}{c_{R+1}\cdots c_{R_{k+1}}}.$$

If $p_R=\Lambda(K,R,B)/4$, then as $R\to\infty$ (cf. (2.9))

$$D(R,\infty)\sim\frac{\lambda(K,R)}{B-1}$$

and

$$c_R\sim\frac{1+2p_R-1/D(R,\infty)}{1-2p_R+1/D(R,\infty)}\sim\exp(4p_R-2/D(R,\infty))\sim\exp\left(\Lambda(K,R,B)-\frac{2(B-1)}{\lambda(K,R)}\right).$$

If $K=1$, then

$$\Lambda(1,R,B)-\frac{2(B-1)}{\lambda(1,R)}\sim\frac{2-B}{R},\quad B\neq 2$$

and

$$\Lambda(1,R,2)-\frac{2}{\lambda(1,R)}=\frac{o(1)}{R},$$

and if $K>1$, then

$$\Lambda(K,R,B)-\frac{2(B-1)}{\lambda(K,R)}\sim\frac{1}{R}.$$

Hence for large k and $R_k\le R\le R_{k+1}$ we have

$$c_{R+1}\cdots c_{R_{k+1}}\sim\exp\left(C\log\frac{R_{k+1}}{R}\right)$$

with some constant C if $K=1$, $B\neq 2$ or $K>1$ and $C=o(1)$ if $K=1$, $B=2$. In view of $\lim_{k\to\infty}R_{k+1}/R_k=1$, for any $\varepsilon>0$, one can choose k large enough such that

$$c_{R+1}\cdots c_{R_{k+1}}\ge 1-\varepsilon,$$

i.e.

$$\xi(R,\infty,\uparrow) \leq \frac{(1+\varepsilon)D(R,\infty)\log\log R}{1-\varepsilon}.$$

Since $\varepsilon > 0$ is arbitrary, (3.23) follows.

To prove the lower bound (3.8), consider an increasing sequence of sites R_k to be determined later. Let

$$\tau_k = \min\{n : X_n = R_k\},$$

the time of the first visit at the site R_k, and define

$$Z(k) := \xi(R_k, \tau_{k+1}).$$

Observe that $\{Z(k),\ k = 1, 2...\}$ are independent. Following the proof of Lemma 3.1 we can conclude that

$$\begin{aligned} &\mathbf{P}(Z(k) \geq L) = (1 + o_{R_k}(1))\times \\ &\times \left[\left(1 - \frac{1}{2}(1 - p(R_k, R_k+1, R_{k+1}))\right)(1 + O((1 - p(R_k, R_k+1, R_{k+1}))p_{R_k}))\right]^{L-1} \end{aligned} \tag{3.25}$$

Based on (2.8) it is easy to calculate that

$$\begin{aligned} D(R_k, R_{k+1}) &= (1 + o_{R_k}(1))\frac{\lambda(K-1, R_k)}{B-1}\log_{K-1} R_k\left(1 - \left(\frac{\log_{K-1} R_k}{\log_{K-1} R_{k+1}}\right)\right) = \\ &= (1 + o_{R_k}(1))\frac{\lambda(K, R_k)}{B-1}\left(1 - \left(\frac{\log_{K-1} R_k}{\log_{K-1} R_{k+1}}\right)\right). \end{aligned} \tag{3.26}$$

Define the sequence R_k by

$$\log_K R_k := k \log Q$$

with some $Q > 1$ (we intentionally forget about the technicalities arising from the fact that the sites should be integers). It is easy to see that with this choice of R_k

$$\frac{\log_{K-1} R_k}{\log_{K-1} R_{k+1}} = \frac{1}{Q}.$$

Let

$$L(k) = 2\frac{\lambda(K, R_k)}{B-1}\frac{Q-1}{Q}\log_{K+1} R_k.$$

From (2.4) we get that

$$\mathbf{P}(Z(k) \geq L(k)) \sim \exp(-\log_{K+1} R_k) = \frac{1}{\log_K R_k} = \frac{1}{k \log Q}.$$

Applying Borel-Cantelli lemma and then letting $Q \to \infty$, we get (3.8). $\square$

Our next issue was to investigate how small could be the local time of our process. More precisely we wanted to know whether it is true that in the transient case there are always infinitely many sites with local time equal to 1. In fact we managed to prove in some sense much more, and in some sense much less. Namely, we prove the following two theorems. Define for $N \geq 2$

$$f(N,R) = f(N,R,\varepsilon) = \frac{1}{\log 2}\left(\sum_{j=2}^{N} \log_j R + \varepsilon \log_N R\right)$$

and

$$g(N,R) = f(N,R,0).$$

Theorem 3.4. *Let* $p_R = \dfrac{\Lambda(1,R,B)}{4}$ *with* $B > 1$ *and let* $N \geq 2$. *Then*

- *with probability* 1 *there exist infinitely many* R *for which*

$$\xi(R+j,\infty) = 1$$

  for each $j = 0,1,2,\ldots,[g(N,R)]$.

- *with probability* 1 *for any* $\varepsilon > 0$ *and* R *large enough there exists an* S

$$R \leq S \leq f(N,R,\varepsilon)$$

  such that

$$\xi(S,\infty) > 1.$$

Let

$$f^*(R,\epsilon) = \frac{(1+\varepsilon)(1-\alpha)\log R}{\log 2} \qquad \text{and} \qquad g^*(R) = f^*(R,0).$$

Theorem 3.5. *Let* $p_R = \dfrac{B}{4R^\alpha}$ $(0 < \alpha < 1)$. *Then*

- *with probability* 1 *there exists infinitely many* R *for which*

$$\xi(R+j,\infty)=1$$

  for each $j=0,1,2,\dots,g^*(R)$.

- *with probability* 1 *for each* R *large enough and* $\varepsilon>0$ *there exists an* S,

$$R\le S\le f^*(R,\varepsilon)$$

  such that

$$\xi(S,\infty)>1.$$

Furthermore, we conjecture that for $p_i \ge B/(4i)$, where $B>1$, with probability 1 there are always infinitely many sites with local time 1. On the other hand, recently James et al. [10] proved that for $p_i \sim \Lambda(2,i,B)$ with $B>1$ with probability 1 there are only finitely many cutpoints, hence finitely many points with local time 1. We note that it can be seen with a similar argument that this is the case for $p_i \sim \Lambda(K,i,B)$ for all $K\ge 2$ as well.

Proof of Theorem 3.4: At first we prove the second statement. Recall the notation of $\lambda(N,R)$ and observe that

$$R2^{g(N,R)}=\lambda(N,R) \quad \text{and} \quad R2^{f(N,R)}=\lambda(N-1,R)(\log_{N-1}R)^{1+\epsilon}. \tag{3.27}$$

Now the proof of the second statement is a trivial consequence of

Lemma 3.4. *For every* $N\ge 2$ *integer as* $R\to\infty$

$$\begin{aligned}
&\mathbf{P}\left\{\bigcap_{j=1}^{f(N,R)}\{\xi(R+j,\infty)=1\}\right\}=\\
&=\prod_{j=1}^{f(N,R)}\left(\frac{1}{2}+\frac{B}{4(R+j)}\right)(1-p(R+f(N,R),R+f(N,R)+1,\infty))=\\
&=(1+o_R(1))\frac{1}{2^{f(N,R)}}\frac{B-1}{R}=(1+o_R(1))\frac{B-1}{\lambda(N-1,R)(\log_{N-1}R)^{1+\varepsilon}}.
\end{aligned}$$

Proof: Obvious by (2.10). □

The proof of the first statement of the theorem is based on the following

Lemma 3.5. *For every $N \geq 2$ integer as $R \to \infty$*

$$\mathbf{P}\left\{\bigcap_{j=1}^{g(N,R)} \{\xi(R+j,\infty)=1\}\right\} = \frac{O(1)}{\lambda(N,R)}, \tag{3.28}$$

$$\begin{aligned} &\mathcal{P} := \mathcal{P}(N,R,S) := \\ &= \mathbf{P}\left\{\bigcap_{j=1}^{g(N,R)} \{\xi(R+j,\infty)=1\} \cap \bigcap_{j=1}^{g(N,S)} \{\xi(S+j,\infty)=1\}\right\} \leq \\ &\leq \begin{cases} \dfrac{(1+o_R(1))(B-1)^2}{\lambda(N,R)\lambda(N,S-R)} & \text{if } S \geq R+g(N,R), \\ \dfrac{O(1)2^R}{2^{S+g(S,N)}}\dfrac{B-1}{S+g(N,S)} & \text{if } R < S < R+g(N,R). \end{cases} \end{aligned} \tag{3.29}$$

Proof: (3.28) follows from Lemma 3.1 and (3.27). In case $R < S < R+g(N,R)$ we have

$$\begin{aligned} \mathcal{P} &= \prod_{i=R}^{S+g(N,S)} \left(\frac{1}{2}+\frac{B}{4i}\right)(1-p(S+g(N,S),S+g(N,S)+1,\infty)) \leq \\ &\leq O(1)\frac{1}{2^{S+g(N,S)-R}}\frac{B-1}{S+g(N,S)}. \end{aligned}$$

In case $S > R+g(N,R)$ we have

$$\begin{aligned} \mathcal{P} &= (1+o_R(1))\frac{1}{2^{g(N,R)}}(1-p(R,R+1,S))\frac{1}{2^{g(N,S)}}(1-p(S,S+1,\infty)) = \\ &= (1+o_R(1))\frac{1}{2^{g(N,R)}\,2^{g(N,S)}}\frac{B-1}{R^B}\frac{1}{R^{1-B}-S^{1-B}}\frac{B-1}{S} = \\ &= (1+o_R(1))\frac{1}{2^{g(N,R)}\,2^{g(N,S)}}\frac{(B-1)^2}{R}\frac{S^{B-2}}{S^{B-1}-R^{B-1}} \leq \\ &\leq (1+o_R(1))\frac{1}{2^{g(N,R)}\,2^{g(N,S-R)}}\frac{(B-1)^2}{R}\frac{1}{(S-R)} \leq \\ &\leq (1+o_R(1))\frac{(B-1)^2}{\lambda(N,R)\lambda(N,S-R)}. \end{aligned}$$

Hence we have the second statement of the lemma. □

Now we turn to the proof of the first statement of the theorem. Let

$$A(R) = \bigcap_{j=1}^{g(N,R)} \{\xi(R+j,\infty)=1\}.$$

Then by (3.28)

$$\sum_{R=1}^{T} \mathbf{P}(A(R)) = O(1)\log_N T \tag{3.30}$$

and

$$\begin{aligned}&\sum_{R=1}^{T}\sum_{S=R+1}^{T} \mathbf{P}(A(R)A(S)) = \\ &= \sum_{R=1}^{T}\sum_{S=R+1}^{R+g(N,R)} \mathbf{P}(A(R)A(S)) + \sum_{R=1}^{T}\sum_{S=R+g(N,R)+1}^{T} \mathbf{P}(A(R)A(S)) =: I+II.\end{aligned}$$

By (3.29) we have

$$\begin{aligned} I &\le O(1)\sum_{R=1}^{T}\sum_{S=R+1}^{R+g(N,R)} \frac{2^R}{2^{S+g(N,S)}}\frac{1}{S+g(N,S)} \le \\ &\le O(1)\sum_{R=1}^{T}\frac{1}{(R+g(N,R))2^{g(N,R)}}\sum_{j=1}^{g(N,R)}\frac{1}{2^j} \le \\ &\le O(1)\sum_{R=1}^{T}\frac{1}{R2^{g(N,R)}} = O(1)\sum_{R=1}^{T}\frac{1}{\lambda(N,R)} \le O(1)(\log_N T)\end{aligned} \tag{3.31}$$

and

$$\begin{aligned} II &\le O(1)\sum_{R=1}^{T}\sum_{S=R+g(N,R)+1}^{T}\frac{1}{\lambda(N,R)}\frac{1}{\lambda(N,S-R)} \le \\ &\le O(1)(\log_N T)^2.\end{aligned} \tag{3.32}$$

By (3.28) and (3.29)

$$\sum_{R=1}^{T} \sum_{S=R+1}^{T} \mathbf{P}(A(R)A(S)) \le O(1)(\log_N T)^2. \tag{3.33}$$

(3.30), (3.33) and the Kochen-Stone Borel–Cantelli lemma (see e.g. Spitzer [18], page 317) imply the first statement with positive probability. Now to finish our proof we need to apply the zero-one law (again in a non-independent set up) as in the proof of Theorem 3.2, observing that for any $n = 1, 2, \ldots$

$$\mathbf{P}(A(R) \text{ i.o.} \mid X_1, X_2, ...X_n) = \mathbf{P}(A(R) \text{ i.o.}).$$

□

Proof of Theorem 3.5: The proof goes along the same line as the proof of Theorem 3.4. The only point which needs a little different approach is the proof of the counterpart of Lemma 3.5. Namely, in the proof of this lemma we need an upper bound for $1-p(R, R+1, S)$, which is equivalent of getting a lower bound for $D(R, S)$. Observe that in Lemma 2.4 we have an asymptotic formula for $D(R, \infty)$. Now to get a lower bound for $D(R, S)$ we need a less precise calculation (the statement of Theorem 3.5 does not depend on B, which was important in Lemma 2.4). It is enough to observe that

$$U_i \ge C \exp\left(-\frac{B^*}{i^\alpha}\right)$$

with an appropriate choice of C and $B^* > B$. After this observation, with some tedious calculation, somewhat similar to Lemma 2.4, we get that

$$D(R, S) \ge CR^\alpha \left(1 - \left(\frac{S}{R}\right)^\alpha \exp\left(C_1(R^{1-\alpha} - S^{1-\alpha})\right)\right). \tag{3.34}$$

It is easy to see

$$D(R, S) \ge C_2 R^\alpha$$

if $S \ge R + R^\alpha / \log R$. On the other hand, if $R < S < R + R^\alpha / \log R$ then it can be seen that

$$D(R, S) > C_3(S - R)$$

and this is enough to carry through the argument in Lemma 3.5. We omit the details. □

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