Joint asymptotic behavior of local and occupation times of random walk in higher dimension

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Abstract: Considering a simple symmetric random walk in dimension $d \geq 3$, we study the almost sure joint asymptotic behavior of two objects: first the local times of a pair of neighboring points, then the local time of a point and the occupation time of the surface of the unit ball around it.

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1. Introduction and main results

Consider a simple symmetric random walk $\{\mathbf{S}_n\}_{n=1}^{\infty}$ starting at the origin $\mathbf{0}$ on the d-dimensional integer lattice $\mathcal{Z}_d$, i.e. $\mathbf{S}_0 = \mathbf{0}$, $\mathbf{S}_n = \sum_{k=1}^{n} \mathbf{X}_k$, $n = 1, 2, \dots$, where $\mathbf{X}_k$, $k = 1, 2, \dots$ are i.i.d. random variables with distribution

$$\mathbf{P}(\mathbf{X}_1 = \mathbf{e}_i) = \frac{1}{2d}, \qquad i = 1, 2, \dots, 2d$$

and $\{\mathbf{e}_1, \mathbf{e}_2, ...\mathbf{e}_d\}$ is a system of orthogonal unit vectors in $\mathcal{Z}_d$ and $\mathbf{e}_{d+j} = -\mathbf{e}_j$, $j = 1, 2, \dots, d$. Define the local time of the walk by

$$\xi(\mathbf{z}, n) := \#\{k : \ 0 < k \le n, \ \mathbf{S}_k = \mathbf{z}\}, \quad n = 1, 2, \dots, \tag{1.1}$$

where $\mathbf{z}$ is any lattice point of $\mathcal{Z}_d$. The maximal local time of the walk is defined as

$$\xi(n) := \max_{\mathbf{z} \in \mathcal{Z}_d} \xi(\mathbf{z}, n). \tag{1.2}$$

Define also the following quantities:

$$\eta(n) := \max_{0 \le k \le n} \xi(\mathbf{S}_k, \infty), \tag{1.3}$$

$$Q(k, n) := \#\{\mathbf{z} : \ \mathbf{z} \in \mathcal{Z}_d, \ \xi(\mathbf{z}, n) = k\}, \tag{1.4}$$

$$\begin{aligned} U(k, n) &:= \#\{j : \ 0 < j \le n, \ \xi(\mathbf{S}_j, \infty) = k, \ \mathbf{S}_j \ne \mathbf{S}_\ell \ (\ell = 1, 2, \dots, j-1)\} \\ &= \#\{\mathbf{z} \in \mathcal{Z}_d : \ 0 < \xi(\mathbf{z}, n) \le \xi(\mathbf{z}, \infty) = k\}. \end{aligned} \tag{1.5}$$

Denote by $\gamma(n) = \gamma(n; d)$ the probability that in the first $n-1$ steps the d-dimensional path does not return to the origin. Then

$$1 = \gamma(1) \ge \gamma(2) \ge ... \ge \gamma(n) \ge ... > 0. \tag{1.6}$$

It was proved in [3] that

Theorem A (Dvoretzky and Erdős [3]) *For* $d \ge 3$

$$\lim_{n \to \infty} \gamma(n) = \gamma = \gamma(\infty; d) > 0, \tag{1.7}$$

and

$$\gamma < \gamma(n) < \gamma + O(n^{1-d/2}). \tag{1.8}$$

Consequently

$$\mathbf{P}(\xi(\mathbf{0},n)=0,\,\xi(\mathbf{0},\infty)>0)=O\left(n^{1-d/2}\right) \tag{1.9}$$

as $n\to\infty$.

So γ is the probability that the d-dimensional simple symmetric random walk never returns to its starting point.

Let $\xi(\mathbf{z},\infty)$ be the total local time at $\mathbf{z}$ of the infinite path in $\mathcal{Z}_d$. Then for $d\geq 3$ (see Erdős and Taylor [4]) $\xi(\mathbf{0},\infty)$ has geometric distribution:

$$\mathbf{P}(\xi(\mathbf{0},\infty)=k)=\gamma(1-\gamma)^k,\qquad k=0,1,2,... \tag{1.10}$$

Erdős and Taylor [4] proved the following strong law for the maximal local time:

Theorem B (Erdős and Taylor [4]) *For* $d\geq 3$

$$\lim_{n\to\infty}\frac{\xi(n)}{\log n}=\lambda\qquad \text{a.s.}, \tag{1.11}$$

where

$$\lambda=\lambda_d=-\frac{1}{\log(1-\gamma)}. \tag{1.12}$$

Following the proof of Erdős and Taylor, without any new idea, one can prove that

$$\lim_{n\to\infty}\frac{\eta(n)}{\log n}=\lambda\qquad \text{a.s.} \tag{1.13}$$

Erdős and Taylor [4] also investigated the properties of $Q(k,n)$. They proved

Theorem C (Erdős and Taylor [4]) *For* $d\geq 3$ *and for any* $k=1,2,\dots$

$$\lim_{n\to\infty}\frac{Q(k,n)}{n}=\gamma^2(1-\gamma)^{k-1}\qquad \text{a.s.} \tag{1.14}$$

Pitt [8] proved (1.14) for general random walk and Hamana [5], [6] proved central limit theorems for $Q(k,n)$.

In [1] we studied the question whether k can be replaced by a sequence $t(n)=t_n\nearrow\infty$ of positive integers in (1.14). Let

$$\psi(n)=\psi(n,B)=\lambda\log n-\lambda B\log\log n. \tag{1.15}$$

Theorem D *Let $d \geq 3$, $\mu(t) := \gamma(1-\gamma)^{t-1}$ and $t_n := [\psi(n, B)]$, $(B > 2)$, where $\psi(n, B)$ is defined by* (1.15). *Then we have*

$$\lim_{n\to\infty} \sup_{t\leq t_n} \left| \frac{Q(t,n)}{n\gamma\mu(t)} - 1 \right| = \lim_{n\to\infty} \sup_{t\leq t_n} \left| \frac{U(t,n)}{n\gamma\mu(t)} - 1 \right| = 0 \qquad \text{a.s.}$$

Here in $\sup_{t\leq t_n}$, t runs through positive integers.

For a set $A \subset \mathcal{Z}_d$ the occupation time of A is defined by

$$\Xi(A, n) := \sum_{\mathbf{z}\in A} \xi(\mathbf{z}, n). \tag{1.16}$$

Consider the translates of A, i.e. $A + \mathbf{u} = \{\mathbf{z} + \mathbf{u} : \mathbf{z} \in A\}$ with $\mathbf{u} \in \mathcal{Z}_d$ and define the maximum occupation time by

$$\Xi^*(A, n) := \sup_{\mathbf{u}\in\mathcal{Z}_d} \Xi(A + \mathbf{u}, n). \tag{1.17}$$

It was shown in [2]

Theorem E *For $d \geq 3$ and for any fixed finite set $A \subset \mathcal{Z}_d$*

$$\lim_{n\to\infty} \frac{\Xi^*(A, n)}{\log n} = c_A \qquad \text{a.s.} \tag{1.18}$$

with some positive constant c_A, depending on A.

Now we present some more notations. For $\mathbf{z} \in \mathcal{Z}_d$ let $T_\mathbf{z}$ be the first hitting time of $\mathbf{z}$, i.e. $T_\mathbf{z} := \min\{i \geq 1 : \mathbf{S}_i = \mathbf{z}\}$ with the convention that $T_\mathbf{z} = \infty$ if there is no i with $\mathbf{S}_i = \mathbf{z}$. Let $T = T_\mathbf{0}$. In general, for a subset A of $\mathcal{Z}_d$, let T_A denote the first time the random walk visits A, i.e. $T_A := \min\{i \geq 1 : \mathbf{S}_i \in A\} = \min_{\mathbf{z}\in A} T_\mathbf{z}$. Let $\mathbf{P}_\mathbf{z}(\cdot)$ denote the probability of the event in the bracket under the condition that the random walk starts from $\mathbf{z} \in \mathcal{Z}_d$. We denote $\mathbf{P}(\cdot) = \mathbf{P}_\mathbf{0}(\cdot)$. Define

$$\gamma_\mathbf{z} := \mathbf{P}(T_\mathbf{z} = \infty). \tag{1.19}$$

Let $\mathcal{S}(r)$ be the surface of the ball of radius r centered at the origin, i.e.

$$\mathcal{S}(r) := \{\mathbf{z} \in \mathcal{Z}_d : \|\mathbf{z}\| = r\},$$

where $\|\cdot\|$ is the Euclidean norm. Denote

$$\Xi(\mathbf{z}, n) := \Xi(\mathcal{S}(1) + \mathbf{z}, n),$$

i.e. the occupation time of the surface of the unit ball centered at $\mathbf{z} \in \mathcal{Z}_d$.

Introduce further

$$p := \mathbf{P}_{\mathbf{e}_1}(T_{\mathcal{S}(1)} < T). \tag{1.20}$$

In words, p is the probability that the random walk, starting from $\mathbf{e}_1$ (or any other points of $\mathcal{S}(1)$), returns to $\mathcal{S}(1)$ before reaching $\mathbf{0}$ (including the case $T_{\mathcal{S}(1)} < T = \infty$).

In particular it was shown in [2]

$$\lim_{n\to\infty} \frac{\sup_{\mathbf{z}\in\mathcal{Z}_d} \Xi(\mathbf{z}, n)}{\log n} = \frac{1}{-\log\left(p + \frac{1}{2d}\right)} =: \kappa \qquad \text{a.s.} \tag{1.21}$$

It is easy to see that Theorem D implies

Consequence 1.1 *With probability 1 there exists a random variable n_0 such that if $n \geq n_0$ then for all $k = 1, 2 \ldots, \psi(n, B)$ there exist*

(i) $\mathbf{z} \in \mathcal{Z}_d$ *such that* $\xi(\mathbf{z}, n) = k$,

(ii) $j \leq n$ *such that* $\xi(\mathbf{S}_j, \infty) = k$.

It would be interesting to investigate the joint behavior of the local time of a point and the occupation time of a set, but in general this seems to be a very complicated question so we will deal only with the following two special cases. We will consider the joint behavior of the local times of two neighboring points, and the local time of a point and the occupation time of a ball of radius 1 centered at the point. Concerning the first question one might like to know whether it is possible that in two neighboring points the local times are simultaneously around $\lambda \log n$. More generally, we might ask whether the pairs of possible values of

$$(\xi(\mathbf{z}, n),\ \xi(\mathbf{z} + \mathbf{e}_i, n)) \tag{1.22}$$

completely fill the lattice points in the set $(\log n)\mathcal{A}$ where $\mathcal{A}$ is defined as

$$\mathcal{A} := \{(x, y) \in \mathcal{Z}_d :\ 0 \leq x \leq \lambda,\ 0 \leq y \leq \lambda\}.$$

The answer for this question turns out to be negative. However we will prove that for

$$\mathcal{B} := \{y \geq 0, x \geq 0 :\ -(x + y)\log(y + x) + x\log x + y\log y - (x + y)\log\alpha \leq 1\}, \tag{1.23}$$

where

$$\alpha := \frac{1 - \gamma}{2 - \gamma}$$

we have

Theorem 1.1. *Let $d \geq 4$. For each $\varepsilon > 0$, with probability 1 there exists an $n_0 = n_0(\varepsilon)$ such that if $n \geq n_0$ then*

(i) $(\xi(\mathbf{z}, n), \xi(\mathbf{z} + \mathbf{e}_i, n)) \in ((1+\varepsilon)\log n)\mathcal{B}, \qquad \forall \mathbf{z} \in \mathcal{Z}_d, \quad \forall i = 1, 2, \ldots, 2d$

(ii) *for any $(k, \ell) \in ((1-\varepsilon)\log n)\mathcal{B} \cap \mathcal{Z}_d$ and for arbitrary $i \in \{1, 2, \ldots, 2d\}$ there exist random $\mathbf{z}_1, \mathbf{z}_2 \in \mathcal{Z}_d$ for which*

$$(\xi(\mathbf{z}_1, n), \xi(\mathbf{z_1} + \mathbf{e}_i, n)) = (k+1, \ell)$$

$$(\xi(\mathbf{z}_2, n), \xi(\mathbf{z_2} + \mathbf{e}_i, n)) = (k, \ell+1).$$

We will first show that without restriction on the dimension we have

Theorem 1.2. *Let $d \geq 3$. For each $\varepsilon > 0$, with probability 1 there exists an $n_0 = n_0(\varepsilon)$ such that if $n \geq n_0$ then*

(i) $(\xi(\mathbf{S}_j, \infty), \xi(\mathbf{S}_j + \mathbf{e}_i, \infty)) \in ((1+\varepsilon)\log n)\mathcal{B}, \qquad \forall j = 0, 1, 2, \ldots, n, \quad \forall i = 1, 2, \ldots, 2d$

(ii) *for any $(k, \ell) \in ((1-\varepsilon)\log n)\mathcal{B} \cap \mathcal{Z}_d$ and for arbitrary $i \in \{1, 2, \ldots, 2d\}$ there exists a random integer $j = j(k, \ell) \leq n$ for which*

$$(\xi(\mathbf{S}_j, \infty), \xi(\mathbf{S}_j + \mathbf{e}_i, \infty)) = (k+1, \ell).$$

Concerning the occupation time of the unit ball, Consequence 1.1 and Theorem E suggest the following

Conjecture 1.1 *For any $\varepsilon > 0$ with probability 1 there exists a random variable $n_0 = n_0(\varepsilon)$ such that if $n \geq n_0$ then for all $k = 1, 2, \ldots, [(1-\varepsilon)\kappa \log n]$ there exists $\mathbf{z} \in \mathcal{Z}_d$ such that $\Xi(\mathbf{z}, n) = k$.*

A simple consequence of our Theorem 1.3 is that Conjecture 1.1 is true. As we indicated above, we are interested in the joint asymptotic behavior of the random sequence

$$(\xi(\mathbf{z}, n), \Xi(\mathbf{z}, n)), \quad \mathbf{z} \in \mathcal{Z}_d$$

as $n \to \infty$. One might ask again whether this random vector will fill out all the lattice points of the triangle $(\log n)\mathcal{C}$, where

$$\mathcal{C} = \{(x, y) \in \mathcal{Z}_d : \ 0 \leq x \leq \lambda, \ x \leq y \leq \kappa\}.$$

As before, it turns out that the above triangle will not be filled. Instead, we will prove the following theorem.

Define the set $\mathcal{D}$ as

$$\mathcal{D} := \{y \geq x \geq 0 : \ -y\log y + x\log(2dx) + (y-x)\log((y-x)/p) \leq 1\}, \tag{1.24}$$

where p was defined in (1.20) and its value in terms of γ is given by (2.5) below.

Theorem 1.3. *Let $d \geq 4$. For each $\varepsilon > 0$ with probability 1 there exists an $n_0 = n_0(\varepsilon)$ such that if $n \geq n_0$ then*

(i) $(\xi(\mathbf{z}, n), \Xi(\mathbf{z}, n)) \in ((1+\varepsilon)\log n)\mathcal{D}, \qquad \forall \mathbf{z} \in \mathcal{Z}_d$

(ii) *for any $(k, \ell) \in ((1-\varepsilon)\log n)\mathcal{D} \cap \mathcal{Z}_d$ there exists a random $\mathbf{z} \in \mathcal{Z}_d$ for which*

$$(\xi(\mathbf{z}, n), \Xi(\mathbf{z}, n)) = (k, \ell + 1).$$

Theorem 1.4. *Let $d \geq 3$. For each $\varepsilon > 0$ with probability 1 there exists an $n_0 = n_0(\varepsilon)$ such that if $n \geq n_0$ then*

(i) $(\xi(\mathbf{S}_j + \mathbf{e}_i, \infty), \Xi(\mathbf{S}_j + \mathbf{e}_i, \infty) \in ((1+\varepsilon)\log n)\mathcal{D}, \quad \forall j = 1, 2, \ldots, n, \quad \forall i = 1, 2, \ldots, 2d$

(ii) *for any $(k, \ell) \in ((1-\varepsilon)\log n)\mathcal{D} \cap \mathcal{Z}_d$ and for arbitrary $i \in \{1, 2, \ldots, 2d\}$ there exists a random integer $j = j(k, \ell) \leq n$ for which*

$$(\xi(\mathbf{S}_j + \mathbf{e}_i, \infty), \Xi(\mathbf{S}_j + \mathbf{e}_i, \infty)) = (k, \ell + 1).$$

Remark 1.1 The condition $d \geq 4$ in Theorem 1.1 and Theorem 1.3 is needed only for the convergence of (4.7) while proving parts (i). The proofs of parts (ii) in both theorems work also for $d = 3$.

2. Preliminary facts and results

Recall the definition of γ, $\gamma_{\mathbf{z}}$, T and $T_{\mathbf{z}}$ in Section 1.

Lemma 2.1. *For $i = 1, 2, \ldots, 2d$*

$$\gamma_{\mathbf{e}_i} = \gamma, \tag{2.1}$$

$$\mathbf{P}(T < T_{\mathbf{e}_i}) = \mathbf{P}(T_{\mathbf{e}_i} < T) = \frac{1-\gamma}{2-\gamma} = \alpha, \tag{2.2}$$

$$\mathbf{P}(T = T_{\mathbf{e}_i} = \infty) = \frac{\gamma}{2-\gamma} = 1 - 2\alpha. \tag{2.3}$$

Proof. By symmetry $\gamma_{\mathbf{e}_i} = \gamma_{\mathbf{e}_1},\ i = 1, 2, \ldots, 2d$. Hence

$$1 - \gamma = \sum_{i=1}^{2d} \mathbf{P}(\mathbf{S}_1 = \mathbf{e}_i)(1 - \gamma_{\mathbf{e}_i}) = \sum_{i=1}^{2d} \frac{1}{2d}(1 - \gamma_{\mathbf{e}_i}) = (1 - \gamma_{\mathbf{e}_1})$$

thus we have (2.1). Furthermore observe that

$$1-\gamma = \mathbf{P}(T<\infty) = \mathbf{P}(T<T_{\mathbf{e_i}}) + \mathbf{P}(T_{\mathbf{e_i}}<T)\mathbf{P}_{\mathbf{e}_i}(T<\infty)$$

and

$$1-\gamma = \mathbf{P}(T_{\mathbf{e}_i}<\infty) = \mathbf{P}(T_{\mathbf{e}_i}<T) + \mathbf{P}(T<T_{\mathbf{e}_i})\mathbf{P}(T_{\mathbf{e}_i}<\infty).$$

Solving this system of equations for $\mathbf{P}(T_{\mathbf{e_i}}<T)$ and $\mathbf{P}(T<T_{\mathbf{e}_i})$, we get (2.2), and (2.3) follows from $\mathbf{P}(T=T_{\mathbf{e}_i}=\infty) = 1-\mathbf{P}(T<T_{\mathbf{e}_i}) - \mathbf{P}(T_{\mathbf{e}_i}<T)$.

Lemma 2.2. *For $i=1,2,\ldots,2d$*

$$\mathbf{P}(\xi(\mathbf{0},\infty)=k, \xi(\mathbf{e}_i,\infty)=\ell) = (1-2\alpha)\binom{k+\ell}{k}\alpha^{k+\ell}, \quad k,\ell=0,1,\ldots \tag{2.4}$$

Proof. By (2.2), the probability of k visits in $\mathbf{0}$ and ℓ visits in $\mathbf{e}_i$ in any particular order is $\alpha^{k+\ell}$. The binomial coefficient in (2.4) is the number of possible orders. Finally, observe that starting from either of the two points, the probability that the walk does not return back to the starting point, nor to the other point is $1-2\alpha$. Hence the lemma follows. □

Recall the definition of p in (1.20).

Lemma 2.3.

$$p = 1-\frac{1}{2d(1-\gamma)}, \tag{2.5}$$

$$\mathbf{P}(\Xi(\mathbf{0},\infty)=j) = \left(1-p-\frac{1}{2d}\right)\left(p+\frac{1}{2d}\right)^{j-1}, \quad j=1,2\ldots, \tag{2.6}$$

$$\mathbf{P}(\xi(\mathbf{0},\infty)=k,\ \Xi(\mathbf{0},\infty)=\ell+1) = \binom{\ell}{k}\left(1-p-\frac{1}{2d}\right)p^{\ell-k}\left(\frac{1}{2d}\right)^{k}, \tag{2.7}$$

$$\ell=0,1,\ldots,\ k=0,1,\ldots,\ell.$$

Proof. Let $Z(A)$ denote the number of visits in the set A up to the first return to zero. Clearly

$$\mathbf{P}(Z(\mathcal{S}(1))=j, T<\infty) = p^{j-1}\frac{1}{2d}, \quad j=1,2,\ldots \tag{2.8}$$

Summing up (2.8) in j, we get

$$1-\gamma = \mathbf{P}(T<\infty) = \sum_{j=1}^{\infty} p^{j-1}\frac{1}{2d} = \frac{1}{2d(1-p)}, \tag{2.9}$$

implying (2.5).

Introduce further

$$\tau = \sum_{j=1}^{\infty} I\{\mathbf{S}_j \in \mathcal{S}(1), \|\mathbf{S}_{j+1}\| > 1\},$$

thus τ is the number of outward excursions from $\mathcal{S}(1)$ to $\mathcal{S}(1)$, including the last incomplete one. Hence

$$\Xi(\mathbf{0}, \infty) = \tau + \xi(\mathbf{0}, \infty).$$

Since p is the probability that the random walk starting from any point of $\mathcal{S}(1)$ returns to $\mathcal{S}(1)$ from outside, while $1/(2d)$ is the probability of the same return through the origin, $p + 1/(2d)$ is the probability that the random walk, starting from any point of $\mathcal{S}(1)$, returns to $\mathcal{S}(1)$ in finite time, (2.6) is immediate. Furthermore, it is easy to see that

$$\mathbf{P}(\xi(\mathbf{0}, \infty) = k,\, \tau = M+1) = \binom{k+M}{k}\left(1 - p - \frac{1}{2d}\right) p^M \left(\frac{1}{2d}\right)^k,$$

implying (2.7). $\square$

Recall and define

$$\gamma_{\mathbf{z}} := \mathbf{P}(T_{\mathbf{z}} = \infty), \qquad \gamma_{\mathbf{z}}(n) := \mathbf{P}(T_{\mathbf{z}} \geq n), \tag{2.10}$$

$$q_{\mathbf{z}} := \mathbf{P}(T < T_{\mathbf{z}}), \qquad q_{\mathbf{z}}(n) := \mathbf{P}(T < \min(n, T_{\mathbf{z}})), \tag{2.11}$$

$$s_{\mathbf{x}} := \mathbf{P}(T_{\mathbf{x}} < T), \qquad s_{\mathbf{z}}(n) := \mathbf{P}(T_{\mathbf{z}} < \min(n, T)). \tag{2.12}$$

Moreover, put

$$p(n) := \mathbf{P}_{\mathbf{e}_1}(T_{S(1)} < \min(n, T)).$$

Similarly to Theorem A, we prove

Lemma 2.4.

$$1 - \gamma_{\mathbf{z}} + \frac{O(1)}{n^{d/2-1}} \leq 1 - \gamma_{\mathbf{z}}(n) \leq 1 - \gamma_{\mathbf{z}}, \tag{2.13}$$

$$q_{\mathbf{z}} + \frac{O(1)}{n^{d/2-1}} \leq q_{\mathbf{z}}(n) \leq q_{\mathbf{z}}, \tag{2.14}$$

$$s_{\mathbf{z}} + \frac{O(1)}{n^{d/2-1}} \leq s_{\mathbf{z}}(n) \leq s_{\mathbf{z}} \tag{2.15}$$

$$p + \frac{O(1)}{n^{d/2-1}} \leq p(n) \leq p, \tag{2.16}$$

and $O(1)$ is uniform in $\mathbf{z}$.

Proof. For the proof of (2.13) see Jain and Pruitt [7].

To prove (2.14) and (2.15), observe that

$$0 \le q_{\mathbf{z}} - q_{\mathbf{z}}(n) = \mathbf{P}(T < T_{\mathbf{z}},\, n \le T < \infty) \le \mathbf{P}(n \le T < \infty) = \gamma(n) - \gamma,$$
$$0 \le s_{\mathbf{z}} - s_{\mathbf{z}}(n) = \mathbf{P}(T_{\mathbf{z}} < T,\, n \le T_{\mathbf{z}} < \infty) \le \mathbf{P}(n \le T_{\mathbf{z}} < \infty) = \gamma_{\mathbf{z}}(n) - \gamma_{\mathbf{z}}.$$

To prove (2.16), introduce $\mathbf{b}_j = \mathbf{e}_1 + \mathbf{e}_j, \;\; j = 1, 2, \ldots, 2d$, then we have

$$0 \le p - p(n) = \mathbf{P}_{\mathbf{e}_1}(n \le T_{\mathcal{S}(1)} < \infty) = \sum_{j=1}^{2d} \mathbf{P}_{\mathbf{e}_1}(\mathbf{S}_1 = \mathbf{b}_j, n \le T_{\mathcal{S}(1)} < \infty). \tag{2.17}$$

Observe that by (2.13), each term in the above sum can be estimated by

$$\mathbf{P}_{\mathbf{e}_1}(\mathbf{S}_1 = \mathbf{b}_j, n \le T_{\mathcal{S}(1)} < \infty) = \frac{1}{2d}\mathbf{P}_{\mathbf{b}_j}(n-1 \le T_{\mathcal{S}(1)} < \infty) = \frac{O(1)}{n^{d/2-1}},$$

proving the lemma. □

Lemma 2.5. *For $i = 1, 2, \ldots, 2d$, $k + \ell > 0$, $n > 0$ we have*

$$\mathbf{P}(\xi(\mathbf{0}, n) = k, \xi(\mathbf{e}_i, n) = \ell) \le \binom{k+\ell}{k}\alpha^{k+\ell}, \tag{2.18}$$

and for $i = 1, 2, \ldots, 2d$, $\ell > 0$, $n > 0$ we have

$$\mathbf{P}(\xi(\mathbf{e}_i, n) = k,\, \Xi(\mathbf{e}_i, n) = \ell) \le \binom{\ell}{k} p^{\ell-k}\left(\frac{1}{2d}\right)^k. \tag{2.19}$$

Proof. To show (2.18), recall that by Lemma 2.1, $q_{\mathbf{e}_i} = s_{\mathbf{e}_i} = \alpha$. The time between consecutive visits to $\mathbf{0}$ or $\mathbf{e}_i$ is less than n, hence using the upper inequalities in (2.14) and (2.15), it is easy to see that the probability of k visits in $\mathbf{0}$ and ℓ visits in $\mathbf{e}_i$ up to time n in any particular order, is less than $\alpha^{k+\ell}$. Now (2.18) is seen by observing that the number of particular orders is the binomial coefficient in (2.18).

Similarly, we can get (2.19) by using (2.16). □

3. The basic equations

It follows from Lemma 2.2 and Stirling formula that the asymptotic relation

$$\log \mathbf{P}(\xi(\mathbf{0}, \infty) = [x \log n],\, \xi(\mathbf{e}_i, \infty) = [y \log n]) \sim -g(x, y) \log n, \quad n \to \infty \tag{3.1}$$

holds for $i \in \{1, 2, \ldots, 2d\}$, $x \geq 0$, $y \geq 0$, where

$$g(x, y) = -(x + y)\log(y + x) + x\log x + y\log y - (x + y)\log \alpha.$$

It follows that $\mathbf{P}(\xi(\mathbf{0}, \infty) = [x \log n], \, \xi(\mathbf{e}_i, \infty) = [y \log n])$ is of order $1/n$ if (x, y) satisfies the basic equation

$$g(x, y) = 1, \quad x \geq 0, \, y \geq 0. \tag{3.2}$$

Observe that $g(x, y)$ is the function defining the set $\mathcal{B}$ in (1.23). The next lemma describes the major properties of the boundary of the set $\mathcal{B}$.

Lemma 3.1.

(i) *For the points* (x, y) *satisfying* (3.2) *we have*

$$x_{max} = y_{max} = \lambda, \tag{3.3}$$

$$(x + y)_{max} = \frac{1}{\log \frac{1}{2\alpha}}, \tag{3.4}$$

when this maximum occurs then $x = y$.

(ii) *If* $x = x_{max} = \lambda$, *then* $y = \lambda(1 - \gamma)$ *and vica versa.*
If $x = 0$, *then* $y = \frac{1}{\log(1/\alpha)}$ *and vica versa.*

(iii) *For a given* x, *the equation* (3.2) *has one solution in* y *for* $x < x_0$, *and for* $x = \lambda$ *and two solutions for* $x_0 \leq x < \lambda$, *where*

$$x_0 = \frac{1}{\log(1/\alpha)}.$$

Proof. Differentiating (3.2) as an implicit function of x, y takes its maximum ($y' = 0$) at $x = \lambda(1 - \gamma)$ and the value of this maximum is $y = \lambda$, which proves the first statements in (i) and (ii).

Similarly, if we maximize $x + y$ as a function of x (i.e. $1 + y' = 0$) then we get that this occurs when $x = y$ and the second part of (i) follows.

Solving (3.2) when $x = 0$ for y, we get the second part of (ii).

Now we turn to the proof of (iii). For given $0 \leq x \leq \lambda$ consider $g(x, y)$ as a function of y. We have

$$\frac{\partial g}{\partial y} = \log \frac{y}{\alpha(x + y)}$$

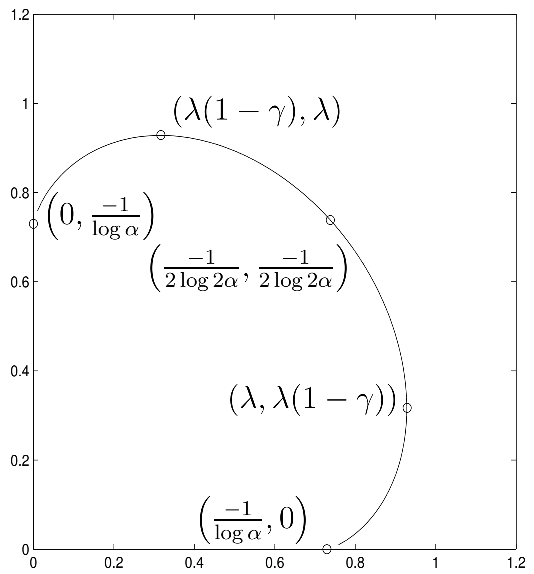

Figure 1: The set $\mathcal{B}$ in the case of the two-point set, $d = 3$.

and this is equal to zero if $y = x(1-\gamma)$. It is easy to see that g takes a minimum here and is decreasing in $(0, x(1-\gamma))$ and increasing in $(x(1-\gamma), \lambda)$. Moreover,

$$\frac{\partial^2 g}{\partial y^2} = \frac{1}{y} - \frac{1}{x+y} > 0,$$

hence g is convex from below. We have for $0 \le x < \lambda$, that this minimum is

$$g(x, x(1-\gamma)) = \frac{x}{\lambda} < 1,$$

and one can easily see that

$$g(x,0) = x\log(1/\alpha) \begin{cases} < 1 \text{ if } x < x_0, \\ = 1 \text{ if } x = x_0, \\ > 1 \text{ if } x > x_0. \end{cases}$$

This shows that equation (3.2) has one solution if $0 \le x < x_0$ and two solutions if $x_0 \le x < \lambda$.

For $x = \lambda$, it can be seen that $y = \lambda(1-\gamma)$ is the only solution of $g(x,y) = 1$.

The proof of Lemma 3.1 is complete. $\Box$

For further reference introduce the following notations to describe the boundary of $\mathcal{B}$: for $x_0 \le x < \lambda$ let $y_{1,\mathcal{B}}(x) < y_{2,\mathcal{B}}(x)$ denote the two solutions and for $0 \le x < x_0$ let $y_{2,\mathcal{B}}(x)$ denote the only solution of (3.2). Define $y_{1,\mathcal{B}}(x) = 0$ for $0 \le x < x_0$ and $y_{1,\mathcal{B}}(\lambda) = y_{2,\mathcal{B}}(\lambda) = \lambda(1-\gamma)$. Then the set $\mathcal{B}$ can be given as

$$\mathcal{B} = \{0 \le x \le \lambda,\, y_{1,\mathcal{B}}(x) \le y \le y_{2,\mathcal{B}}(x)\}.$$

For further discussion of properties of the set $\mathcal{B}$ see Section 6.

Concerning similar description of the set $\mathcal{D}$ belonging to the other problem, it follows from (2.7) of Lemma 2.3 and Stirling formula that the asymptotic relation

$$\log \mathbf{P}(\xi(\mathbf{0},\infty) = [x \log n],\, \Xi(\mathbf{0},\infty) = [y \log n]) \sim -f(x,y)\log n, \quad n \to \infty \tag{3.5}$$

holds for $0 \le x \le y$, where

$$f(x,y) = -y\log y + x\log x + (y-x)\log(y-x) + x\log(2d) + (y-x)\log(1/p).$$

It follows that $\mathbf{P}(\xi(\mathbf{0},\infty) = [x\log n],\, \Xi(\mathbf{0},\infty) = [y\log n])$ is of order $1/n$ if (x,y) satisfies the basic equation

$$f(x,y) = 1, \qquad 0 \le x \le y. \tag{3.6}$$

Lemma 3.2.

(i) *For the maximum values of x, y, satisfying* (3.6)*, we have*

$$x_{\max} = \frac{1}{\log(2d(1-p))} = \lambda, \tag{3.7}$$

$$y_{\max} = \frac{-1}{\log(p + \frac{1}{2d})} = \kappa. \tag{3.8}$$

(ii) *If $x = x_{\max} = \lambda$, then $y = \lambda/(1-p)$. If $y = y_{\max} = \kappa$, then $x = \kappa/(2dp+1)$. If $x = 0$, then $y = 1/\log(1/p)$.*

(iii) *For given x the equation* (3.6) *has one solution in y for $0 \le x < 1/\log(2d)$ and for $x = \lambda$, and two solutions in y for $1/\log(2d) \le x < \lambda$.*

Proof. (i) First consider x as a function of y satisfying (3.6). We seek the maximum, where the derivative $x'(y) = 0$. Differentiating (3.6) and putting $x' = 0$, a simple calculation leads to

$$-\log y + \log(y-x) + \log(1/p) = 0,$$

i.e.

$$y = x/(1-p).$$

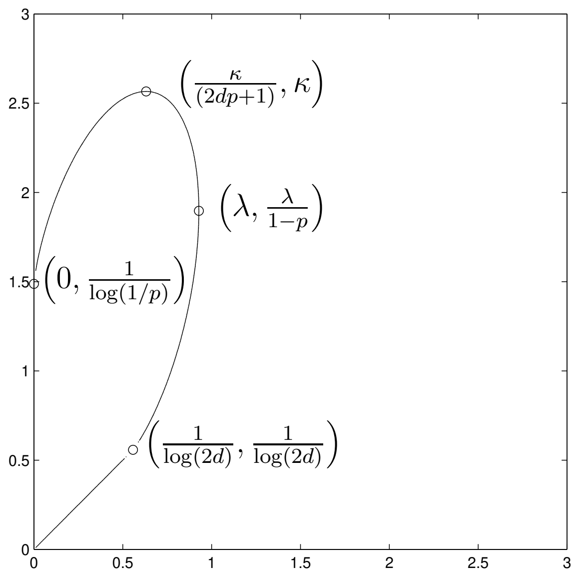

Figure 2: The set $\mathcal{D}$ in the case of the unit ball, $d = 3$.

It can be seen that this is the value of y when x takes its maximum. Substituting this into (3.6), we get

$$x_{\max} = \frac{1}{\log(2d(1-p))} = \lambda,$$

verifying (3.7).

Next consider y as a function of x and maximize y subject to (3.6). Again, differentiating (3.6) with respect to x and putting $y' = 0$, we get

$$-\log(y - x) + \log x - \log(1/p) + \log(2d) = 0$$

from which $x = y/(1 + 2pd)$. Substituting in (3.6) we get $y_{\max} = \kappa$.

This completes the proof of Lemma 3.2(i) and the first two statements in Lemma 3.2(ii). An easy calculation shows that if $x = 0$, then $y = 1/\log(1/p)$.

Now we turn to the proof of Lemma 3.2(iii). For given $0 \le x \le \lambda$ consider $f(x, y)$ as a function of y. We have

$$\frac{\partial f}{\partial y} = \log \frac{y - x}{py}$$

and this is equal to zero if $y = x/(1-p)$. It is easy to see that f takes a minimum here and is decreasing if $y < x/(1-p)$ and increasing if $y > x/(1-p)$. Moreover,

$$\frac{\partial^2 f}{\partial y^2} = \frac{1}{y-x} - \frac{1}{y} > 0,$$

hence f is convex from below. We have for $0 < x < \lambda$, that this minimum is

$$f\left(x, \frac{x}{1-p}\right) = x\log((1-p)2d)) = \frac{x}{\lambda} < 1,$$

and

$$f(x,0) = x\log(2d)\begin{cases} < 1 \text{ if } x < 1/\log(2d), \\ = 1 \text{ if } x = 1/\log(2d), \\ > 1 \text{ if } x > 1/\log(2d). \end{cases}$$

This shows that equation (3.6) has one solution if $0 \le x < 1/\log(2d)$ and two solutions if $1/\log(2d) \le x < \lambda$.

For $x = \lambda$, it can be seen that $y = \lambda/(1-p)$ is the only solution of $f(x,y) = 1$.

The proof of Lemma 3.2 is complete. □

For further reference once again introduce the following notations to describe the boundary of $\mathcal{D}$: for $1/\log(2d) \le x < \lambda$ let $y_{1,\mathcal{D}}(x) < y_{2,\mathcal{D}}(x)$ denote the two solutions and for $0 \le x < 1/\log(2d)$ let $y_{2,\mathcal{D}}(x)$ denote the only solution of (3.6). Define $y_{1,\mathcal{D}}(x) = x$ for $0 \le x < 1/\log(2d)$ and $y_{1,\mathcal{D}}(\lambda) = y_{2,\mathcal{D}}(\lambda) = \lambda/(1-p)$. Then the set $\mathcal{D}$ can be given as

$$\mathcal{D} = \{0 \le x \le \lambda,\ y_{1,\mathcal{D}}(x) \le y \le y_{2,\mathcal{D}}(x)\}.$$

For further discussion of properties of the set $\mathcal{D}$ see Section 6.

4. Proof of Theorems, Parts (i)

In this section we prove parts (i) of the theorems in the following order: Theorem 1.2(i), Theorem 1.1(i), Theorem 1.4(i), Theorem 1.3(i). In the proofs the constant c may vary from line to line.

Proof of Theorem 1.2(i).

We say that $\mathbf{S}_j$ $(j = 0, 1, \ldots)$ is new (cf. [3]) if either $j = 0$, or $j \ge 1$ and

$$\mathbf{S}_m \ne \mathbf{S}_j, \text{ for } m = 1, 2, \ldots, j-1.$$

Let A_j be the event that $\mathbf{S}_j$ is new.

Consider the reverse random walk starting from $\mathbf{S}_j$, i.e. $\mathbf{S}'_r := \mathbf{S}_{j-r} - \mathbf{S}_j$, $r = 0, 1, \ldots, j$ and also the forward random walk $\mathbf{S}''_r := \mathbf{S}_{j+r} - \mathbf{S}_j$, $r = 0, 1, 2, \ldots$ Then $\{\mathbf{S}'_0, \mathbf{S}'_1, \ldots, \mathbf{S}'_j\}$ and $\{\mathbf{S}''_0, \mathbf{S}''_1, \ldots\}$ are independent and so are their respective local times ξ' and ξ". One can easily see that

$$\xi(\mathbf{S}_j, j) = \xi'(\mathbf{0}, j) + 1, \quad \xi(\mathbf{S}_j + \mathbf{e}_i, j) = \xi'(\mathbf{e}_i, j)$$

and

$$\xi(\mathbf{S}_j, \infty) - \xi(\mathbf{S}_j, j) = \xi"(\mathbf{0}, \infty), \quad \xi(\mathbf{S}_j + \mathbf{e}_i, \infty) - \xi(\mathbf{S}_j + \mathbf{e}_i, j) = \xi"(\mathbf{0} + \mathbf{e}_i, \infty),$$

hence by Lemmas 2.2 and 2.5

$$\mathbf{P}(\xi(\mathbf{S}_j, \infty) = k, \xi(\mathbf{S}_j + \mathbf{e}_i, \infty) = \ell, A_j)$$

$$= \mathbf{P}(\xi'(\mathbf{0}, j) = 0, \xi"(\mathbf{0}, \infty) = k - 1, \xi'(\mathbf{e}_i, j) + \xi"(\mathbf{e}_i, \infty) = \ell)$$

$$= \sum_{\ell_1=0}^{\ell} \mathbf{P}(\xi'(\mathbf{0}, j) = 0, \xi'(\mathbf{e}_i, j) = \ell_1)\mathbf{P}(\xi"(\mathbf{0}, \infty) = k - 1, \xi"(\mathbf{e}_i, \infty) = \ell - \ell_1)$$

$$\leq \sum_{\ell_1=0}^{\ell} \alpha^{\ell_1} \binom{k - 1 + \ell - \ell_1}{\ell - \ell_1} \alpha^{k-1+\ell-\ell_1} = \alpha^{k+\ell-1} \sum_{\ell_1=0}^{\ell} \binom{k - 1 + \ell - \ell_1}{\ell - \ell_1} = \binom{k + \ell}{\ell} \alpha^{k+\ell-1}.$$

Let $(k, \ell) \notin ((1 + \varepsilon) \log n)\mathcal{B}$. Since $g(cx, cy) = cg(x, y)$ for any $c > 0$, we conclude from (3.1) that

$$\mathbf{P}(\xi(\mathbf{S}_j, \infty) = k, \xi(\mathbf{S}_j + \mathbf{e}_i, \infty) = \ell, A_j) \leq \frac{c}{n^{1+\varepsilon}}$$

and using this and (1.10)

$$\mathbf{P}(\xi(\mathbf{S}_j, \infty), \xi(\mathbf{S}_j + \mathbf{e}_i, \infty)) \notin ((1 + \varepsilon) \log n)\mathcal{B}, A_j)$$

$$\leq \sum_{\substack{(k,\ell) \notin ((1+\varepsilon) \log n)\mathcal{B} \\ k \leq (1+\varepsilon)\lambda \log n \\ \ell \leq (1+\varepsilon)\lambda \log n}} \mathbf{P}(\xi(\mathbf{S}_j, \infty) = k, \xi(\mathbf{S}_j + \mathbf{e}_i, \infty) = \ell, A_j)$$

$$+ \sum_{k > (1+\varepsilon)\lambda \log n} \mathbf{P}(\xi(\mathbf{S}_j, \infty) = k, A_j) + \sum_{\ell > (1+\varepsilon)\lambda \log n} \mathbf{P}(\xi(\mathbf{S}_j + \mathbf{e}_i, \infty) = \ell, A_j)$$

$$\leq \frac{c \log^2 n}{n^{1+\varepsilon}} + 2 \sum_{k > (1+\varepsilon)\lambda \log n} (1 - \gamma)^k \leq \frac{c}{n^{1+\varepsilon/2}}. \tag{4.1}$$

Hence selecting a subsequence $n_r = r^{4/\varepsilon}$ we have

$$\mathbf{P}(\cup_{j \leq n_{r+1}} \cup_{i=1}^{2d} \{(\xi(\mathbf{S}_j, \infty), \xi(\mathbf{S}_j + \mathbf{e}_i, \infty)) \notin ((1 + \varepsilon) \log n_r)\mathcal{B}\})$$

$$= \mathbf{P}(\cup_{j\le n_{r+1}} \cup_{i=1}^{2d} \{(\xi(\mathbf{S}_j,\infty),\xi(\mathbf{S}_j+\mathbf{e}_i,\infty)) \notin ((1+\varepsilon)\log n_r)\mathcal{B}\} \cap A_j) \le \frac{c}{n_r^{\varepsilon/2}}.$$

Borel-Cantelli lemma implies that with probability 1 for all large r and for all $j \le n_{r+1}$, $i \le 2d$ we have

$$(\xi(\mathbf{S}_j,\infty),\xi(\mathbf{S}_j+\mathbf{e}_i,\infty)) \in ((1+\varepsilon)\log n_r)\mathcal{B}.$$

It follows that with probability 1 there exists n_0 such that if $n \ge n_0$ then

$$(\xi(\mathbf{S}_j,\infty),\xi(\mathbf{S}_j+\mathbf{e}_i,\infty)) \in ((1+\varepsilon)\log n)\mathcal{B}$$

for all $i = 1,2,\ldots,2d$, $j \le n$.

This proves (i) of Theorem 1.2. □

Proof of Theorem 1.1(i).

Introduce the following notation:

$$\xi(\mathbf{z},(n,\infty)) := \xi(\mathbf{z},\infty) - \xi(\mathbf{z},n) \tag{4.2}$$

Fix $i \in \{1,2,\ldots,2d\}$ and define the following events for $j \le n$.

$$\begin{aligned}
B(j,n) &:= \{(\xi(\mathbf{S}_j,n),\xi(\mathbf{S}_j+\mathbf{e}_i,n)) \notin ((1+\varepsilon)\log n)\mathcal{B}\}, &(4.3)\\
B^*(j,n) &:= \{(\xi(\mathbf{S}_j,j),\xi(\mathbf{S}_j+\mathbf{e}_i,j)) \notin ((1+\varepsilon)\log n)\mathcal{B}\}, &(4.4)\\
C(j,n) &:= \{\mathbf{S}_m \ne \mathbf{S}_j, \mathbf{S}_m \ne \mathbf{S}_j+\mathbf{e}_i,\ m = j+1,\ldots,n\}, &(4.5)\\
D(j,n) &:= \{\xi(\mathbf{S}_j,(n,\infty))>0\} \cup \{\xi(\mathbf{S}_j+\mathbf{e}_i,(n,\infty))>0\}. &(4.6)
\end{aligned}$$

Considering again the reverse random walk starting from $\mathbf{S}_j$, i.e. $\mathbf{S}'_r = \mathbf{S}_{j-r} - \mathbf{S}_j$, $r = 0,1,\ldots,j$ we have

$$\xi(\mathbf{S}_j,j) = \xi'(\mathbf{0},j)+1, \qquad \xi(\mathbf{S}_j+\mathbf{e}_i,j) = \xi'(\mathbf{e}_i,j),$$

where ξ' is the local time of the random walk $\mathbf{S}'$.

By (2.18) of Lemma 2.5 and (3.1), if $(k,\ell) \notin ((1+\varepsilon)\log n)\mathcal{B}$, then

$$\mathbf{P}(\xi'(\mathbf{0},j) = k-1, \xi'(\mathbf{e}_i,j) = \ell) \le \frac{k}{k+\ell}\binom{k+\ell}{k}\alpha^{k+\ell-1} \le \frac{c}{n^{1+\varepsilon}}.$$

Hence, as in (4.1), we have

$$\mathbf{P}(B^*(j,n)) \le \frac{c}{n^{1+\varepsilon/2}}.$$

Observe that

$$B(j,n)C(j,n)D(j,n) = B^*(j,n)C(j,n)D(j.n).$$

Furthermore $\{\mathbf{S}'_r,\ r=0,1,\ldots,j\}$ and $\{\mathbf{S}_m-\mathbf{S}_j,\ m=j,j+1,\ldots\}$ are independent. Hence

$$\mathbf{P}(B(j,n)C(j,n)D(j,n)) = \mathbf{P}(B^*(j,n))P(C(j,n)D((j,n)).$$

Combining these with Theorem A implies

$$\mathbf{P}(B(j,n)C(j,n)D(j,n)) \le \frac{c}{n^{1+\varepsilon/2}(n-j+1)^{d/2-1}},$$

consequently for $d \ge 4$

$$\sum_{n=1}^{\infty}\sum_{j=1}^{n}\mathbf{P}(B(j,n)C(j,n)D(j,n)) < \infty. \tag{4.7}$$

Hence with probability 1, there exists n_0 such that for $n \ge n_0$ the event $\overline{B}(j,n) \cup \overline{C}(j,n) \cup \overline{D}(j,n)$ occurs. We may assume that n_0 satisfies also the requirement in Theorem 1.2(i). If $\overline{B}(j,n)$ occurs, then

$$(\xi(\mathbf{S}_j,n),\xi(\mathbf{S}_j+\mathbf{e}_i,n)) \in ((1+\varepsilon)\log n)\mathcal{B}.$$

If $\overline{D}(j,n)$ occurs, then

$$(\xi(\mathbf{S}_j,n),\xi(\mathbf{S}_j+\mathbf{e}_i,n)) = (\xi(\mathbf{S}_j,\infty),\xi(\mathbf{S}_j+\mathbf{e}_i,\infty)) \in ((1+\varepsilon)\log n)\mathcal{B}$$

by Theorem 1.2(i).

Now consider $\mathbf{z} \in \mathcal{Z}_d$ such that $\xi(\mathbf{z},n)+\xi(\mathbf{z}+\mathbf{e}_i,n) > 0$, but arbitrary otherwise and let L be the time of the last visit to $\{\mathbf{z},\mathbf{z}+\mathbf{e}_i\}$ before n, i.e. $L := \max\{m \le n : \mathbf{S}_m \in \{\mathbf{z},\mathbf{z}+\mathbf{e}_i\}\}$. Then $\overline{B}(L,n) \cup \overline{C}(L,n) \cup \overline{D}(L,n)$ occurs for $n \ge n_0$. Since $\overline{C}(L,n)$ cannot occur, we have that $\overline{B}(L,n) \cup \overline{D}(L,n)$ occurs. If $\mathbf{S}_L = \mathbf{z}$, this implies

$$(\xi(\mathbf{S}_L,n),\xi(\mathbf{S}_L+\mathbf{e}_i,n)) = (\xi(\mathbf{z},n),\xi(\mathbf{z}+\mathbf{e}_i,n)) \in ((1+\varepsilon)\log n)\mathcal{B}.$$

If $\mathbf{S}_L = \mathbf{z}+\mathbf{e}_i$, then applying the above procedure using the unit vector $-\mathbf{e}_i$ we get that

$$(\xi(\mathbf{S}_L,n),\xi(\mathbf{S}_L-\mathbf{e}_i,n)) = (\xi(\mathbf{z}+\mathbf{e}_i,n),\xi(\mathbf{z},n)) \in ((1+\varepsilon)\log n)\mathcal{B}.$$

By symmetry of the set $\mathcal{B}$ this implies also

$$(\xi(\mathbf{z},n),\xi(\mathbf{z}+\mathbf{e}_i)) \in ((1+\varepsilon)\log n)\mathcal{B}.$$

Since $i \in \{1, 2, \ldots, 2d\}$ is arbitrary, this completes the proof of Theorem 1.1(i). $\square$

Proof of Theorem 1.4(i).

The proof is similar to that of Theorem 1.2(i). Let $\mathbf{z} \in \mathcal{Z}_d$ and consider the unit ball centered at $\mathbf{z}$. Let now A_j be the event that the random walk hits this unit ball first at time j. Under this condition $(\xi(\mathbf{z}, \infty), \Xi(\mathbf{z}, \infty))$ has the (unconditional) distribution of $(\xi(\mathbf{0}, \infty), \Xi(\mathbf{0}, \infty))$. Hence if $(k, \ell) \notin ((1+\varepsilon)\log n)\mathcal{D}$, then by using (3.5)

$$\mathbf{P}(\xi(\mathbf{z}, \infty) = k, \Xi(\mathbf{z}, \infty) = \ell, A_j) \leq \frac{c}{n^{1+\varepsilon}}.$$

The same way as in the proof of Theorem 1.2(i) we can show the following estimation, with the modification that whenever we have a summation by ℓ, λ should be replaced by κ and instead of using (1.10) we apply (2.6).

$$\mathbf{P}((\xi(\mathbf{z}, \infty), \Xi(\mathbf{z}, \infty)) \notin ((1+\varepsilon)\log n)\mathcal{D}), A_j) \leq \frac{c}{n^{1+\varepsilon/2}}.$$

For n_r as in the proof of Theorem 1.2(i), one gets similarly

$$\mathbf{P}(\cup_{j \leq n_{r+1}} \cup_{i=1}^{2d} \{(\xi(\mathbf{S}_j + \mathbf{e}_i, \infty), \Xi(\mathbf{S}_j + \mathbf{e}_i, \infty)) \notin ((1+\varepsilon)\log n_r)\mathcal{B}\})$$

$$= \mathbf{P}(\cup_{j \leq n_{r+1}} \cup_{i=1}^{2d} \{(\xi(\mathbf{z} + \mathbf{e}_i, \infty), \Xi(\mathbf{z} + \mathbf{e}_i, \infty)) \notin ((1+\varepsilon)\log n_r)\mathcal{B}\} \cap A_j) \leq \frac{c}{n_r^{\varepsilon/2}}$$

and we can complete the proof by using Borel-Cantelli lemma. $\square$

Proof of Theorem 1.3(i).

The proof is similar to that of Theorem 1.1(i).

Introduce the following notation:

$$\Xi(\mathbf{z}, (n, \infty)) := \Xi(\mathbf{z}, \infty) - \Xi(\mathbf{z}, n). \tag{4.8}$$

Define $\Gamma = \Gamma_i := \{\mathbf{e}_i + \mathcal{S}(1)\}$. For $i \in \{1, \ldots, 2d\}$ introduce, as before, the following events for $j \leq n$.

$$B(j, n) := \{(\xi(\mathbf{S}_j + \mathbf{e}_i, n), \Xi(\mathbf{S}_j + \mathbf{e}_i, n)) \notin ((1+\varepsilon)\log n)\mathcal{D}\}, \tag{4.9}$$

$$B^*(j, n) := \{(\xi(\mathbf{S}_j + \mathbf{e}_i, j), \Xi(\mathbf{S}_j + \mathbf{e}_i, j)) \notin ((1+\varepsilon)\log n)\mathcal{D}\}, \tag{4.10}$$

$$C(j, n) := \{\mathbf{S}_m \notin \mathbf{S}_j + \Gamma,\ m = j+1, \ldots, n\}, \tag{4.11}$$

$$D(j, n) := \{\Xi(\mathbf{S}_j + \mathbf{e}_i, (n, \infty)) > 0\}. \tag{4.12}$$

Considering again the reverse random walk starting from $\mathbf{S}_j$, i.e. $\mathbf{S}'_r = \mathbf{S}_{j-r} - \mathbf{S}_j$, $r = 0, 1, \dots, j$ we remark

$$\xi(\mathbf{S}_j + \mathbf{e}_i, j) = \xi'(\mathbf{e}_i, j), \qquad \Xi(\mathbf{S}_j + \mathbf{e}_i, j) = \Xi'(\mathbf{e}_i, j) - 1,$$

where Ξ' is the occupation time of the unit ball of the random walk $\mathbf{S}'$.

>From this we can follow the proof of Theorem 1.1(i), using (2.19) and (3.5) instead of (2.18) and (3.1) and applying Theorem 1.4(i) instead of Theorem 1.2(i). $\Box$

5. Proof of Theorems, Parts (ii)

In this Section we prove parts (ii) of the Theorems.

Proof of Theorem 1.1(ii) and Theorem 1.2(ii).

Without loss of generality we give the proof for $i = 1$. Define the two-point set $\Upsilon := \{\mathbf{0}, \mathbf{e}_1\}$. We say that $\mathbf{S}_j$ $(j = 1, 2, 3 \dots)$ is Υ-new if either $j = 1$, or $j \geq 2$ and

$$\mathbf{S}_m \notin \mathbf{S}_j + \Upsilon, \quad (m = 1, 2, \dots, j-1).$$

Lemma 5.1. *Let ζ_n denote the number of Υ-new points up to time n. Then*

$$\lim_{n\to\infty} \frac{\zeta_n}{n} = 1 - 2\alpha \quad \text{a.s.}$$

Proof. Define

$$Z_j = \begin{cases} 1 \text{ if } \mathbf{S}_j \text{ is } \Upsilon\text{−new} \\ 0 \text{ otherwise} \end{cases}$$

Then $\zeta_n = \sum_{j=1}^n Z_j$ and hence

$$\mathbf{E}(\zeta_n) = \sum_{j=1}^n P(Z_j = 1),$$

$$\mathbf{E}(\zeta_n^2) = \mathbf{E}\left(\sum_{j=1}^n \sum_{i=1}^n Z_j Z_i\right) = \mathbf{E}\left(\sum_{j=1}^n Z_j\right) + 2\mathbf{E}\left(\sum_{j=1}^n \sum_{i=1}^{j-1} Z_j Z_i\right)$$
$$\leq n + 2\sum_{j=1}^n \sum_{i=1}^{j-1} \mathbf{P}(Z_i = 1) P(Z_{j-i} = 1).$$

Considering the reverse random walk from $\mathbf{S}_i$ to $\mathbf{S}_0 = 0$, we see that the event $\{Z_i = 1\}$ is equivalent to the event that the reversed random walk starting from any point of Υ does not return to Υ up to time i. Using Lemma 2.1 and 2.4 we get

$$\mathbf{P}(Z_i = 1) = 1 - q_{\mathbf{e}_1}(i) - s_{\mathbf{e}_1}(i) = 1 - 2\alpha + O(i^{1-d/2}).$$

Hence

$$\begin{aligned}\mathbf{E}(\zeta_n^2) &\le n + 2\sum_{j=1}^{n}\sum_{i=1}^{j-1}\left(1 - 2\alpha + O(i^{1-d/2})\right)\left(1 - 2\alpha + O((j-i)^{1-d/2})\right)\\ &= n(n-1)\,(1-2\alpha)^2 + O(n^{3/2}),\end{aligned}$$

thus

$$Var(\zeta_n) = O(n^{3/2}).$$

By Chebyshev's inequality

$$P\left(|\zeta_n - n\,(1-2\alpha)| > \varepsilon n\right) \le O\left(\frac{1}{\sqrt{n}}\right).$$

Considering the subsequence $n_k = k^3$, and using Borel-Cantelli lemma and the monotonicity of ζ_n, we obtain the lemma. □

Lemma 5.2. *For each $\delta > 0$, there exist a subsequence n_r and r_0 such that if $r \ge r_0$ then for any $(k, \ell) \in ((1-\delta)\log n_r)\mathcal{B} \cap \mathcal{Z}_d$ there exists a random integer $j_r = j_r(k, \ell) \le n_r$ for which*

$$(\xi(\mathbf{S}_{j_r}, n_r), \xi(\mathbf{S}_{j_r} + \mathbf{e}_1, n_r)) = (\xi(\mathbf{S}_{j_r}, \infty), \xi(\mathbf{S}_{j_r} + \mathbf{e}_1, \infty)) = (k+1, \ell).$$

Proof. Let $\{a_n\}$ and $\{b_n\}$ ($a_n \log n \ll b_n \ll n$) be two sequences to be chosen later. Define

$$\theta_1 = \min\{j > b_n : \mathbf{S}_j \text{ is } \Upsilon\text{−new}\},$$

$$\theta_m = \min\{j > \theta_{m-1} + b_n : \mathbf{S}_j \text{ is } \Upsilon\text{−new},\} \quad m = 2, 3, \ldots$$

and let ζ_n' be the number of θ_m points up to time $n - b_n$. Obviously $\zeta_n'(b_n + 1) \ge \zeta_n$, hence $\zeta_n' \ge \zeta_n/(b_n+1)$ and it follows from Lemma 5.1 that for $c < 1-2\alpha$, we have with probability 1 that $\zeta_n' > u_n := [cn/(b_n+1)]$ except for finitely many n.

For $1 \le i \le u_n$ let

$$\rho_0^i = 0, \qquad \rho_h^i = \min\{j > \rho_{h-1}^i : \mathbf{S}_{\theta_i + j} \in \Upsilon\}, \quad h = 1, 2, \ldots$$

For a fixed pair of integers (k,ℓ) define the following events:

$$\begin{aligned}
A_i &= \{\xi(\mathbf{S}_{\theta_i},\theta_i+\rho^i_{k+\ell})=k+1, \xi(\mathbf{S}_{\theta_i}+\mathbf{e}_1,\theta_i+\rho^i_{k+\ell})=\ell,\\
&\qquad \rho^i_h-\rho^i_{h-1}\le a_n,\ h=1,\dots,k+\ell,\ \mathbf{S}_j\notin \mathbf{S}_{\theta_i}+\Upsilon,\ j=\theta_i+\rho^i_{k+\ell}+1,\dots,\theta_i+b_n\},\\
B_i &= \{\mathbf{S}_j\notin\mathbf{S}_{\theta_i}+\Upsilon,\ j>\theta_i+b_n\},\\
C_n &= A_1B_1+\overline{A_1}A_2B_2+\overline{A_1A_2}A_3B_3+\ldots+\overline{A_1}\ldots\overline{A_{u_n-1}}A_{u_n}B_{u_n}.
\end{aligned}$$

Note that if $(k,\ell)\in((1-\delta)\log n)\mathcal{B}$, then $k+\ell\le c\log n$ for some constant c, hence $\rho^i_h-\rho^i_{h-1}\le a_n$, $h=1,\dots,k+\ell$ implies $\rho^i_{k+\ell}\le(k+\ell)a_n\le ca_n\log n\le b_n$ and so the events A_i are well defined and are independent, since A_i depends only on the part of random walk between θ_i and θ_{i+1}. More precisely, the events $A_1,\dots,A_{i-1},A_iB_i$ are independent. Moreover, $\mathbf{P}(A_i)=\mathbf{P}(A_1)$ and $\mathbf{P}(A_iB_i)=\mathbf{P}(A_1B_1)$, $i=2,3,\dots$ Hence we have

$$\mathbf{P}(C_n)=\mathbf{P}(A_1B_1)\sum_{j=0}^{u_n-1}(1-\mathbf{P}(A_1))^j=\frac{\mathbf{P}(A_1B_1)}{\mathbf{P}(A_1)}(1-(1-\mathbf{P}(A_1))^{u_n}.$$

$$\mathbf{P}(\overline{C_n})\le 1-\frac{\mathbf{P}(A_1B_1)}{\mathbf{P}(A_1)}+e^{-u_n\mathbf{P}(A_1)}$$

$$\begin{aligned}
\mathbf{P}(A_1B_1)&=\mathbf{P}(D\cap\{\mathbf{S}_j\notin\Upsilon,\ j=\rho_{k+\ell}+1,\rho_{k+\ell}+2,\dots\})=(1-2\alpha)\mathbf{P}(D),\\
\mathbf{P}(A_1)&=\mathbf{P}(D\cap\{\mathbf{S}_j\notin\Upsilon,\ j=\rho_{k+\ell}+1,\dots,b_n\})=(1-2\alpha+O(b_n^{1-d/2}))\mathbf{P}(D),
\end{aligned}\tag{5.1}$$

where

$$\rho_0=0,\qquad \rho_h=\min\{j>\rho_{h-1}:\mathbf{S}_j\in\Upsilon\},\quad h=1,2,\dots,$$

$$D=\{\xi(\mathbf{0},\rho_{k+\ell})=k,\xi(\mathbf{e}_1,\rho_{k+\ell})=\ell,\rho_h-\rho_{h-1}\le a_n,\ h=1,\dots,k+\ell\}.$$

In (5.1) we used that by Lemmas 2.1, 2.4 and remembering that $q_{\mathbf{e}_1}=s_{\mathbf{e}_1}=\alpha$, we have

$$\mathbf{P}(D\cap\{\mathbf{S}_j\notin\Upsilon,\ j=\rho_{k+\ell}+1,\dots,b_n\})$$

$$\le\mathbf{P}(D)(1-q_{\mathbf{e}_1}(b_n-(k+\ell)a_n)-s_{\mathbf{e}_1}(b_n-(k+\ell)a_n))=\mathbf{P}(D)(1-2\alpha+O(b_n^{1-d/2})).$$

Consequently,

$$\frac{\mathbf{P}(A_1B_1)}{\mathbf{P}(A_1)}=1+O(b_n^{1-d/2}),$$

therefore

$$\mathbf{P}(\overline{C_n})\le O(b_n^{1-d/2})+e^{-cn\mathbf{P}(A_1)/b_n}.$$

Choosing $b_n = n^{\delta/2}$, $a_n = n^{\delta/4}$, we prove

$$\mathbf{P}(A_1) \geq \frac{c}{n^{1-\delta}}. \tag{5.2}$$

Using (2.14) and (2.15) of Lemma 2.4 for $\mathbf{z} = \mathbf{e}_1$ we get

$$\mathbf{P}(A_1) \geq (1-2\alpha)\binom{k+\ell}{\ell}\left(\alpha + O(a_n^{1-d/2})\right)^{k+\ell} \geq c\binom{k+\ell}{\ell}\alpha^{k+\ell},$$

since if $(k,\ell) \in (\log n)\mathcal{B}$, then $k+\ell = O(\log n)$. Now (5.2) follows from Stirling formula, similarly to (3.1).

Using (5.2) we can verify that $\sum_r \mathbf{P}(\overline{C}_{n_r}) < \infty$ for $n_r = r^\rho$ with $\rho\delta(d-2) > 4$.

By Borel-Cantelli lemma, with probability 1, C_{n_r} occurs for all but finitely many r. This completes the proof of Lemma 5.2. $\Box$

On choosing $\delta = \varepsilon/2$, we can see for $n_r \leq n < n_{r+1}$

$$((1-\varepsilon)\log n)\mathcal{B} \subset ((1-\varepsilon/2)\log n_r)\mathcal{B}$$

for large enough r and since $\xi(\mathbf{S}_{j_r}, n)$ and $\xi(\mathbf{S}_{j_r} + \mathbf{e}_1, n)$ do not change for $n \geq n_r$, we have the Theorem 1.2(ii) and the first statement of Theorem 1.1(ii). The second statement in this Theorem follows by symmetry. $\Box$

Proof of Theorem 1.3(ii) and Theorem 1.4(ii).

The proof in this subsection is almost the same as in the previous one, so we skip some details. Without loss of generality, the proof is given for $i = 1$. Let $\Gamma = \Gamma_1$ as defined in the proof of Theorem 1.3(i), i.e. Γ is the unit ball centered at $\mathbf{e}_1$. $\mathbf{S}_j$ ($j = 1, 2, 3\ldots$) is called Γ-new if either $j = 1$, or $j \geq 2$ and

$$\mathbf{S}_m \notin \mathbf{S}_j + \Gamma, \quad (m = 1, 2, \ldots, j-1).$$

Lemma 5.3. *Let ν_n denote the number of Γ-new points up to time n. Then*

$$\lim_{n\to\infty} \frac{\nu_n}{n} = 1 - p - \frac{1}{2d} \quad \text{a.s.}$$

Proof. Define

$$Z_j = \begin{cases} 1 & \text{if } \mathbf{S}_j \text{ is } \Gamma\text{−new} \\ 0 & \text{otherwise} \end{cases}$$

Then $\nu_n = \sum_{j=1}^n Z_j$.

Considering the reverse random walk from $\mathbf{S}_i$ to $\mathbf{S}_0 = 0$, we see that the event $\{Z_i = 1\}$ is equivalent to the event that the reversed random walk starting from any point of $S(1)$ does not return to $S(1)$ up to time i. Using Lemma 2.4 we get

$$P(Z_i = 1) = 1 - p(i) - \frac{1}{2d} = 1 - p - \frac{1}{2d} + O(i^{1-d/2}).$$

The rest of the argument is identical with that of Lemma 5.1.

Lemma 5.4. *For each $\delta > 0$, there exist a subsequence n_r and r_0 such that if $r \geq r_0$ then for any $(k, \ell) \in ((1-\delta)\log n_r)\mathcal{D} \cap \mathcal{Z}_d$ there exists a random integer $j_r = j_r(k, \ell) \leq n_r$ for which*

$$(\xi(\mathbf{S}_{j_r} + \mathbf{e}_1, n_r), \Xi(\mathbf{S}_{j_r} + \mathbf{e}_1, n_r)) = (\xi(\mathbf{S}_{j_r} + \mathbf{e}_1, \infty), \Xi(\mathbf{S}_{j_r} + \mathbf{e}_1, \infty)) = (k, \ell + 1).$$

Proof. Let $\{a_n\}$ and $\{b_n\}$ ($a_n \log n \ll b_n \ll n$) be two sequences to be chosen later. Define

$$\theta_1 = \min\{j > b_n : S_j \,\text{is}\, \Gamma\text{−new}\},$$

$$\theta_m = \min\{j > \theta_{m-1} + b_n : S_j \,\text{is}\, \Gamma\text{−new}\}, \quad m = 2, 3, \ldots$$

and let ν_n' be the number of θ_m points up to time $n - b_n$. Obviously $\nu_n'(b_n + 1) \geq \nu_n$, hence $\nu_n' \geq \nu_n/(b_n + 1)$ and it follows from Lemma 5.3 that for $c < 1 - p - \frac{1}{2d}$, we have with probability 1 that $\nu_n' > u_n := cn/(b_n + 1)$ except for finitely many n.

Let

$$\rho_0^i = 0, \qquad \rho_h^i = \min\{j > \rho_{h-1}^i : \mathbf{S}_{\theta_i + j} \in \Gamma\}, \quad h = 1, 2, \ldots$$

For a fixed pair of integers (k, ℓ) define the following events:

$$\begin{aligned}
A_i &= \{\xi(\mathbf{S}_{\theta_i} + \mathbf{e}_1, \theta_i + \rho_\ell^i) = k, \Xi(\mathbf{S}_{\theta_i} + \mathbf{e}_1, \theta_i + \rho_\ell^i) = \ell + 1, \\
&\quad\ \rho_h^i - \rho_{h-1}^i \leq a_n,\, h = 1, \ldots, \ell,\, \mathbf{S}_j \notin \mathbf{S}_{\theta_i} + \Gamma,\, j = \theta_i + \rho_\ell^i + 1, \ldots, \theta_i + b_n\}, \\
B_i &= \{\mathbf{S}_j \notin \mathbf{S}_{\theta_i} + \Gamma,\, j > \theta_i + b_n\}, \\
C_n &= A_1 B_1 + \overline{A_1} A_2 B_2 + \overline{A_1 A_2} A_3 B_3 + \ldots + \overline{A_1} \ldots \overline{A_{u_n - 1}} A_{u_n} B_{u_n}.
\end{aligned}$$

Similarly to the proof of Lemma 5.2, $\mathbf{P}(A_i) = \mathbf{P}(A_1)$ and $\mathbf{P}(A_i B_i) = \mathbf{P}(A_1 B_1)$, $i = 2, 3, \ldots$ and

$$\mathbf{P}(C_n) = \mathbf{P}(A_1 B_1) \sum_{j=0}^{u_n - 1} (1 - \mathbf{P}(A_1))^j = \frac{\mathbf{P}(A_1 B_1)}{\mathbf{P}(A_1)} (1 - (1 - \mathbf{P}(A_1))^{u_n},$$

$$\mathbf{P}(\overline{C_n}) \le 1 - \frac{\mathbf{P}(A_1B_1)}{\mathbf{P}(A_1)} + e^{-u_n\mathbf{P}(A_1)}.$$

$$\mathbf{P}(A_1B_1) = \binom{\ell}{k}\left(1 - p - \frac{1}{2d}\right)(p(a_n))^{\ell-k}\left(\frac{1}{2d}\right)^k,$$

$$\mathbf{P}(A_1) \le \binom{\ell}{k}\left(1 - p(b_n - \ell a_n) - \frac{1}{2d}\right)(p(a_n))^{\ell-k}\left(\frac{1}{2d}\right)^k.$$

By Lemma 2.4

$$\frac{\mathbf{P}(A_1B_1)}{\mathbf{P}(A_1)} = 1 + O(b_n^{1-d/2}),$$

therefore

$$\mathbf{P}(\overline{C_n}) \le O(b_n^{1-d/2}) + e^{-cn\mathbf{P}(A_1)/b_n}.$$

Choosing $b_n = n^{\delta/2}$, $a_n = n^{\delta/4}$, we can prove similarly to (5.2)

$$\mathbf{P}(A_1) \ge \frac{1}{n^{1-\delta}}$$

and verify that $\sum_r \mathbf{P}(\overline{C}_{n_r}) < \infty$ for $n_r = r^\rho$ with $\rho\delta(d-2) > 4$.

By Borel-Cantelli lemma, with probability 1, C_{n_r} occurs for all but finitely many r. This completes the proof of Lemma 5.4. $\Box$

On choosing $\delta = \varepsilon/2$, we can see for $n_r \le n < n_{r+1}$

$$((1-\varepsilon)\log n)\mathcal{D} \subset ((1-\varepsilon/2)\log n_r)\mathcal{D}$$

for large enough r and since $\xi(\mathbf{S}_{j_r} + \mathbf{e}_1, n)$ and $\Xi(\mathbf{S}_{j_r} + \mathbf{e}_1, n)$ do not change for $n \ge n_r$, we have the statements (ii) of both Theorems 1.3 and 1.4. $\Box$

6. Further discussions

Observe that the following points are on the curve $g(x,y) = 1$ (see Figure 1):

$$\left(0, \frac{1}{\log(1/\alpha)}\right), \quad \left(\frac{1}{\log(1/\alpha)}, 0\right),$$

$$(\lambda, \lambda(1-\gamma)), \quad (\lambda(1-\gamma), \lambda),$$

$$\left(\frac{1}{2\log(1/(2\alpha))}, \frac{1}{2\log(1/(2\alpha))}\right).$$

In the following discussion we are having almost sure statements, which we will not be emphasize over and over again.

Our Theorem 1.1 shows that there are points $\mathbf{z}_n$ with

$$\xi(\mathbf{z}_n, n) = 0, \qquad \text{and} \qquad \xi(\mathbf{z}_n + \mathbf{e}_1, n) \sim \frac{\log n}{\log(1/\alpha)}.$$

On the other hand, if for a point $\mathbf{z}_n$,

$$\xi(\mathbf{z}_n, n) > (1+\varepsilon)\frac{\log n}{\log(1/\alpha)},$$

then for all of its neighbors we have $\xi(\mathbf{z}_n + \mathbf{e}_i, n) > c \log n$ for some $c > 0$. Moreover, if $\xi(\mathbf{z}_n, n) \sim \lambda \log n$ then for all of its neighbors $\xi(\mathbf{z}_n + \mathbf{e}_i, n) \sim \lambda(1-\gamma)\log n$. Roughly speaking if a point has nearly maximal local time, it essentially determines the local time of its neighbors, and hence the occupation time of the surface of the unit ball around it.

For the maximal occupation time of neighboring pairs we can obtain

$$\lim_{n\to\infty} \frac{\sup_{\mathbf{z}\in\mathcal{Z}_d}(\xi(\mathbf{z}, n) + \xi(\mathbf{z}+\mathbf{e}_i, n))}{\log n} = \frac{1}{\log \frac{1}{2\alpha}},$$

and for $\mathbf{z}_n$, where the sup is attained, we have, as $n \to \infty$,

$$\xi(\mathbf{z}_n, n) \sim \xi(\mathbf{z}_n + \mathbf{e}_i, n) \sim \frac{\log n}{2\log\frac{1}{2\alpha}}.$$

It is easy to calculate the maximal local time difference between two neighboring points.

$$\lim_{n\to\infty} \frac{\sup_{\mathbf{z}\in\mathcal{Z}_d}(\xi(\mathbf{z}, n) - \xi(\mathbf{z}+\mathbf{e}_i, n))}{\log n} = \frac{1}{\log \frac{1+\sqrt{1-4\alpha^2}}{2\alpha}},$$

and for z_n where the sup is attained, we have, as $n \to \infty$,

$$\xi(\mathbf{z}_n, n) \sim \frac{1+\sqrt{1-4\alpha^2}}{2\sqrt{1-4\alpha^2}} \frac{\log n}{\log\frac{1+\sqrt{1-4\alpha^2}}{2\alpha}}, \qquad \xi(\mathbf{z}_n + \mathbf{e}_i, n) \sim \frac{1-\sqrt{1-4\alpha^2}}{2\sqrt{1-4\alpha^2}} \frac{\log n}{\log\frac{1+\sqrt{1-4\alpha^2}}{2\alpha}}.$$

Considering now the joint behavior of the local time of a point and the occupation time of the surface of the unit ball around it, observe that the following points are on the curve $f(x, y) = 1$ (see Figure 2):

$$\left(0, \frac{1}{\log(1/p)}\right), \left(\frac{1}{\log(2d)}, \frac{1}{\log(2d)}\right), \left(\frac{\kappa}{2dp+1}, \kappa\right), \left(\lambda, \frac{\lambda}{1-p}\right).$$

As a conclusion of Theorem 1.3 we have that there are points $\mathbf{z}_n$ with

$$\xi(\mathbf{z}_n, n) = 0 \qquad \text{and} \qquad \Xi(\mathbf{z}_n, n) \sim \frac{\log n}{\log(1/p)}.$$

On the other hand, if for a point $\mathbf{z}_n$

$$\Xi(\mathbf{z}_n, n) > (1+\varepsilon)\frac{\log n}{\log(1/p)},$$

then for its center we have $\xi(\mathbf{z}_n, n) > c\log n$ for some $c > 0$. Moreover, if $\xi(\mathbf{z}_n, n) \sim \lambda \log n$, then for the unit ball

$$\Xi(\mathbf{z}_n, n) \sim \frac{\lambda \log n}{1-p}.$$

Roughly speaking if a point has nearly maximal local time, it essentially determines the occupation time of the surface of the unit ball around it.

Observe that from (2.5) it follows that $\lambda/(1-p) = 2d\lambda(1-\gamma)$, hence we may conclude that for a ball having maximal local time at the center, the occupation time of the surface is $2d$ times the "deterministic" local time of a point having a neighbor with maximal local time. Consequently, all surface points of a unit ball having maximal local time at the center, have approximately the same local time. Moreover, if the occupation time of the surface of a unit ball is around the maximal value, i.e. $\Xi(\mathbf{z}_n, n) \sim \kappa \log n$, then for the local time of its center we have

$$\xi(\mathbf{z}_n, n) \sim \frac{\kappa \log n}{2dp+1}.$$

Finally we conclude that even though it is natural that we can find unit balls having the same occupation time of the surface as the local time of its center, the fact that it is also possible when this common value is fairly big is quite surprising. Namely it is possible that

$$\xi(\mathbf{z}_n, n) \sim \Xi(\mathbf{z}_n, n) \sim \frac{\log n}{\log(2d)}.$$

With a little extra computation one can easily calculate (asymptotically) the maximal weight of the unit ball;

$$w(\mathbf{z}, n) := \xi(\mathbf{z}, n) + \Xi(\mathbf{z}, n), \quad w(n) := \sup_{\mathbf{z}\in\mathcal{Z}_d} (\xi(\mathbf{z}, n) + \Xi(\mathbf{z}, n)).$$

This was already done in [2]. However from Theorem 1.3 we get the following observation as well: for $d \geq 4$ if we know that either one of the three quantities of $\xi(\mathbf{z}, n)$, $\Xi(\mathbf{z}, n)$ or $w(\mathbf{z}, n)$

is (asymptotically) maximal, then this maximal value uniquely determines the values of the other two (asymptotically). For completeness here are the numerical results;

$$\lim_{n\to\infty} \frac{w(n)}{\log n} = -\frac{1}{\log\left(\frac{p}{2} + \sqrt{\frac{p^2}{4} + \frac{1}{2d}}\right)} =: C \qquad \text{a.s.}$$

Whenever $w(\mathbf{z}_n, n) \sim C \log n$, then

$$\xi(\mathbf{z}_n, n) \sim \frac{C}{2+A} \log n, \quad \text{and} \quad \Xi(\mathbf{z}_n, n) \sim C\frac{1+A}{2+A} \log n,$$

where

$$A = dp^2 + \sqrt{d^2p^4 + 2dp^2}.$$

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