ON THE NUMBER OF CUTPOINTS OF THE TRANSIENT NEAREST NEIGHBOR RANDOM WALK ON THE LINE

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Abstract: We consider transient nearest neighbor random walks on the positive part of the real line. We give criteria for the finiteness of the number of cutpoints and strong cutpoints. Examples and open problems are presented.

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1. Introduction

Let $X_0 = 0,\ X_1, X_2, \ldots$ be a Markov chain with

$$\begin{aligned} E_i &:= \mathbf{P}(X_{n+1} = i+1 \mid X_n = i) = 1 - \mathbf{P}(X_{n+1} = i-1 \mid X_n = i) \\ &= \begin{cases} 1 & \text{if} \quad i = 0 \\ 1/2 + p_i & \text{if} \quad i = 1, 2, \ldots, \end{cases} \end{aligned} \tag{1.1}$$

where $-1/2 < p_i < 1/2,\ i = 1, 2, \ldots$.

Theorem A ([2], page 74) *Let X_n be a Markov chain with transition probabilities given in* (1.1). *Define*

$$U_i := \frac{1 - E_i}{E_i} = \frac{1/2 - p_i}{1/2 + p_i}, \qquad i = 1, 2, \ldots. \tag{1.2}$$

Then X_n is transient if and only if

$$\sum_{k=1}^{\infty} \prod_{i=1}^{k} U_i < \infty. \tag{1.3}$$

In case $p_i \geq 0$ the sequence $\{X_i\}$ describes the motion of a particle which starts at zero, moves over the nonnegative integers and going away from 0 with a larger probability than to the direction of 0. We suppose throughout this paper that $0 \leq p_i < 1/2$, $i = 1, 2, \ldots$.

In [3] we introduced the quantities

$$D(m, n) := \begin{cases} 0 & \text{if } n = m, \\ 1 & \text{if } n = m + 1, \\ 1 + \sum_{j=1}^{n-m-1} \prod_{i=1}^{j} U_{m+i} & \text{if } n \geq m + 2 \end{cases} \tag{1.4}$$

and

$$\lim_{n \to \infty} D(m, n) =: D(m). \tag{1.5}$$

Clearly (1.3) implies that if the walk is transient then $D(m)$ is finite for all $m = 1, 2, \ldots$.

The properties of this Markov chain, often called birth and death chain were extensively studied. Some of these results are mentioned e.g. in [3]. Our main concern in that paper was to study the local time of $\{X_n\}$, defined by

$$\xi(x, n) := \#\{k : 0 \leq k \leq n,\ X_k = x\}, \quad x = 0, 1, 2, \ldots, \quad n = 1, 2, \ldots \tag{1.6}$$

and

$$\xi(x) := \lim_{n\to\infty} \xi(x,n). \tag{1.7}$$

The first topic in that paper was to find upper class results for the local time.

Theorem B *Assume that $p_R \to 0$ as $R \to \infty$. Then with probability* 1 *we have*

$$\xi(R) \le 2(1+\varepsilon)D(R)\log R \tag{1.8}$$

for any $\varepsilon > 0$ if R is large enough.

Moreover,

$$\xi(R) \ge MD(R) \quad \text{i.o. a.s.} \tag{1.9}$$

for any $M > 0$.

The next question was how small can the local time be. In particular, we studied the number of sites R where $\xi(R) = 1$. We found that the answer heavily depends on the sequence $\{p_R\}_{R=1}^{\infty}$.

We will say that the NN walk X_n^* is *slower* than X_n (or equivalently, X_n is *quicker* than X_n^*) if

$$p_R^* \le p_R \quad \text{for all } R = 1, 2, \ldots \tag{1.10}$$

It is obvious that the quicker is X_n, the more sites with local time equal to 1 will occur.

Remark 1. In (1.10) the required inequality could be relaxed to hold for all but finitely many R only, since finitely many p_R have no effect on the asymptotic behavior of the walk. The same remark applies throughout the paper, when we require certain properties of the $\{p_R\}$ system.

Introduce the following notations:

$$\begin{aligned}
\Lambda(1,i,B) &= \frac{B}{i},\\
\Lambda(2,i,B) &= \frac{1}{i} + \frac{B}{i\log i},\\
&\ldots,\\
\Lambda(K,i,B) &= \frac{1}{i} + \frac{1}{i\log i} + \ldots + \frac{B}{i\log i\log\log i\ldots\log_{K-1} i}.
\end{aligned}$$

In [3] we proved the following

Fact 1 *If for any $K = 1, 2, \ldots$*

$$p_i = \frac{\Lambda(K,i,B)}{4},$$

then the Markov chain $\{X_n\}$ is recurrent if $B \leq 1$ and transient if $B > 1$.

In the spirit of Remark 1 above, it is enough if p_i takes the value given above with finitely many exceptions, but assuming that $0 \leq p_i < 1/2$ for all $i = 1, 2, \ldots$.

We proved in [3] that if $p_i = \dfrac{\Lambda(1, i, B)}{4}$ with $B > 1$, then we not only have infinitely many sites with local time 1, but we have infinitely many increasing runs of sites each having local time 1. More precisely we have

Theorem C *Let $\{X_n\}$ be an NN random walk with $p_R = \dfrac{\Lambda(1, R, B)}{4} = \dfrac{B}{4R}$ and $B > 1$. Then with probability 1 there exist infinitely many R for which*

$$\xi(R + j) = 1$$

for each $j = 0, 1, 2, \ldots, \left[\dfrac{\log\log R}{\log 2}\right]$.

However, if X_n^* is transient but slower than X_n in Theorem C, then one might ask whether it still has infinitely many sites with local time 1. It turns out that this is not always true. James et al. [7] proved a surprising result which implies the following

Theorem D *If $\{X_n\}$ is an NN random walk with $p_R = \dfrac{\Lambda(2, R, B)}{4}$ and $B > 1$, then with probability 1 X_n has only finitely many sites R with $\xi(R) = 1$.*

In fact, they formulated their results in terms of cutpoints. Call the site R a *cutpoint* if for some k, we have $X_k = R$ and $\{X_0, X_1 \ldots X_k\}$ is disjoint from $\{X_{k+1}, X_{k+2} \ldots\}$, i.e. $X_i \leq R$, $i = 0, 1, \ldots, k$, $X_k = R$ and $X_i > R$, $i = k + 1, k + 2, \ldots$

The original version of Theorem C in [7] reads as follows.

Theorem D* *If $\{X_n\}$ is an NN random walk with*

$$\frac{c_1}{k(\log k)^\beta} \leq U_1 U_2 \ldots U_k \leq \frac{c_2}{k(\log k)^\beta}$$

for some $\beta > 1$ and positive constants c_1, c_2, then $\{X_n\}$ is transient and has only finitely many cutpoints a.s.

Cutpoints and related intersection problems for more general stochastic processes have been investigated extensively in the literature, starting with Dvoretzky et al. [4], Erdős and Taylor [5]. A nice summary of this topic is given by Lawler [8].

For usual random walk (sums of i.i.d. random variables) we mention the following general result of James and Peres [6], where the definition of cutpoint is somewhat different from above. S_k is called a cutpoint there if $p(S_i, S_j) = 0$ for all (i, j) such that $0 \leq i < k < j$, where $p(x, y)$ is the one-step transition probability from x to y.

Theorem E *Any transient random walk $\{S_k\}$ with bounded increments on the lattice Z^d has infinitely many cutpoints a.s.*

To formulate our main result, we introduce the following definitions.

Call the site R a *strong cutpoint* if for some k, we have $X_k = R$, $X_i < R$, $i = 0, 1, \ldots, k-1$ and $X_i > R$, $i = k+1, k+2, \ldots$. Observe that R is a strong cutpoint if and only if $\xi(R) = 1$. Clearly every strong cutpoint is a cutpoint.

In this paper we give a criteria for a transient NN random walk which determines whether the number of cutpoints (or strong cutpoints) is finite or infinite almost surely.

Theorem 1.1. *Let $X_0 = 0, X_1, X_2, \ldots$ be a transient Markov chain with transition probability E_i as in* (1.1) *and $0 \le p_i < 1/2$, $i = 1, 2, \ldots$ Let $D(n)$, $n = 1, 2, \ldots$ be as in* (1.5).

- *If*
$$\sum_{n=2}^{\infty} \frac{1}{D(n) \log n} < \infty, \tag{1.11}$$
then $\{X_n\}$ has finitely many cutpoints almost surely.

- *If $D(n) \le \delta n \log n$ $(n \ge n_0)$ for some $\delta > 0$ and*
$$\sum_{n=2}^{\infty} \frac{1}{D(n) \log n} = \infty,$$
then $\{X_n\}$ has infinitely many strong cutpoints almost surely.

Remark 2. Observe that if the sum in (1.11) is finite then $\{X_n\}$ has finitely many strong cutpoints. On the other hand, if the same sum is divergent and $D(n) \le \delta n \log n$, then $\{X_n\}$ has infinitely many cutpoints as well.

Remark 3. The condition $D(n) \le \delta n \log n$ of the second statement is a technical one, most probably it can be removed.

The above mentioned technical condition prevent us to establish the following

Conjecture 1.1 *The number of cutpoints is finite if and only if the number of strong cutpoints is finite.*

In Section 2 we will present some preliminary results. Sections 3 and 4 are devoted to prove Theorem 1.1. In Section 5 we give some examples and open problems.

2 Preliminary results

For $0 \le a \le b \le c$ integers define

$$p(a,b,c) := \mathbf{P}(\min\{j:\ j>m,\ X_j=a\} < \min\{j:\ j>m,\ X_j=c\} \mid X_m=b),$$

i.e. $p(a,b,c)$ is the probability that a particle starting from b hits a before c.

Lemma A *For* $0 \le a \le b \le c$

$$p(a,b,c) = 1 - \frac{D(a,b)}{D(a,c)}.$$

Especially, for $n = 1,2,\dots$ *we have*

$$p(0,1,n) = 1 - \frac{1}{D(0,n)}, \quad p(n,n+1,\infty) = 1 - \frac{1}{D(n)}. \tag{2.1}$$

It is easy to see that

$$D(n) = 1 + D(n+1)U_{n+1},$$

$$U_n = \frac{D(n-1)-1}{D(n)}, \quad p_n = \frac{1}{2}\,\frac{D(n)-D(n-1)+1}{D(n)+D(n-1)-1}, \quad n = 1,2,\dots \tag{2.2}$$

Then observe that for $n \ge m+2$

$$\begin{aligned} D(m,n) &= 1 + \sum_{j=1}^{n-m-1}\prod_{i=1}^{j} U_{m+i} = D(m) - \sum_{j=n-m}^{\infty}\prod_{i=1}^{j} U_{m+i} = D(m) - U_{m+1}\dots U_n D(n) \\ &= D(m) - \frac{(D(m)-1)\dots(D(n-1)-1)}{D(m+1)\dots D(n-1)} = D(m)\left(1 - \prod_{i=m}^{n-1}\left(1 - \frac{1}{D(i)}\right)\right). \end{aligned} \tag{2.3}$$

We also define the number of upcrossings by

$$\xi(R,n,\uparrow) := \#\{k:\ 0 \le k \le n,\ X_k = R,\ X_{k+1} = R+1\}. \tag{2.4}$$

$$\xi(R,\uparrow) := \lim_{n\to\infty} \xi(R,n,\uparrow). \tag{2.5}$$

It was shown in [3] that
Lemma B *For* $R = 0,1,2,\dots$

$$\mathbf{P}(\xi(R) = L) = \frac{1+2p_R}{2D(R)}\left(1 - \frac{1+2p_R}{2D(R)}\right)^{L-1}, \quad L = 1,2,\dots \tag{2.6}$$

$$\mathbf{P}(\xi(R,\uparrow)=L)=\frac{1}{D(R)}\left(1-\frac{1}{D(R)}\right)^{L-1}, \quad L=1,2,\ldots \tag{2.7}$$

It is easy to see that R is a cutpoint if and only if $\xi(R,\uparrow)=1$. Recall that R is a strong cutpoint if and only if $\xi(R)=1$.

Denote the (random) set of

- cutpoints by C
- strong cutpoints by C^S

Observe that

$$C^S \subseteq C. \tag{2.8}$$

We present the following exact probabilities.

Lemma 2.1. *For $k=1,2,\ldots$ we have*

$$\mathbf{P}(k\in C)=1-p(k,k+1,\infty)=\frac{1}{D(k)}, \tag{2.9}$$

$$\mathbf{P}(k\in C^S)=\frac{1+2p_k}{2D(k)}, \tag{2.10}$$

$$\mathbf{P}(j\in C,k\in C)=\frac{1}{D(j,k+1)D(k)}, \quad j<k, \tag{2.11}$$

$$\mathbf{P}(j\in C^S,k\in C^S)=\left(\frac{1}{2}+p_j\right)\left(\frac{1}{2}+p_k\right)\frac{1}{D(j,k)D(k)}, \quad j<k. \tag{2.12}$$

Proof. The statements (2.9) and (2.10) follow from Lemma B.

To show (2.11), we have to observe that after the first arrival to $j+1$ the walk has to arrive to $k+1$ without hitting j, and from $k+1$ it must not return to k at all. Formally, by Lemma A

$$\mathbf{P}(j\in C,k\in C)=(1-p(j,j+1,k+1))(1-p(k,k+1,\infty))=\frac{1}{D(j,k+1)D(k)}.$$

To show (2.12), observe that after the first hit of j the walk has to go to $j+1$. Then from $j+1$ it has to hit k before it goes back to j. From k it has to go to $k+1$, and from $k+1$ it must not return to k at all. Hence again by Lemma A,

$$\mathbf{P}(j\in C^S,k\in C^S)=\left(\frac{1}{2}+p_j\right)(1-p(j,j+1,k))\left(\frac{1}{2}+p_k\right)(1-p(k,k+1,\infty))$$

$$= \left(\frac{1}{2} + p_j\right)\left(\frac{1}{2} + p_k\right)\frac{1}{D(j,k)D(k)}.$$

This completes the proof of the Lemma. □

Lemma 2.2. *For any positive non-decreasing function $G(x)$, $x \geq 0$, the following two sums*

$$\sum_{n=2}^{\infty} \frac{1}{G(n)\log n} \qquad\qquad \sum_{n=2}^{\infty} \frac{1}{G([n\log n])}$$

are equiconvergent.

Proof. Under the condition of the Lemma we have

$$\int_n^{n+1} \frac{dx}{G(x)\log x} \leq \frac{1}{G(n)\log n} \leq \int_{n-1}^{n} \frac{dx}{G(x)\log x},$$

consequently

$$\int_2^{\infty} \frac{dx}{G(x)\log x} \leq \sum_{n=2}^{\infty} \frac{1}{G(n)\log n} \leq \frac{1}{G(2)\log 2} + \int_2^{\infty} \frac{dx}{G(x)\log x}.$$

Similarly,

$$\int_2^{\infty} \frac{dx}{G(x\log x)} \leq \sum_{n=2}^{\infty} \frac{1}{G([n\log n])} \leq \frac{1}{G(1)} + \frac{1}{G(3)} + \int_2^{\infty} \frac{dx}{G(x\log x)}.$$

It remains to show that the integrals

$$\int^{\infty} \frac{dx}{G(x)\log x} \quad \text{and} \quad \int^{\infty} \frac{dx}{G(x\log x)}$$

are equiconvergent. This can be shown by using the substitution $x = y\log y$ in the first integral above to get

$$\int^{\infty} \frac{dx}{G(x)\log x} = \int^{\infty} \frac{1+\log y}{\log y + \log\log y}\,\frac{dy}{G(y\log y)}.$$

Clearly we have

$$c_1 \int^{\infty} \frac{dy}{G(y\log y)} \leq \int^{\infty} \frac{1+\log y}{\log y + \log\log y}\,\frac{dy}{G(y\log y)} \leq c_2 \int^{\infty} \frac{dy}{G(y\log y)}$$

with some $0 < c_1 < c_2$, hence the Lemma. □

Lemma 2.3. *If X_n and X_n^* are two NN random walks such that X_n^* is slower than X_n then*

$$D(n) \le D^*(n).$$

Proof. If X_n^* is slower than X_n then

$$U_i = \frac{1/2 - p_i}{1/2 + p_i} \le \frac{1/2 - p_i^*}{1/2 + p_i^*} = U_i^*$$

Hence from the definition of $D(n)$ we get that

$$D(n) \le D^*(n), \tag{2.13}$$

proving our Lemma. □

We will need the following Lemma from Polfeldt [9], which is a particular case of his theorem.

Lemma C *Let $S(x)$ be a slowly varying function such that*

$$\lim_{x\to\infty} \frac{(\log S(x))'}{(\log\log x)'} = -\infty$$

and

$$\lim_{x\to\infty} \frac{(\log S(x))'}{(\log x)'} \log L(x) = -1$$

for some normalized differentiable slowly varying function $L(x)$. Then

$$\int_x^\infty \frac{S(t)}{t}\,dt \sim S(x)\log L(x), \quad \text{as} \quad x \to \infty.$$

Recall (see Bingham et al. [1] page 12-15) that a slowly varying function $H(x)$ can be represented as

$$H(x) = a(x)\exp\left(\int_b^x \frac{\varepsilon(t)}{t}\,dt\right),$$

where $a(x) \to a \neq 0$, $\varepsilon(x) \to 0$ as $x \to \infty$. If $a(x) = a$, then H is normalized.

Moreover, a differentiable slowly varying function is normalized if and only if

$$\frac{xH'(x)}{H(x)} \to 0 \quad \text{as} \quad x \to \infty.$$

3 Proof of the convergent part

We follow the ideas of [7]. We have $\mathbf{P}(j \in C | k \in C) = 1/D(j, k+1)$ for $j < k$. Observe that it is also the probability that

$$\mathbf{P}(j \in C | k \in C, F_{k+1}) \quad j < k, \tag{3.1}$$

where F_{k+1} is any event determined by the future of the walk after it reaches $k+1$ for the first time. Let $C_{j,k}$ be the set of cutpoints in $(2^j, 2^k]$ and $A_{j,k} := |C_{j,k}|$ the number of cutpoints in $(2^j, 2^k]$. Define

$$a_m := \mathbf{P}(A_{m,m+1} > 0) \tag{3.2}$$

and

$$b_m := \min_{k \in (2^m, 2^{m+1}]} \sum_{i=1}^{2^{m-1}} \frac{1}{D(k-i, k+1)} \tag{3.3}$$

On the event that $A_{m,m+1} > 0$, let ℓ_m be the largest cutpoint in $C_{m,m+1}$. We want to give a lower bound for the expected number of cutpoints in $(2^{m-1}, 2^{m+1}]$ by conditioning on the last cutpoint in $(2^m, 2^{m+1}]$, if there is one:

$$\begin{aligned} \sum_{j=2^{m-1}+1}^{2^{m+1}} \mathbf{P}(j \in C) &= \mathbf{E}(A_{m-1,m+1}) \\ &\geq a_m \mathbf{E}(A_{m-1,m+1} |\, A_{m,m+1} > 0) \\ &= a_m \mathbf{E}(\mathbf{E}(A_{m-1,m+1} |\, A_{m,m+1} > 0, \ell_m)) \\ &\geq a_m\, b_m. \end{aligned} \tag{3.4}$$

It is readily seen that if $p_i \geq 0$, $i = 1, 2, \ldots$, then $U_i \leq 1$, $i = 1, 2, \ldots$ and hence

$$D(m, n) \leq n - m,$$

and so

$$b_m \geq \sum_{i=1}^{2^{m-1}} \frac{1}{i+1} \geq c\, m \tag{3.5}$$

with some $c > 0$.

Hence with constants c not necessarily the same on each appearance,

$$\sum_{m=1}^{\infty} \mathbf{P}(A_{m,m+1}) = \sum_{m=1}^{\infty} a_m \leq \sum_{m=1}^{\infty} \frac{1}{b_m} \sum_{j=2^{m-1}+1}^{2^{m+1}} \mathbf{P}(j \in C)$$

$$\leq \sum_{m=1}^{\infty} \frac{c}{m} \sum_{j=2^{m-1}+1}^{2^{m+1}} \frac{1}{D(j)} \leq c \sum_{m=1}^{\infty} \sum_{j=2^{m-1}+1}^{2^{m+1}} \frac{1}{D(j)\log j} \leq c \sum_{n=2}^{\infty} \frac{1}{D(n)\log n} < \infty,$$

and by Borel-Cantelli lemma only finitely many of the events $A_{m,m+1}$ occurs with probability 1, which proves the convergent part of Theorem 1.1. $\square$

4 Proof of the divergent part

Let $m_k = [k \log k]$ and

$$A_k = \{\xi(m_k) = 1\}.$$

We prove that $\mathbf{P}(A_k \text{ i.o.}) = 1$ which implies the divergent part of Theorem 1.1. By Lemma 2.1

$$\mathbf{P}(A_k) = \frac{1 + 2p_{m_k}}{2D(m_k)} \geq \frac{1}{2D([k \log k])},$$

so by Lemma 2.2,

$$\sum_k \mathbf{P}(A_k) = \infty.$$

For $n > m$ we have

$$\mathbf{P}(\xi(m) = 1, \, \xi(n) = 1) = \left(\frac{1}{2} + p_m\right) \frac{1}{D(m,n)} \left(\frac{1}{2} + p_n\right) \frac{1}{D(n)} \leq \frac{1}{D(m)D(n)H(m,n)}$$

with

$$H(m,n) = \frac{D(m,n)}{D(m)}.$$

It follows from (2.3) that

$$H(m,n) = 1 - \left(1 - \frac{1}{D(m)}\right) \cdots \left(1 - \frac{1}{D(n-1)}\right)$$

$$\geq 1 - \exp\left(-\frac{1}{D(m)} - \ldots - \frac{1}{D(n-1)}\right).$$

Let $\varepsilon > 0$ and for given k we split the set $\{\ell > k\}$ into 2 parts. Let

$$\ell_1 = \min\left\{\ell > k : \sum_{i=m_k}^{m_\ell - 1} \frac{1}{D(i)} \geq \log \frac{1+\varepsilon}{\varepsilon}\right\}. \tag{4.1}$$

- (1) $\ell \geq \ell_1$,
- (2) $k < \ell < \ell_1$.

In case (1), using that $H(m,n)$ is increasing in n for fixed m, we have for $\ell \geq \ell_1$

$$\mathbf{P}(A_k\, A_\ell) = \frac{\mathbf{P}(A_k)\mathbf{P}(A_\ell)}{H(m_k, m_\ell)} \leq \frac{\mathbf{P}(A_k)\mathbf{P}(A_\ell)}{H(m_k, m_{\ell_1})} \leq (1+\varepsilon)\mathbf{P}(A_k)\mathbf{P}(A_\ell).$$

In the case $\ell \in (2)$ we use the inequality $1 - e^{-u} \geq cu$ for $0 \leq u \leq \log((1+\varepsilon)/\varepsilon)$ with some $c > 0$ to get

$$\mathbf{P}(A_k\, A_\ell) \leq \frac{\mathbf{P}(A_k)\mathbf{P}(A_\ell)}{H(m_k, m_\ell)} \leq \frac{\mathbf{P}(A_k)\mathbf{P}(A_\ell)}{c\sum_{i=m_k}^{m_\ell - 1}\frac{1}{D(i)}} \leq c\mathbf{P}(A_k)\mathbf{P}(A_\ell)\frac{D(m_\ell)}{m_\ell - m_k}.$$

Here and in what follows c, c_i denote some positive constants, the values of which might change from line to line.

So we have for $\ell \in (2)$

$$\mathbf{P}(A_k\, A_\ell) \leq \frac{c\mathbf{P}(A_k)}{\ell \log \ell - k \log k},$$

$$\sum_{\ell=k+1}^{\ell_1 - 1} \mathbf{P}(A_k\, A_\ell) \leq c\mathbf{P}(A_k) \sum_{\ell=k+1}^{\ell_1 - 1} \frac{1}{\ell \log \ell - k \log k}$$

$$\leq c\mathbf{P}(A_k)\frac{1}{\log k}\sum_{\ell=k+1}^{\ell_1 - 1}\frac{1}{\ell - k} \leq c\mathbf{P}(A_k)\frac{\log \ell_1}{\log k}.$$

Now we show that

$$\frac{\log \ell_1}{\log k} \leq \gamma \tag{4.2}$$

with some positive constant γ depending only on ε. We know from (4.1) that for $\ell \in (2)$ we have

$$\sum_{i=m_k}^{m_\ell - 1} \frac{1}{D(i)} < \log \frac{1+\varepsilon}{\varepsilon}. \tag{4.3}$$

We show that this implies that for large k we have $\ell < k^\gamma$ with $\gamma > (1+\varepsilon/\varepsilon)^\delta$. If we assume the contrary that $\ell \geq k^\gamma$, then

$$\sum_{i=m_k}^{m_\ell - 1} \frac{1}{D(i)} \geq \frac{1}{\delta}\sum_{i=m_k}^{m_\ell - 1}\frac{1}{i \log i} \sim \frac{1}{\delta}(\log\log(m_\ell - 1) - \log\log m_k)$$

$$\sim \frac{1}{\delta}\log\frac{\log(\ell\log\ell)}{\log(k\log k)} \geq \frac{1}{\delta}\log\gamma > \log\frac{1+\varepsilon}{\varepsilon}$$

which contradicts to (4.3). Hence $\ell_1 - 1 \leq k^\gamma$, implying (4.2).

Consequently,

$$\sum_{\ell\in(2)} \mathbf{P}(A_k\, A_\ell) \leq c\mathbf{P}(A_k)$$

Assembling these estimations, we have

$$\sum_{k=1}^{N}\sum_{\ell=k+1}^{N} \mathbf{P}(A_k\, A_\ell) \leq (1+\varepsilon)\sum_{k=1}^{N}\sum_{\ell=k+1}^{N} \mathbf{P}(A_k)\mathbf{P}(A_\ell) + c\sum_{k=1}^{N}\mathbf{P}(A_k)$$

Since $\varepsilon > 0$ is arbitrary, Borel-Cantelli lemma implies $\mathbf{P}(A_k \,\text{i.o.}) = 1$. $\square$

5 Conclusions and open problems

Our results are formulated in terms of the sequence $\{D(\cdot)\}$ but it would be much more natural to formulate them in terms of the sequence $\{p_i\}$. Even though we have an explicit expression of $D(\cdot)$ in terms of $\{p_i\}$, usually it is not easy to see the asymptotics of $D(\cdot)$ and whether the sum in Theorem 1.1 is convergent or divergent by looking at $\{p_i\}$ only. Therefore we want to give some examples.

In [3] we have shown

Example 1. *If $p_k = B/4k$ with $B > 1$, then*

$$D(i) \sim \frac{i}{B-1}$$

as $i \to \infty$. Consequently, by Theorem 1.1 *there are infinitely many strong cutpoints a.s.*

It was shown also in [3]

Example 2. *If $p_k = \Lambda(K,k,B)/4$ with $K \geq 2$ and $B > 1$, then*

$$D(i) \sim \frac{i\log i\log\log i\ldots\log_{K-1} i}{B-1}$$

as $i \to \infty$. Consequently, by Theorem 1.1 *we have finitely many cutpoints a.s.*

Recall that the case $K = 2$ corresponds to Theorem D.

Of course, if the NN walk is quicker than the walk in Example 1, (e.g. $p_k = ck^{-\alpha}$ with $\alpha < 1$), then we have infinitely many strong cutpoints a.s. On the other hand, if the NN walk is slower than the walk in Example 2, then we have finitely many cutpoints a.s.

The above two examples show that the jump from finitely many to infinitely many cutpoints is for

$$p_k = \frac{1}{4}\left(\frac{1}{k} + \frac{1}{kf(k)}\right)$$

with some $f(k) \to \infty$. It is not hard to show that if $f(k) = (\log k)^\alpha$ with $0 < \alpha < 1$, then we still have finitely many cutpoints a.s. Now we show a more precise result which implies this one.

Theorem 5.1. *Let $\{X_n\}$ be an NN random walk with*

$$p_k = \frac{1}{4}\left(\frac{1}{k} + \frac{1}{k(\log\log k)^\beta}\right),$$

then we have finitely many cutpoints a.s. if $\beta > 1$ and infinitely many strong cutpoints a.s. if $\beta \le 1$.

Proof. Let

$$r_k = \prod_{i=1}^{k} U_i, \quad t_k = \sum_{i=k}^{\infty} r_i, \qquad k = 1, 2, \ldots$$

Then it is easy to see that $D(k) = t_k/r_k$.

For $k \to \infty$ we obtain

$$U_k = \frac{1-2p_k}{1+2p_k} = \exp(-4p_k + O(p_k^2))$$

and

$$\begin{aligned} r_k &= \exp(-4\sum_{i=1}^{k} p_i + O(\sum_{i=1}^{k} p_i^2)) = \exp\left(-\sum_{i=3}^{k}\left(\frac{1}{i} + \frac{1}{i(\log\log i)^\beta}\right) + O(1)\right) \\ &= \exp\left(-\int_3^k \left(\frac{1}{u} + \frac{1}{u(\log\log u)^\beta}\right) du + O(1)\right). \end{aligned}$$

Hence

$$\frac{c_1}{k}\exp\left(-\int_3^k \frac{du}{u(\log\log u)^\beta}\right) \le r_k \le \frac{c_2}{k}\exp\left(-\int_3^k \frac{du}{u(\log\log u)^\beta}\right) \tag{5.1}$$

with some positive constants c_1, c_2. Consequently,

$$c_1\sum_{j=k}^{\infty}\frac{1}{j}\exp\left(-\int_3^j \frac{du}{u(\log\log u)^\beta}\right) \le t_k \le c_2\sum_{j=k}^{\infty}\frac{1}{j}\exp\left(-\int_3^j \frac{du}{u(\log\log u)^\beta}\right). \tag{5.2}$$

Moreover,

$$\sum_{j=k}^{\infty} \frac{1}{j} \exp\left(-\int_3^j \frac{du}{u(\log\log u)^\beta}\right) = \int_k^\infty \frac{1}{y} \exp\left(-\int_3^y \frac{du}{u(\log\log u)^\beta}\right) + O(1).$$

To find the asymptotics of the above integral, we will apply Lemma C, with

$$S(y) = \exp\left(-\int_3^y \frac{du}{u(\log\log u)^\beta}\right)$$

and

$$L(x) = e^{(\log\log x)^\beta}.$$

Choosing $S(\cdot)$ and $L(\cdot)$ as above, all the conditions of Lemma C are met and we conclude that

$$\int_k^\infty \frac{1}{y} \exp\left(-\int_3^y \frac{du}{u(\log\log u)^\beta}\right) \sim (\log\log k)^\beta \exp\left(-\int_3^k \frac{du}{u(\log\log u)^\beta}\right).$$

From (5.1) and (5.2) we obtain

$$c_1\, k\, (\log\log k)^\beta \le D(k) = \frac{t_k}{r_k} \le c_2\, k\, (\log\log k)^\beta.$$

This combined with Theorem 1.1 proves Theorem 5.1. □

Lemma 2.4 easily implies the following

Corollary 5.1 *If $\{X_n\}$ is an NN random walk with*

$$p_k \le \frac{1}{4}\left(\frac{1}{k} + \frac{1}{k(\log\log k)^\beta}\right)$$

and $\beta > 1$, then X_n has finitely many cutpoints a.s.

On the other hand, if $\{X_n\}$ is an NN random walk with

$$p_k \ge \frac{1}{4}\left(\frac{1}{k} + \frac{1}{k(\log\log k)^\beta}\right)$$

and $\beta \le 1$, then X_n has infinitely many strong cutpoints a.s.

Now we present some related open problems.

- (1) It would be interesting to know whether Theorem 1.1 also holds for the number of sites with $\xi(R) = a$ or $\xi(R, \uparrow) = a$ for any fixed integer $a > 1$, i.e. whether we have the same criteria for $\{\xi(R) = a\}$ and $\{\xi(R, \uparrow) = a\}$ to occur infinitely often almost surely for any positive integer a.

- (2) Call the site R a *weak cutpoint* if for some k, we have $X_k = R$, $X_i \leq R$, $i = 0, 1, \ldots, k-1$ and $X_i \geq R$, $i = k+1, k+2, \ldots$. One would like to know whether Theorem 1.1 can be extended for the number of weak cutpoints.

- (3) It would be interesting to know whether Theorem 1.1 holds for cutpoints with a given local time, i.e. for $\{\xi(R) = a,\ \xi(R, \uparrow) = 1\}$, or in general $\{\xi(R) = a,\ \xi(R, \uparrow) = b\}$ infinitely often almost surely, with positive integers a, b.

- (4) Theorem B gives limsup behavior of the local time. One might ask how does it change if we want to consider the limsup of the local time restricted to the cutpoints.

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