On Vervaat and Vervaat-error type processes for partial sums and renewals

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Abstract. We study the asymptotic behaviour of stochastic processes that are generated by sums of partial sums of i.i.d. random variables and their renewals. We conclude that these processes cannot converge weakly to any nondegenerate random element of the space $D[0,1]$. On the other hand we show that their properly normalized integrals as Vervaat-type stochastic processes converge weakly to a squared Wiener process. Moreover, we also deal with the asymptotic behaviour of the deviations of these processes, the so-called Vervaat-error type processes.

Keywords. Partial sums, renewals, Vervaat and Vervaat-error type processes, Wiener process, strong and weak approximations, weak convergence.

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1 Introduction

Let $X_1, X_2, \ldots$ be an i.i.d. sequence with d.f. $F(x) = P(X_1 \le x)$, $-\infty < x < \infty$, and

(i) $EX_1 = \mu > 0$;

(ii) $0 < \operatorname{Var}(X_1) = \sigma^2$ $(< \infty)$;

(iii) $EX_1^4 < \infty$.

Set $S_n := X_1 + \cdots + X_n$ $(n = 1, 2, \ldots)$, $S_0 := 0$. For $t \ge 0$ define pointwise the corresponding renewal (or first passage time) process as $N(t) := \min\{n \ge 1 \;:\; S_n > t\}$. We further introduce the "average" and "standardized" processes

$$S_n(t) := S_{[nt]}/(n\mu) \qquad (t \ge 0); \tag{1.1}$$

$$s_n(t) := n^{1/2}\sigma^{-1}\mu\Big(S_n(t) - t\Big) = n^{-1/2}\sigma^{-1}\Big(S_{[nt]} - n\mu t\Big) \qquad (t \ge 0); \tag{1.2}$$

$$N_n(t) := N(n\mu t)/n \qquad (t \ge 0); \tag{1.3}$$

$$r_n(t) := n^{1/2}\sigma^{-1}\mu\Big(N_n(t) - t\Big) = n^{-1/2}\sigma^{-1}(\mu N(n\mu t) - n\mu t) \qquad (t \ge 0), \tag{1.4}$$

where $[x]$ denotes the integer part of x.

Throughout the paper $\|\cdot\|$ denotes the sup-norm on $D[0,1]$, i.e., for $f \in D[0,1]$, $\|f\| := \sup_{0 \le t \le 1} |f(t)|$.

We first collect some well-known facts concerning the above processes.

Theorem A *On a large enough probability space, for $X_1, X_2, \ldots$ satisfying* (i), (ii) *and* (iii) *above, one can construct a standard Wiener process $\{W(t) :\; 0 \le t < \infty\}$, such that, the processes s_n and r_n can be simultaneously approximated by W as follows:*

$$\lim_{n\to\infty} n^{1/4}\|s_n - W_n\| = 0 \quad \text{a.s.} \tag{1.5}$$

and

$$\limsup_{n\to\infty} n^{1/4}(\log n)^{-1/2}(\log\log n)^{-1/4}\|r_n + W_n\| = 2^{1/4}\sigma^{1/2}\mu^{-1/2} \quad \text{a.s.}, \tag{1.6}$$

where $W_n(t) := n^{-1/2}W(nt)$.

The result (1.5) is due to Komlós *et al.* (1976), and that of (1.6) is due to Horváth (1984). For more details and further developments we refer to Chapter 2 of Csörgő and Horváth (1993).

Consider now the stochastic process

$$R_n^*(t) := s_n(t) + r_n(t), \qquad 0 \le t \le 1, \tag{1.7}$$

which can be viewed as the remainder term in the Bahadur-Kiefer type representation $r_n = -s_n + R_n^*$ of the renewal process r_n in terms of the partial sums process s_n. In particular, Bahadur (1966), Kiefer (1967, 1970) introduced and studied the stochastic process

$$R_n(t) := \alpha_n(t) + \beta_n(t), \qquad 0 \le t \le 1,$$

that is known in the literature as the Bahadur–Kiefer process, where

$$\begin{aligned}\alpha_n(t) &:= n^{1/2}(F_n(t)-t), \qquad 0\le t\le 1,\\ \beta_n(t) &:= n^{1/2}(F_n^{-1}(t)-t), \qquad 0\le t\le 1,\end{aligned}$$

the uniform empirical and quantile processes, with F_n^{-1} being the left-continuous inverse of the right-continuously defined empirical distribution function F_n of the independent uniform (0,1) random variables $U_1,\dots,U_n$, $n\ge 1$.

In this regard, we summarize the most relevant results of Kiefer (1967, 1970) in the following theorem.

Theorem B *For every fixed* $t\in(0,1)$, *we have*

$$n^{1/4}R_n(t) \xrightarrow{\mathcal{D}} (t(1-t))^{1/4}\mathcal{N}(|\widetilde{\mathcal{N}}|)^{1/2}, \quad n\to\infty, \tag{1.8}$$

$$\limsup_{n\to\infty}\frac{n^{1/4}|R_n(t)|}{(\log\log n)^{3/4}} = (t(1-t))^{1/4}\frac{2^{5/4}}{3^{3/4}} \qquad \text{a.s.}, \tag{1.9}$$

where $\mathcal{N}$ *and* $\widetilde{\mathcal{N}}$ *are independent standard normal variables and* $\xrightarrow{\mathcal{D}}$ *denotes convergence in distribution. Also,*

$$\lim_{n\to\infty} n^{1/4}(\log n)^{-1/2}\frac{\|R_n\|}{(\|\alpha_n\|)^{1/2}} = 1 \qquad \text{a.s.} \tag{1.10}$$

As to (1.10), Kiefer (1970) announced it but proved only convergence in probability (cf. Theorem 1A, and the two sentences right after, in Kiefer, 1970). The upper bound for the almost sure convergence in (1.10) was proved by Shorack (1982), and the lower bound by Deheuvels and Mason (1990).

Concerning similar known results for the process R_n^*, we summarize them in the next theorem.

Theorem C *Under the assumptions* (i), (ii), (iii) *we have for every fixed* $t\in(0,1]$

$$n^{1/4}R_n^*(t) \xrightarrow{\mathcal{D}} t^{1/4}\sigma^{1/2}\mu^{-1/2}\mathcal{N}(|\widetilde{\mathcal{N}}|)^{1/2}, \quad n\to\infty. \tag{1.11}$$

where, as in Theorem B, $\mathcal{N}$ *and* $\widetilde{\mathcal{N}}$ *are independent standard normal random variables, and*

$$\lim_{n\to\infty} n^{1/4}(\log n)^{-1/2}\|R_n^*\|/\|r_n\|^{1/2} = \sigma^{1/2}\mu^{-1/2} \quad \text{a.s.} \tag{1.12}$$

For the result in (1.11) we refer to Csörgő and Horváth (1993, Theorem 2.1.5) and for that of (1.12) to Deheuvels and Mason (1990, Theorem 1B).

As a consequence of (1.12), on account of

$$\|r_n\| \xrightarrow{\mathcal{D}} \|W\| \qquad \text{(cf., e.g., Vervaat (1972))}, \tag{1.13}$$

as $n\to\infty$, we have (cf. Deheuvels and Mason (1990))

$$n^{1/4}(\log n)^{-1/2}\|R_n^*\| \xrightarrow{\mathcal{D}} \sigma^{1/2}\mu^{-1/2}\|W\|. \tag{1.14}$$

Based on (1.11) and (1.14), the next conclusion is immediate.

Corollary 1.1 *Given the assumptions* (i), (ii), (iii), *the statement*

$$a_n R_n^* \xrightarrow{\mathcal{D}} Y, \quad n \to \infty, \tag{1.15}$$

cannot hold true in the space $D[0,1]$ *(endowed with the Skorokhod topology) for any non-degenerate random element* Y *of* $D[0,1]$ *with any normalizing sequence* $\{a_n\}$.

The respective results of (1.11) and (1.12) are based on a strong invariance principle for the Bahadur-Kiefer type process $\{R_n^*(t),\, 0 \le t \le 1;\, n = 1, 2, \ldots\}$ that was explicitly stated in Csörgő and Horváth (1993, p. 43) as follows.

Theorem D *On assuming the conditions* (i), (ii), (iii), *on the probability space of* Theorem A, *as* $n \to \infty$, *we have*

$$R_n^*(t) = n^{-1/2}\left(W(nt) - W\left(nt - \frac{\sigma}{\mu}W(nt)\right)\right) + o(n^{-1/4}) \qquad \text{a.s.,} \tag{1.16}$$

uniformly in $t \in [0,1]$.

Corollary 1.1 and Theorem 3 of Vervaat (1972) serve as motivation for studying the following Vervaat-type process for partial sums and renewals:

$$\begin{aligned} V_n(t) &:= \int_0^t \left\{\left(S_n(s) - s\right) + \left(N_n(s) - s\right)\right\} ds \\ &= \int_0^t M_n(s)\,ds, \qquad 0 \le t \le 1, \end{aligned} \tag{1.17}$$

where $M_n(s) := n^{-1/2}\sigma\mu^{-1}R_n^*(s),\ s \ge 0$.

So far we have been dealing with renewal processes of partial sums under the conditions (i), (ii) and (iii), that is to say we had general renewal processes in mind. We will now see that the just introduced Vervaat-type process V_n will be well-behaving asymptotically as $n \to \infty$. However the asymptotic behaviour of V_n will be different when it is based on general renewal processes, as compared to it being based on ordinary renewal processes, i.e., when in addition to (i), (ii), (iii), we also assume that X_1 is positive. Consequently, in our Section 2 we will deal with Vervaat-type processes for ordinary renewals, while Section 3 will be devoted to studying such processes for general renewals. Moreover, in Section 4 we will also be studying the asymptotic behaviour of Vervaat error-type processes that, based on the respective results of Sections 2 and 3, will be defined there (cf. (4.1)).

2 The Vervaat-type process for ordinary renewals

In addition to the conditions (i), (ii), (iii) of Section 1, in this Section we assume also the condition

(iv) $\quad P(X_1 > 0) = 1.$

Theorem 2.1 *Assume the conditions* (i), (ii), (iii) and (iv). *Then, as* $n \to \infty$

(a)
$$\frac{n}{\log\log n}\left\|V_n - \frac{1}{2}\bar{S}_n^2\right\| \to 0 \quad \text{a.s.}$$

and

(b)
$$n\left\|V_n - \frac{1}{2}\bar{S}_n^2\right\| \xrightarrow{P} 0,$$

where $\bar{S}_n(t) := S_n(t) - t$.

Proof. Given condition (iv) it can be checked (cf. Figure 1 below) that V_n has the following representation:

$$V_n(t) = A_n(t) - \bar{S}_n(t)\bar{N}_n(t) - \frac{1}{2}\bar{N}_n^2(t), \tag{2.1}$$

where

$$A_n(t) := \int_{N_n(t)}^{t} \Big(\bar{S}_n(s) - \bar{S}_n(t)\Big)ds, \tag{2.2}$$

$$\bar{S}_n(t) := S_n(t) - t = (n\mu)^{-1}\Big(S_{[nt]} - n\mu t\Big), \tag{2.3}$$

$$\bar{N}_n(t) := N_n(t) - t = n^{-1}\Big(N(n\mu t) - nt\Big), \tag{2.4}$$

for all $0 \le t \le 1$.

In order to check the above representation, we note that

$$N(n\mu t) = k, \qquad \text{for} \quad \frac{S_{k-1}}{n\mu} \le t < \frac{S_k}{n\mu}.$$

Also, by calculations, we arrive at

$$\begin{aligned}
&\int_0^t \Big(S_n(s) + N_n(s)\Big)ds = t^2 + \int_t^{N_n(t)} (t - S_n(s))ds \\
&= \int_{N_n(t)}^{t} \Big\{(S_n(s) - s) - (S_n(t) - t)\Big\}ds - \int_{N_n(t)}^{t} (t-s)ds - (S_n(t) - t)(N_n(t) - t) + t^2.
\end{aligned}$$

Consequently, we conclude (2.1) (cf. also Figure 1) as follows:

$$V_n(t) = \int_0^t \Big((S_n(s) - s) + (N_n(s) - s)\Big)ds = A_n(t) - \frac{1}{2}\bar{N}_n^2(t) - \bar{S}_n(t)\bar{N}_n(t) + t^2 - t^2.$$

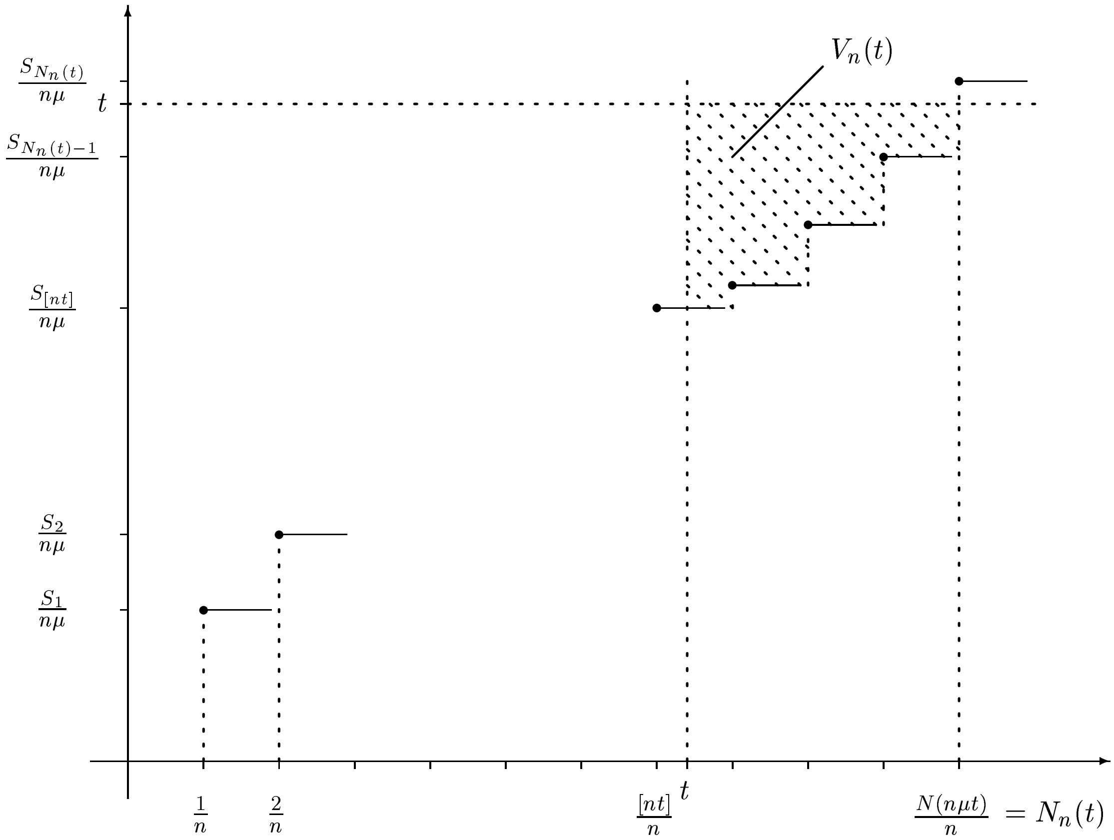

Figure 1

As an immediate consequence in our proof, we get

$$V_n(t) - \frac{1}{2}\bar{S}_n^2(t) = A_n(t) - \frac{1}{2}\Big(\bar{S}_n(t) + \bar{N}_n(t)\Big)^2 \tag{2.5}$$
$$= A_n(t) - \frac{1}{2}M_n^2(t).$$

Now, in view of (1.6), (1.12) and the law of the iterated logarithm for a Wiener process,

$$\|M_n^2\| = O\Big(n^{-3/2}(\log n)(\log\log n)^{1/2}\Big) \quad \text{a.s.,} \tag{2.6}$$

and, as a consequence of (1.14),

$$\|M_n^2\| = O_P\Big(n^{-3/2}\log n\Big). \tag{2.7}$$

Next, we show that A_n is the dominating process on the right-hand side of (2.5). In (2.2) put $s = t - u(t - N_n(t)) = t + u\bar{N}_n(t)$. Then we get

$$A_n(t) = -\bar{N}_n(t)\int_0^1 \Big(\bar{S}_n(t + u\bar{N}_n(t)) - \bar{S}_n(t)\Big)du. \tag{2.8}$$

On account of (1.5),

$$\frac{n\mu}{\sigma}\bar{S}_n(t) = W(nt) + o(n^{1/4}) \quad \text{a.s.,} \tag{2.9}$$

uniformly in $t \in [0,1]$. A combination of (2.8) and (2.9) yields

$$(2.10)\quad \frac{n\mu}{\sigma} A_n(t) = -\bar{N}_n(t) \int_0^1 \Big(W(nt + un\bar{N}_n(t)) - W(nt)\Big) du + o\Big(n^{-1/4}(\log\log n)^{1/2}\Big) \quad \text{a.s.},$$

uniformly in $t \in [0,1]$, since

$$n\bar{N}_n(t) = O\Big((n \log\log n)^{1/2}\Big) \quad \text{a.s.},$$

uniformly in $t \in [0,1]$.

This rate in (2.10) can also be replaced by $o_P(n^{-1/4})$, on account of

$$n\bar{N}_n(t) = O_P(n^{1/2}),$$

uniformly in $t \in [0,1]$.

Moreover, by Theorem 1.2.1 of Csörgő and Révész (1981),

$$(2.11)\qquad W(nt + un\bar{N}_n(t)) - W(nt) = O\Big(n^{1/4}(\log n)^{1/2}(\log\log n)^{1/4}\Big) \quad \text{a.s.}$$

as well as

$$(2.12)\qquad W(nt + un\bar{N}_n(t)) - W(nt) = O_P\Big(n^{1/4}(\log n)^{1/2}\Big),$$

uniformly in $u, t \in [0,1]$.

In view of (1.6), we also have

$$(2.13)\qquad \frac{\mu}{\sigma} n\bar{N}_n(t) = -W(nt) + O\Big(n^{1/4}(\log n)^{1/2}(\log\log n)^{1/4}\Big) \quad \text{a.s.}$$

As a consequence of (2.10)–(2.13), we arrive at

$$(2.14)\quad \frac{n\mu}{\sigma} A_n(t) = \frac{\sigma W(nt)}{n\mu} \int_0^1 \Big(W(nt + un\bar{N}_n(t)) - W(nt)\Big) du + o\Big(n^{-1/4}(\log\log n)^{1/2}\Big) \quad \text{a.s.},$$

uniformly in $t \in [0,1]$, where this rate can also be replaced by $o_P(n^{-1/4})$.

On making use of (2.13) in combination with Theorem 1.2.1 of Csörgő and Révész (1981), we conclude the following strong and weak invariance principles for $A_n(t)$ of (2.2):

$$\begin{aligned}
(2.15)\, A_n(t) &= \frac{\sigma W(nt)}{n\mu} \int_0^1 \frac{\sigma}{n\mu}\Big(W\Big(nt - u\frac{\sigma}{\mu}W(nt)\Big) - W(nt)\Big) du + o\Big(n^{-5/4}(\log\log n)^{1/2}\Big) \quad \text{a.s.} \\
&= \frac{\sigma}{n\mu} \int_0^{\frac{\sigma}{n\mu}W(nt)} (W(nt - nx) - W(nt)) dx + o\Big(n^{-5/4}(\log\log n)^{1/2}\Big) \quad \text{a.s.} \\
&= \int_0^{Y_n(t)} (Y_n(t-x) - Y_n(t)) dx + o\Big(n^{-5/4}(\log\log n)^{1/2}\Big) \quad \text{a.s.},
\end{aligned}$$

uniformly in $t \in [0,1]$, where $Y_n(t) := \frac{\sigma}{n\mu} W(nt)$, and this a.s. rate can also be replaced by $o_P\Big(n^{-5/4}\Big)$.

Arguing similarly as in (2.11) and (2.12), we also have

$$\begin{aligned}
(2.16)\qquad A_n(t) &= O\Big(n^{-1/2}(\log\log n)^{1/2}\ n^{-3/4}(\log n)^{1/2}(\log\log n)^{1/4}\Big) \quad \text{a.s.} \\
&= O\Big(n^{-5/4}(\log n)^{1/2}(\log\log n)^{3/4}\Big) \quad \text{a.s.},
\end{aligned}$$

and

$$A_n(t) = O_P\Big(n^{-5/4}(\log n)^{1/2}\Big), \tag{2.17}$$

uniformly in $t \in [0,1]$.

Thus, in view of (2.5), (2.6) and (2.16) we conclude (a) of Theorem 2.1, while (2.5), (2.7) and (2.17) result in (b) of Theorem 2.1. □

As an immediate consequence of Theorem 2.1 and Theorem A, we get

Corollary 2.1 *Assume the conditions* (i), (ii), (iii) and (iv). *Then, as $n \to \infty$, on the probability space of* Theorem A *with W_n as in* (1.5), *we have*

$$\text{(a)} \qquad \frac{1}{\log\log n}\bigg\| nV_n - \frac{1}{2}\Big(\frac{\sigma}{\mu}W_n\Big)^2 \bigg\| \to 0 \quad \text{a.s.}$$

and

$$\text{(b)} \qquad \bigg\| nV_n - \frac{1}{2}\Big(\frac{\sigma}{\mu}W_n\Big)^2 \bigg\| \xrightarrow{P} 0.$$

As a consequence of (b) of Corollary 2.1, one concludes also

$$nV_n \xrightarrow{\mathcal{D}} Z, \quad n \to \infty, \tag{2.18}$$

in the space $C[0,1]$ (endowed with the uniform topology), where $Z(t) := \frac{1}{2}\Big(\frac{\sigma}{\mu}W(t)\Big)^2$, $t \in [0,1]$.

The strong invariance principle (a) of Corollary 2.1 in turn implies Strassen (1964)-type laws of the iterated logarithm for V_n as follows.

Corollary 2.2 *Assume the conditions* (i), (ii), (iii) and (iv). *Then, as $n \to \infty$, the set*

$$\left\{\frac{\mu^2 nV_n}{\sigma^2 \log\log n},\ n \geq 3\right\} \tag{2.19}$$

is relatively compact in $C[0,1]$, equipped with the sup-norm $\|\cdot\|$. Furthermore, the set of all limit points of the functions in the set (2.19) *almost surely coincides with the set $\{f^2 : f \in \mathcal{S}\}$, where $\mathcal{S}$ is the Strassen class of all absolutely continuous functions f on $[0,1]$ such that $f(0) = 0$ and $\|f'\|_2 \leq 1$, where $\|\cdot\|_p$ denotes the L_p-norm, $p \geq 1$.*

As examples of consequences of Corollary 2.2 à la Strassen (1964), we mention the following results.

Corollary 2.3 *Under the conditions* (i), (ii), (iii) and (iv) *we have*

$$\limsup_{n\to\infty} \frac{n\|V_n\|}{\log\log n} = \frac{\sigma^2}{\mu^2} \qquad \text{a.s.}$$

and

$$\limsup_{n\to\infty} \frac{n\|V_n\|_1}{\log\log n} = \frac{4\sigma^2}{\mu^2\pi^2} \qquad \text{a.s.}$$

3 The Vervaat process for general renewals

In this section we study the case of general renewals for which assumption (iv) of Section 2 may not necessarily be true. In order to do this, we need to introduce some further notations and auxiliary results:

Let ν_i denote the i-th (strong) ascending ladder index of the partial sum sequence $\{S_n\}_{n=0,1,\ldots}$, i.e., $\nu_0 = 0$, and, recursively,

$$\nu_i := \min\Big\{n > \nu_{i-1} \,:\, S_n - S_{\nu_{i-1}} > 0\Big\}. \tag{3.1}$$

We note that, under assumptions (i), (ii), (iii), $\{\nu_i - \nu_{i-1}\}_{i=1,2,\ldots}$ is an i.i.d. sequence of random variables with

$$E\nu_1^4 < \infty. \tag{3.2}$$

Moreover, the sequence $\{S_{\nu_i} - S_{\nu_{i-1}}\}_{i=1,2,\ldots}$ of ladder heights is also an i.i.d. sequence of random variables with

$$ES_{\nu_1}^4 < \infty \tag{3.3}$$

(cf., e.g., Gut (1988, Sections III. 2-3)). By definition,

$$N(t) = \nu_i, \quad \text{for } S_{\nu_{i-1}} \le t < S_{\nu_i} \quad (i = 1, 2, \ldots;\, t \ge 0),$$

i.e.,

$$N(n\mu s) = \nu_i, \quad \text{for } \frac{S_{\nu_{i-1}}}{n\mu} \le s < \frac{S_{\nu_i}}{n\mu} \quad (i = 1, 2, \ldots;\, s \ge 0). \tag{3.4}$$

Let $N(n\mu t) = \nu_\ell$, i.e.,

$$\ell = \min\Big\{j : S_{\nu_j} > n\mu t\Big\}, \tag{3.5}$$

that is, $\ell = N_H(n\mu t)$, where N_H denotes the (ordinary) renewal process corresponding to the (ladder height) sequence $S_{\nu_1}, S_{\nu_2} - S_{\nu_1}, \ldots$. This will play a crucial role in the calculations below.

We first note that we continue to use the same definition for the Vervaat process V_n for general renewals as in (1.17). Namely, we study the integral

$$V_n(t) = \int_0^t M_n(s)\, ds = \int_0^t \Big(\bar{S}_n(s) + \bar{N}_n(s)\Big)\, ds,$$

where $\bar{S}_n$ and $\bar{N}_n$ are as in (2.3) and (2.4), respectively, i.e.,

$$\bar{S}_n(t) = S_n(t) - t = (n\mu)^{-1}(S_{[nt]} - n\mu t),$$

$$\bar{N}_n(t) = N_n(t) - t = n^{-1}(N(n\mu t) - nt),$$

with $S_n(t)$ and $N_n(t)$ as in (1.1) and (1.3), respectively.

Consider now the following figure.

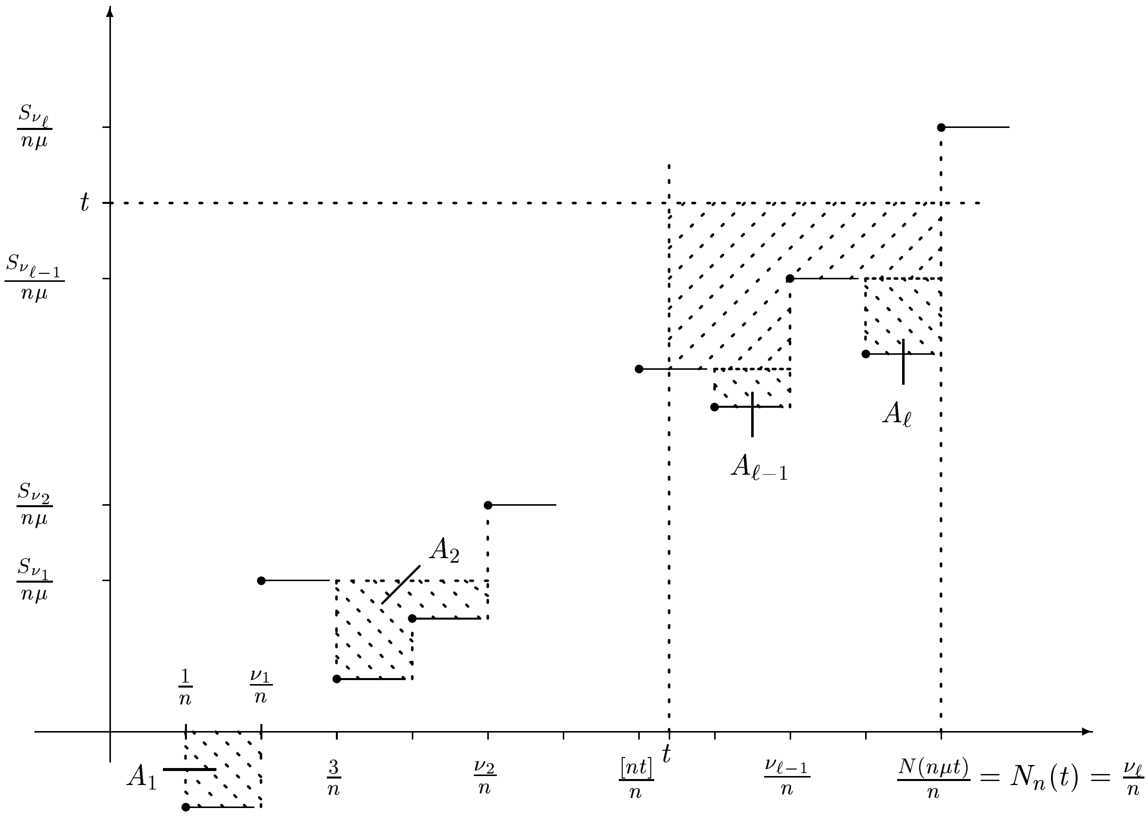

Figure 2

Note that, in contrast to the case of the ordinary renewal process (cf. Figure 1), the (random) areas $A_1, A_2, \ldots$ (cf. Figure 2) are additional in the present case of the Vervaat process for general renewals. Thus, in computing the above integral V_n, these areas are to be taken into consideration. Our next lemma accomplishes this via establishing the following new representation for V_n in the general case that will now replace that of (2.5).

Lemma 3.1 *Under the conditions* (i), (ii), (iii) *we have*

$$V_n(t) = A_n(t) + B_n(t) - \frac{1}{2}M_n^2(t) + \frac{1}{2}\bar{S}_n^2(t), \tag{3.6}$$

with A_n *as in* (2.2), $M_n = \bar{S}_n + \bar{N}_n$ *and* B_n *is defined by*

$$B_n(t) := \frac{1}{n^2\mu} \sum_{i=1}^{N_H(n\mu t)} \left(\sum_{j=1}^{\nu_i - \nu_{i-1} - 1} (\nu_i - \nu_{i-1} - j) X_{\nu_{i-1}+j} \right), \tag{3.7}$$

with $N_H(n\mu t) := \ell$ *of* (3.5).

Proof. We first show that

$$\int_0^t \bar{N}_n(s)ds + \int_0^{N_n(t)} \bar{S}_n(s)ds = -\frac{1}{2}\bar{N}_n^2(t) + B_n(t), \tag{3.8}$$

We have

$$\int_0^t (N_n(s) - s)ds + \int_0^{N_n(t)} (S_n(s) - s)ds = \int_0^t N_n(s)ds + \int_0^{N_n(t)} S_n(s)ds - \frac{t^2}{2} - \frac{N_n^2(t)}{2}. \tag{3.9}$$

By (3.4) and the definition of N_n we arrive at

$$\begin{aligned}\int_0^t N_n(s)ds &= \frac{1}{n}\sum_{i=1}^{\ell} \int_{\frac{S_{\nu_{i-1}}}{n\mu}}^{\frac{S_{\nu_i}}{n\mu}} \nu_i \; ds - \frac{1}{n}\int_t^{\frac{S_{\nu_\ell}}{n\mu}} \nu_\ell \; ds \\ &= \frac{1}{n^2\mu}\sum_{i=1}^{\ell} \nu_i\Big(S_{\nu_i} - S_{\nu_{i-1}}\Big) - \frac{1}{n}\nu_\ell\Big(\frac{S_{\nu_\ell}}{n\mu} - t\Big).\end{aligned} \tag{3.10}$$

Also,

$$\begin{aligned}\int_0^{N_n(t)} S_n(s)ds &= \sum_{i=1}^{\nu_\ell} \int_{\frac{i-1}{n}}^{\frac{i}{n}} S_n(s)ds = \frac{1}{n^2\mu}\sum_{i=1}^{\nu_\ell} S_{i-1} \\ &= \frac{1}{n^2\mu}\sum_{i=1}^{\ell}\sum_{j=1}^{\nu_i-\nu_{i-1}} \Big(S_{\nu_{i-1}+j-1} - S_{\nu_{i-1}}\Big) + \frac{1}{n^2\mu}\sum_{i=1}^{\ell}(\nu_i - \nu_{i-1})S_{\nu_{i-1}}.\end{aligned} \tag{3.11}$$

Now, for any real sequences $\{a_i\}_{i=0,1,\ldots}$ and $\{b_i\}_{i=0,1,\ldots}$, it can easily be checked that

$$\sum_{i=1}^{n} a_i(b_i - b_{i-1}) + \sum_{i=1}^{n} b_{i-1}(a_i - a_{i-1}) = a_n b_n - a_0 b_0.$$

Hence, since $\nu_0 = 0 = S_{\nu_0}$,

$$\sum_{i=1}^{\ell} \nu_i\Big(S_{\nu_i} - S_{\nu_{i-1}}\Big) = \nu_\ell S_{\nu_\ell} - \sum_{i=1}^{\ell}(\nu_i - \nu_{i-1})S_{\nu_{i-1}}. \tag{3.12}$$

Thus, on combining (3.9) - (3.12), we obtain

$$\begin{aligned}&\int_0^t \bar{N}_n(s)ds + \int_0^{N_n(t)} \bar{S}_n(s)ds \\ &= \frac{1}{n^2\mu}\sum_{i=1}^{\ell}\sum_{j=1}^{\nu_i-\nu_{i-1}-1} \Big(S_{\nu_{i-1}+j} - S_{\nu_{i-1}}\Big) - \frac{t^2}{2} - \frac{N_n^2(t)}{2} + t\frac{\nu_\ell}{n} \\ &= -\frac{1}{2}(N_n(t) - t)^2 + \frac{1}{n^2\mu}\sum_{i=1}^{\ell}\left(\sum_{j=1}^{\nu_i-\nu_{i-1}-1} (\nu_i - \nu_{i-1} - j)X_{\nu_{i-1}+j}\right).\end{aligned} \tag{3.13}$$

This, in turn, proves (3.8).

As a consequence of (3.8) and the definition of $A_n(t)$ we now conclude our claim in (3.6) as follows.

$$
\begin{aligned}
V_n(t) &= \int_0^t \Big(\bar{S}_n(s) + \bar{N}_n(s)\Big)\,ds \\
&= \int_0^t \bar{N}_n(s)ds + \int_0^{N_n(t)} \bar{S}_n(s)ds + \int_{N_n(t)}^t \bar{S}_n(s)ds \\
&= -\frac{1}{2}\bar{N}_n^2(t) + B_n(t) + A_n(t) - \bar{S}_n(t)\bar{N}_n(t) \\
&= A_n(t) + B_n(t) - \frac{1}{2}M_n^2(t) + \frac{1}{2}\bar{S}_n^2(t).
\end{aligned}
$$

□

For the sake of stating and proving an analogue of Theorem 2.1 in case of the general renewal process, we need further notations.

Consider the summands

(3.14) $$D_i := \sum_{j=2}^{\nu_i-\nu_{i-1}} (\nu_i - \nu_{i-1} - j)X_{\nu_{i-1}+j-1} \quad i = 1, 2, \ldots$$

in the definition of B_n (cf. 3.7). We note that the just introduced random variables D_i are i.i.d. with mean $\mu_D := ED_1$ and finite second moment $ED_1^2 < \infty$. The latter is implied by (3.2) and (3.3), on applying the Cauchy-Schwarz inequality when computing ED_1^2. Put also $\mu_H := ES_{\nu_1}$, the so-called expected ladder height (cf., e.g., Gut (1988, Chap. IV, Theorem 2.7)).

The extension of our Theorem 2.1 to the general renewal situation now reads as follows.

Theorem 3.1 *Assume the conditions* (i), (ii), (iii). *Then, as* $n \to \infty$,

(a) $$\frac{1}{\log\log n}\Big|\Big|nV_n - \frac{n}{2}\bar{S}_n^2 - \Big(\frac{\mu_D}{\mu_H}\Big)id\Big|\Big| \to 0 \quad \text{a.s.}$$

(b) $$\Big|\Big|nV_n - \frac{n}{2}\bar{S}_n^2 - \Big(\frac{\mu_D}{\mu_H}\Big)id\Big|\Big| \xrightarrow{P} 0\ ,$$

where $id(t) := t$, $t \in [0, 1]$.

Proof. For B_n of Lemma 3.1 we conclude the following asymptotic representation. As $n \to \infty$, by the law of the iterated logarithm, we have almost surely

(3.15)
$$
\begin{aligned}
B_n(t) &= \frac{1}{n^2\mu}\sum_{i=1}^{N_H(n\mu t)} D_i \\
&= \frac{1}{n^2\mu}\Big(\sum_{i=1}^{N_H(n\mu t)} D_i - n\mu t\frac{\mu_D}{\mu_H}\Big) + \frac{1}{n}\Big(\frac{\mu_D}{\mu_H}\Big)t \\
&= \frac{1}{n}\Big(\frac{\mu_D}{\mu_H}\Big)t + O\Big(\frac{(\log\log n)^{1/2}}{n^{3/2}}\Big),
\end{aligned}
$$

uniformly in $t \in [0,1]$. Consequently, via (2.6)/(2.7), (2.16)/(2.17), Lemma 3.1 and (3.15) we conclude the respective two statements of Theorem 3.1. $\square$

Remark 3.1 Via the definition of the ladder index ν_1, we have

$$X_1, X_1 + X_2, \ldots, X_1 + \cdots + X_{\nu_1 - 1} \leq 0, \; X_1 + \cdots + X_{\nu_1} > 0.$$

Hence (cf. (3.14)) we have also

$$D_1 = \sum_{j=1}^{\nu_1 - 1} (\nu_1 - j) X_j \leq 0.$$

If, however,

$$\text{(iv}'\text{)} \qquad P(X_1 \geq 0) = 1,$$

then

$$\text{(3.16)} \qquad P(D_1 = 0) = 1.$$

On the other hand, if

$$\text{(v)} \qquad P(X_1 < 0) > 0,$$

then we have

$$\text{(3.17)} \qquad P(D_1 < 0) > 0,$$

as well.

This shows that the results of Section 2 also remain true under assumption (iv') replacing (iv), for then $B_n = 0$ a.s.

As an immediate consequence of Theorem 3.1 and Theorem A, we conclude

Corollary 3.1. *Assume the conditions* (i), (ii), (iii). *Then, as $n \to \infty$, on the probability space of* Theorem A *with W_n as in* (1.5), *we have*

$$\text{(a)} \qquad \frac{1}{\log \log n} \left\| nV_n - \frac{1}{2} \left(\frac{\sigma}{\mu} W_n \right)^2 - \left(\frac{\mu_D}{\mu_H} \right) id \right\| \to 0 \quad \text{a.s.}$$

and

$$\text{(b)} \qquad \left\| nV_n - \frac{1}{2} \left(\frac{\sigma}{\mu} W_n \right)^2 - \left(\frac{\mu_D}{\mu_H} \right) id \right\| \xrightarrow{P} 0,$$

As a consequence of (b) of Corollary 3.1, we conclude also

$$\text{(3.18)} \qquad nV_n \xrightarrow{\mathcal{D}} \widetilde{Z}, \quad n \to \infty,$$

in the space $C[0,1]$ (endowed with the uniform topology), where $\widetilde{Z}(t) := \frac{1}{2}\left(\frac{\sigma}{\mu} W(t)\right)^2 + \left(\frac{\mu_D}{\mu_H}\right) t$, $t \in [0,1]$.

The latter conclusion in turn also allows us a Vervaat (1972)-type proof of Corollary 1.1. Namely, if the statement (1.15) were true, then the sequence in there could be replaced by $\{n\}$, due to the convergence of types theorem (cf. Lemma 1 on p. 253 of Feller (1971)). But then one would also have the equality in distribution

$$\int_0^t Y(s)ds \stackrel{\mathcal{D}}{=} \frac{1}{2}\Big(\frac{\sigma}{\mu}W(t)\Big)^2 + \Big(\frac{\mu_D}{\mu_H}\Big)t, \qquad t \in [0,1],$$

which is impossible, since the Wiener process W is a.s. nowhere differentiable. Consequently, the relation (1.15) cannot hold true.

In concluding this section, it may be of interest to note that, while it is true that the asymptotic behaviour of the Bahadur-Kiefer-type process R_n^* of (1.12) with ordinary and general renewals is the same pointwise and in sup-norm, their integrated form V_n behaves differently as $n \to \infty$. Namely, in Corollary 3.1 and (3.18) the squared Wiener process appears with a drift term which is not present in Corollary 2.1 and (2.18).

4 Vervaat-error type processes for partial sums and renewals

Theorem 2.1 and Theorem 3.1 suggest the study of the asymptotic behaviour of the following Vervaat-error type process:

(4.1) $$Q_n(t) := V_n(t) - \frac{1}{2}\bar{S}_n^2(t) - \Big(\frac{\mu_D}{\mu_H}\Big)t, \qquad 0 \le t \le 1.$$

We note that the drift term $(\mu_D/\mu_H)t$ is identically zero in case of an ordinary renewal process (cf. Theorem 2.1).

Define the stochastic process Z_n by

(4.2) $$Z_n(t) := \int_0^{Y_n(t)} \Big(Y_n(t-x) - Y_n(t)\Big)dx, \qquad 0 \le t \le 1,$$

where $Y_n(t) := \frac{\sigma}{n\mu}W(nt)$.

We have the following representations for Q_n in terms of Z_n.

Proposition 4.1 *Assume the conditions* (i), (ii), (iii)*. Then, as $n \to \infty$, on the probability space of* Theorem A, *we have*

(4.3) $$Q_n(t) = Z_n(t) + o\Big(n^{-5/4}(\log\log n)^{1/2}\Big) \quad \text{a.s.,}$$

and

(4.4) $$Q_n(t) = Z_n(t) + o_P\Big(n^{-5/4}\Big),$$

uniformly in $t \in [0,1]$.

Proof. In view of (3.6), via (2.15), (3.15), and (2.6), we conclude (4.3). Similarly, and again in view of (3.6), via (2.15) with the error term $o_P(n^{-5/4})$, (3.15), and (2.7), we arrive at (4.4) as well.

Theorem 4.1 *Assume* (i), (ii), (iii). *Then, for* $0 < t \le 1$ *fixed, as* $n \to \infty$

$$n^{5/4} Q_n(t) \xrightarrow{\mathcal{D}} \left(\frac{1}{3}\right)^{1/2} \left(\frac{\sigma}{\mu}\right)^{5/2} t^{3/4} \mathcal{N} |\widetilde{\mathcal{N}}|^{3/2}, \tag{4.5}$$

where $\mathcal{N}$ *and* $\widetilde{\mathcal{N}}$ *are independent standard normal random variables.*

We note in passing that the result of (4.5) parallels that of (2.5) of Theorem 2.1 of Csáki *et al.* (2002), which deals with a similar problem concerning the Vervaat-error process of the empirical and quantile processes. For further related results along these lines we refer to Csörgő and Zitikis (1999).

Proof. Given (4.4) of Proposition 4.1, it is clear that the pointwise asymptotic distribution of $Q_n(t)$ for any fixed $t \in (0,1]$ can be established via that of $Z_n(t)$. Clearly, from (4.2) we obtain

$$Z_n(t) = \frac{\sigma W(nt)}{n\mu} \int_0^1 \frac{\sigma}{n\mu} \Big(W(nt - s\frac{\sigma}{\mu} W(nt)) - W(nt) \Big) ds. \tag{4.6}$$

Let

$$\begin{aligned} W_1(u) &:= (nt)^{-1/2} (W(nt - unt) - (1-u) W(nt)) + u\mathcal{N}^*, \quad 0 \le u \le 1, \\ W_2(u) &:= (nt)^{-1/2} (W(nt + unt) - W(nt)), \quad 0 \le u \le 1, \end{aligned}$$

where $\mathcal{N}^*$ is a standard normal random variable, independent of $W(\cdot)$. It is obvious that for any fixed n and t, the processes $\{W_1(u),\, 0 \le u \le 1\}$ and $\{W_2(u),\, 0 \le u \le 1\}$ are independent standard Wiener processes, which are also independent of $W(nt)$.

Consider $Z_n(t)$ of (4.6), and first assume that $W(nt) > 0$. Put $unt = s(\sigma/\mu) W(nt)$ in (4.6). Thus we obtain

$$Z_n(t) = \frac{t\sigma}{n\mu} \int_0^{\frac{\sigma W(nt)}{\mu nt}} (W(nt - unt) - W(nt))\, du$$

$$= \frac{t\sigma}{\mu} \left(\frac{t}{n}\right)^{1/2} \int_0^{\frac{\sigma W(nt)}{\mu nt}} W_1(u)\, du - \frac{t\sigma}{\mu n} \int_0^{\frac{\sigma W(nt)}{\mu nt}} u W(nt)\, du - \frac{t\sigma}{\mu} \left(\frac{t}{n}\right)^{1/2} \int_0^{\frac{\sigma W(nt)}{\mu nt}} u\mathcal{N}^*\, du$$

$$= \frac{t\sigma}{\mu} \left(\frac{t}{n}\right)^{1/2} \int_0^{\frac{\sigma W(nt)}{\mu nt}} W_1(u)\, du + O_P(n^{-3/2}).$$

Assuming now that $W(nt) < 0$, on putting $unt = -s(\sigma/\mu) W(nt)$ in (4.6), we get

$$Z_n(t) = \frac{t\sigma}{\mu} \left(\frac{t}{n}\right)^{1/2} \int_0^{\frac{-\sigma W(nt)}{\mu nt}} W_2(u)\, du.$$

Hence, for each fixed $t \in (0,1]$, $Z_n(t)$ has the following representation

$$Z_n(t) = Z_n^*(t) + O_P(n^{-3/2}), \tag{4.7}$$

where

$$Z_n^*(t) := \frac{t\sigma}{\mu} \left(\frac{t}{n}\right)^{1/2} \int_0^{\left|\frac{\sigma W(nt)}{\mu nt}\right|} W^*(u)\, du,$$

and $W^*(u) := W_1(u) I\{W(nt) > 0\} + W_2(u) I\{W(nt) < 0\}$ is a standard Wiener process, independent of $W(nt)$ for each fixed n and t.

It is well-known that for any fixed $T > 0$,

$$\int_0^T W^*(u)\,du \overset{\mathcal{D}}{=} N\Big(0, \frac{T^3}{3}\Big) \overset{\mathcal{D}}{=} \frac{T^{3/2}}{3^{1/2}} N(0,1).$$

Consequently, by independence of $W(nt)$ and $\{W^*(u),\, 0 \le u \le 1\}$ for each fixed n and t,

$$Z_n^*(t) \overset{\mathcal{D}}{=} \Big(\frac{1}{3}\Big)^{1/2} \Big(\frac{\sigma}{\mu}\Big)^{5/2} \frac{1}{n^2} \bigg(\frac{|W(nt)|}{(nt)^{1/2}}\bigg)^{3/2} (nt)^{3/4} \mathcal{N}. \tag{4.8}$$

Hence for each fixed n and t we conclude

$$Z_n^*(t) \overset{\mathcal{D}}{=} n^{-5/4} \Big(\frac{1}{3}\Big)^{1/2} \Big(\frac{\sigma}{\mu}\Big)^{5/2} t^{3/4} |\widetilde{\mathcal{N}}|^{3/2} \mathcal{N}, \tag{4.9}$$

where $\mathcal{N}$ and $\widetilde{\mathcal{N}}$ are independent standard normal random variables.

Consequently, on account of (4.7), as $n \to \infty$, we obtain

$$n^{5/4} Z_n(t) \overset{\mathcal{D}}{\to} \Big(\frac{1}{3}\Big)^{1/2} \Big(\frac{\sigma}{\mu}\Big)^{5/2} t^{3/4} \mathcal{N} |\widetilde{\mathcal{N}}|^{3/2}, \tag{4.10}$$

for each fixed $t \in (0, 1]$.

This in view of (4.4) of Proposition 4.1 also completes the proof of (4.5) of Theorem 4.1. $\square$

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