

## **RESEARCH RESULTS OF THE HUN-REN ALFRÉD RÉNYI INSTITUTE OF MATHEMATICS IN 2025**

### **Research in algebra**

They showed that there exists a low-temperature phase in which the maximal-entropy random walk is Benjamini–Schramm continuous on expander graph sequences. They studied the bundling phenomenon in hardcore, positive-density, ergodic line processes; for a discrete-lattice analogue of this, they gave effective entropy estimates. They showed that in hyperbolic space, for a low-temperature Poisson-factor potential, the  $q$ -transform exists. They studied the transience of degree-biased random walks on Barabási-Albert trees.

The separating Noether number of a finite abelian group over the complex base field has a purely combinatorial characterization. In applications (for example in signal processing) the case of the real (or rational) base field is relevant, therefore the above-mentioned characterization of the separating Noether number was extended for the case of not necessarily algebraically closed base fields. Furthermore, an algorithm was implemented in the SageMath mathematical software, that can compute the separating Noether number for small, not necessarily abelian groups (working over an algebraically closed base field).

One approach to construct quantum groups is via appropriate noncommutative deformations of the coordinate rings of classical groups. The coordinate ring of the cones of the classical groups appears in this process. The combinatorial and representation theoretic study of these coordinate rings was initiated. The direct product of two copies of a classical group acts naturally on the coordinate ring of its cone, and the decomposition into the direct sum of irreducibles of this representation was determined.

There exist a positive constant  $\epsilon$ , such that if  $S$  a finite simple group of rank  $r$  and  $A$  a generating subset of size at least two, then the order of product of two conjugates of  $A$  is at least an  $(1+\epsilon)$ -th power of the order of  $A$ , where the conjugating element is in a polynomial power  $A$ . They proved this conjecture for most classical groups in case when  $A$  contains a transvection. It was also proved that the Skew Product Theorem holds for a random conjugating element.

They proved theorems on the base size and minimal degree of transitive permutation groups. They also showed that in many non-solvable finite groups there do not exist two conjugacy classes consisting of non-trivial elements whose sizes are coprime.

### **Research in applied mathematics**

They studied percolation models on Euclidean lattices, nonamenable and finite graphs, relevant to statistical mechanics, measured group theory, and network science. This included interlacements and other Poisson Zoo models, indistinguishability of clusters, phase transitions in graph-bootstrap percolation, the local structure of random spanning trees, and a theory for weighted amenability. They organized an international workshop on Activated Random Walks, yielding several results for the 1D case. They also worked on random Schrödinger operators and the related model of Uniform Random Walks on different graphs.

They derived convergence rate estimations in weighted total variation norm from the respective estimates in Wasserstein metric, using Fourier analytic tools. They continued work on the ergodic theory of Markov chains in random environments, in queuing systems and stochastic volatility models, in particular. They studied convergence rates for distributed averaging algorithms and investigated the effect of various types of perturbations to a Markov chain on a graph, with acceleration opportunities in mind. They worked on some non-convex optimization problems coming from optimal investment models with price impact.

They investigated the mathematical structure of the internal representations of transformer-based language models, with particular emphasis on the representation of relations. They showed that, in many cases, relations can be described by trilinear functions that can be

realized tensor networks incorporating dimensionality reduction. The paper presenting these results was accepted as a spotlight at the highly prestigious NeurIPS conference.

In the area of higher-order Fourier analysis, they studied algorithms based on spectral decompositions. In particular, in the quadratic case they developed efficient algorithms that decompose functions into structured and noise components. The related comprehensive manuscript has been completed and submitted.

In the course of a machine-assisted search for Helly-type problems, they discovered an unexpected connection with quadratic programming, which led to new proof techniques. The results were published in the Q1 journal *Linear Algebra and its Applications*. In their study of Hadamard matrices, they also proved an algebraic conjecture using Gröbner basis techniques.

### **Methods of analysis and arithmetics**

Kuznetsov's formula relates a spectral sum of the product of two Fourier coefficients of cusp forms of weight 0 to a sum of Kloosterman sums. In this year they proved a formula which relates the product of not 2, but 4 Fourier coefficients of weight  $1/2$  cusp forms to arithmetic objects: to the class numbers of pairs of integral quadratic forms with given discriminants and codiscriminant.

One of the most important problems in mathematics is the Riemann hypothesis for various  $L$ -functions. As a partial solution, we are looking for zero-free regions. The classical theorem of Tatzawa concerns zero-free regions of Dirichlet  $L$ -functions. In this year they generalized it for Rankin-Selberg  $L$ -functions.

They determined the orbits of orthogonal groups over finite fields of odd characteristics acting on symmetric matrices. The description includes a complete classification and the number of elements in each of the orbits. The result can also be used to estimate symplectic Kloosterman sums.

Earlier they determined the magnitude of how many vertices the convex hull of the lattice points inside a disk of radius  $R$  has. In this year they solved the similar question about the convex hull of the lattice points over the positive branch of the hyperbola  $xy = N$ .

They proved results in various branches of mathematical analysis, such as Fourier analysis, approximation theory, optimal transport and dimension theory.

In Fourier analysis, they proved general results about Delsarte's extremal problems: strong linear duality, the existence of extremizers, and a tiling-type relation between the primal and dual extremal objects. They gave applications of the results in connection with weak tilings and 1-avoiding sets. They applied the related concept of the geometric fractional chromatic number to unit distance graphs of the plane, and also investigated the concept of density in a very general setting. In approximation theory, they proved Bernstein-type and Markov-type inequalities for multivariate polynomials. In connection with optimal transports, they studied isometries of Wasserstein spaces and their rigidity for quantum bits and Euclidean spheres. In dimension theory, they introduced new Hausdorff-type dimensions, and described their fundamental properties.

### **Problems in discrete mathematics**

A new fundamental game theoretic equilibrium concept was introduced. The most centralized graphs and trees were determined on  $n$  vertices. Star decompositions and independence ratio of random regular graphs were investigated. Turán type theorems were proven for graphs and hypergraphs. Ramsey type results were obtained for posets. A strengthening of Erdős-Ko-Rado theorem was given. New intersection conditions in extremal set theory were investigated. Languages in NP defined by a finite set of linearly ordered finite graphs were studied. Extensive research in convex geometry was performed. Additive representation functions and Diophantine equations were investigated.

The paper for Milestone 3 has been published, in which, beyond results on edge-ordered graphs, the conjectured optimal bound on the maximum number of intersection-reverse sequences was established, with numerous consequences. Within the framework of Milestone 4, results were obtained on the Turán number of planar graphs.

An upper bound was given on the necessary size of spanner graphs for planar metrics and abstract trees; this work was presented at the SODA conference and has already been cited by several papers.

The article promised as the fifth milestone is still under review, but several of their publications dealt with the theory of submodular functions. They characterized the matroids of graphings as the limit of the matroids of graph sequences of bounded degree. They realized that infinite-alternating set functions are precisely the infinite generalizations of finite "coverage" functions.

In their analysis of epidemic spread, they examined how awareness, behavior, and network structure shape the spread processes and how they can lead to nonlinear or paradoxical effects.

They succeeded in providing constant-time competitive algorithms for several special cases of the Matroid Secretary problem. They developed a fixed-parameter algorithm for finding highly Steiner rooted-connected orientations.

### **Research in mathematical methodology**

The Didactics Group has begun implementing a four-year research program aimed at developing, researching, and widely disseminating discovery learning in mathematics education. The **This Is How We Would Teach** project was launched in 2024 by the Rényi Institute and other organizations with the aim of building a nationwide network of mathematics teachers, developing a culture of pedagogy, and providing professional support to teachers. Teaching aids and lesson plans organized by topic and grade level are being developed to help revitalize everyday mathematics education and strengthen the professional skills of young teachers. The materials produced so far are available to everyone at <https://itm.renyi.hu/>.

### **Research in topology and geometry**

They improved the Crossing Lemma for the pair-crossing number and the odd-crossing number. In a special case they verified the 35 years old conjecture of Lescure and Meyniel that a complete  $k$ -vertex graph can be immersed in every  $n$ -vertex,  $k$ -chromatic graph  $G$  such that edges are represented by edge-disjoint paths. They reduced the unit distance problem to a conjecture regarding rigidity of graphs, which seems to be much easier to deal with. They proved a far-reaching generalization of a theorem of Raimi.

Concerning Minkowski-type differential equations, they solved the so called  $L_p$  Minkowski problem with the "geometric flow" method. They managed to characterize the case of equality in the Liakopoulos inequality.

They proved the following: (1) every countably compact  $T_3$  Delta-space is compact; (2) the existence of a dense-in-itself Baire Delta-space implies the existence of an irresolvable Baire space, and hence also the consistency of a measurable cardinal. They proved that the minimal cardinality of a regular space without a Noetherian base is the first strongly inaccessible cardinal.

In set-theoretic topology, they established a consistency theorem characterizing which cardinal sequences of length below  $\omega_3$  may occur as the cardinal sequences of scattered Boolean spaces. They also sharpened their earlier result on the complexity of Haar-null sets by determining the exact complexity class, namely the analytic-inductive class.

They developed a new lattice homology theory, which assigns bigraded homology modules to integrally closed ideals and submodules.

They showed the complete embedded topological invariance of the analytic lattice cohomology of irreducible complex analytic plane curve singularities. They studied the rank 3 representations of the Painlevé equations and described the corresponding character varieties as affine cubic surfaces. They also determined the elliptic pencils describing their Hitchin systems and proved that Fourier-Laplace transformation establishes hyperKähler isometries with the moduli spaces of their standard rank 2 representations. They continued the construction of exotic smooth manifolds with free involution. They studied the topology of complex surfaces that lie on the Bogomolov–Miyazaki–Yau line. They published a paper on links in the 3-sphere which do not bound disjoint discs in a punctured  $S^2 \times S^2$ .

### **Erdős Center**

#### **Organizing thematic semesters, Workshops and Summer Schools on various current topics in mathematics**

In 2025, the Erdős Center organized two thematic semesters of exceptionally high level, one on Statistical Physics and Probability in the Spring, and one on Complex Manifolds in the Fall. Both thematic semesters include two workshops and two summer schools, and visits by leading visitors from all around the world. In the Summer of 2025, they organized 4 focused weeks.

*Thematic semester from February to June, 2025.*

#### **Probability and Statistical Physics**

Organizers: Bálint Tóth (HUN-REN Rényi Institute/University of Bristol), Gábor Pete (HUN-REN Rényi Institute)

##### Events:

School on Disordered media, Workshop on Disordered media, School on Random matrices, Workshop on random matrices.

*Thematic semester from August to December, 2025.*

#### **Analysis and Geometry of Complex Manifolds**

Organizers: László Lempert (Purdue University), Gábor Székelyhidi (NorthWestern University, Boston), Tamás Darvas (University of Maryland), Róbert Szőke (HUN-REN Rényi Institute)

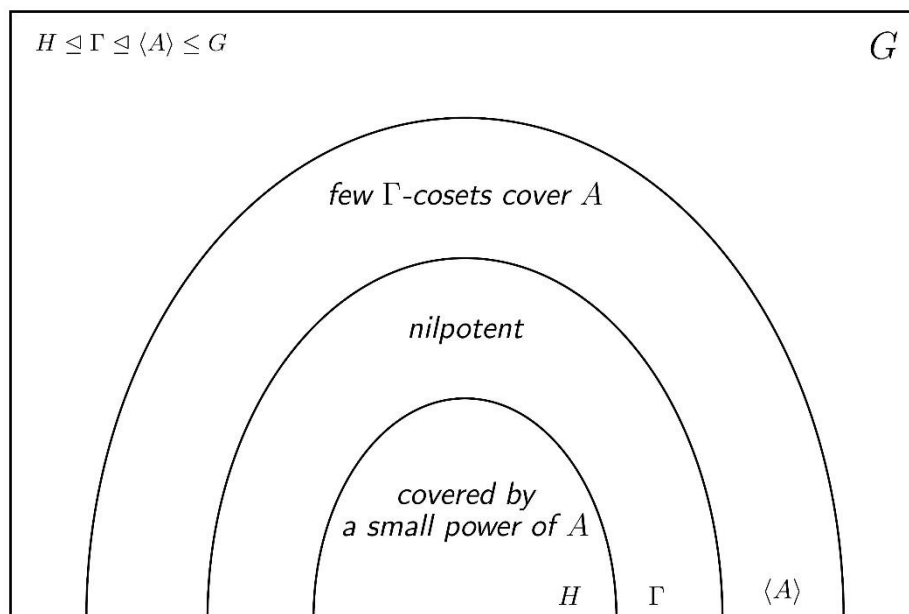
##### Events:

School on „Invitation to Complex Geometry”, School on „Singular canonical Kähler manifolds”, Workshop on „Singular canonical Kähler manifolds”, Workshop on „Almost complex and non-Kähler manifolds”.

## OUTSTANDING ACHIEVEMENT IN 2025

### Growth in linear groups

The most fundamental operations we use are addition and multiplication. The sum and product of two natural numbers are also natural numbers. In fact, we can define a similar two-variable operation on any set. If we include negative numbers and assign to any integer its opposite sign, the resulting set of numbers, including zero, forms a group with respect to addition. Similarly, rational numbers form a group with respect to multiplication and taking reciprocals. For example, the rotations of the Rubik's cube form a finite group, where two rotation sequences are “multiplied” by performing them one after the other. A new direction in the theory of finite “indecomposable” groups, called simple groups, stems from an unexpected result that has become known as the “Product Theorem”. The result establishes a growth property of subsets of simple groups that has become a useful tool in other areas of mathematics. This result was obtained independently and simultaneously by Pyber-Szabó and the Fields Medalist Terence Tao and his colleagues. The Fields Medalist Bourgain and a colleague showed how, building on this new growth property of simple groups, a significant portion of the graphs belonging to groups can be shown to be so-called expander graphs. Expanders can be used in many areas of mathematics. Expander graphs are networks that, no matter how we cut them into two parts, there are many edges between the two parts. The researchers of the HUN-REN Rényi Institute in a 2025 article [Eberhard S, Murphy B, [Pyber L](#), [Szabó E](#): Growth in linear groups, *Duke Mathematical Journal* 174:3 pp. 403-448, 46 p. (2025)] proved a conjecture of Helfgott on the structure of non-growing subsets of simple groups. Duke is among the six leading journals in mathematics. Their new theorem includes the Product Theorem as a special case. As an application of their methods, they also show that the diameter of sufficiently quasirandom finite linear groups is poly-logarithmic, a step towards constructing new expanders.



The structure of non-growing subsets