Applications of the logarithmic capacity in one-dimensional problems concerning the Bergman kernel and metric

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Potential Theory – basics

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Define $I(\mu) := \int_K p_\mu(z) d\mu(z)$. For a Borel set $E \subset \mathbb{C}$ we define

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The above measure μ_K is called the *equilibrium measure of K*.

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The following properties hold

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- (*subadditivity*) if $B = \bigcup_{j=1}^{N} B_j$, B_j is Borel and $\mathrm{diam}(B) \leq d$ then

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- $c(\triangle(z,r)) = c(\partial\triangle(z,r)) = r$, $c([a,b]) = |b-a|/4$.

Frostman's theorem

A point $z_0 \in \partial D$ is called *regular* (w.r.t. Dirichlet problem) if there are a neighborhood U of z_0 and a negative subharmonic function u defined on $U \cap D$ such that $\lim_{D \ni z \to z_0} u(z) = 0$.

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Theorem

(Frostman) For a non-polar K we have $p_{\mu_K} \geq \log c(K)$ and $p_{\mu_K} = \log c(K)$ on $K \setminus F$ where F is an F_σ-polar subset of ∂K. Moreover, if the point $z \in \partial K$ is regular for the Dirichlet problem for the unbounded connected component of $\mathbb{C} \setminus K$ then $p_{\mu_K}(z) = \log c(K)$.

Wiener's criterion

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(Wiener's criterion) Let D be a bounded domain in $\mathbb{C}$, $z_0 \in \partial D$, $\theta \in (0,1)$. Denote

$$A_j(z_0) := \{z \in \mathbb{C} \setminus D : \theta^{j+1} \leq |z - z_0| < \theta^j\}. \tag{3}$$

Then z_0 is regular iff $\sum_{j=1}^{\infty} \frac{-j}{\log c(A_j(z_0))} = \infty$.

The Green function

For a domain $D \subset \mathbb{C}^n$, $p, z \in D$ we define *the pluricomplex Green function with the logarithmic pole at* p as $g_D(p, z) := \sup\{u(z)\}$, where the supremum is taken over all $u \in PSH(D)$, $u < 0$ and such that $\limsup_{w \to p}(u(w) - \log ||w - p||) < \infty$.

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Bergman functions

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If $(\phi_j)_j$ is a complete orthonormal system of $L^2_h(D)$ then

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The following formula holds

$$K_D(z) = \sup\{|f(z)|^2 : f \in L^2_h(D), ||f|| \leq 1\}. \tag{6}$$

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One may express β_D in another way. Namely, if D is bounded then

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Bergman functions – continued

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In dimension one we may analoguously introduce β-exhaustiveness of a bounded domain.

(Pluri)Potential theory meets Bergman functions

The following is a full characterization of extendability of L^2_h holomorphic function in the sense of the Riemann removability theorem for bounded holomorphic functions.

Theorem

Let D be a domain in $\mathbb{C}$, $z_0 \in \partial D$. Then z_0 is a removable singularity for $L^2_h(D)$ (i. e. there is an open neighborhood U of z_0 such that any function from $L^2_h(D)$ extends analytically to $D \cup U$) iff there is an open neighborhood U of z_0 such that $U \setminus D$ is polar.

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Let D be a domain in $\mathbb{C}$. Then $\frac{2}{\pi}\frac{\partial^2 g_D}{\partial w \partial \bar{z}}(w,z) = K_D(w,z)$, $w, z \in D$, $w \neq z$.

Suita conjecture: potential theory & Bergman kernel once more

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$$e^{-2nt}\lambda(\{g_D(w,\cdot)<t\})\to\lambda(I_D^A(w)) \text{ as } t\to-\infty.$$

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This led to SCV version of the Suita conjecture.

Theorem

(Błocki-Zwonek, 2015) Let D be a bounded pseudoconvex domain in $\mathbb{C}^n$. Then

$$K_D(w)\geq\frac{1}{\lambda(I_D^A(w))}.$$

SCV Suita conjecture - remarks

Remark 1. For $n = 1$ one has $\lambda(I_D^A(w)) = \pi / A_D(w)^2$.

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3. In the case D is not pseudoconvex but smooth we have no lower estimate as above (Nikolov).

Conjecture For D pseudoconvex and $w \in D$ the function

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The problems behind the proof of the Suita conjecture

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is convex on $(-\infty, 0]$. Fornæss however constructed a counterexample to this (already for $n = 1$).

Theorem The conjecture is true for $n = 1$.

Suita conjecture – continued

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($A_D^2 \leq \pi K_D$ being a quantitative version of $\Leftarrow$).

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Remark The proof of the optimal lower bound $F_D \geq 1$ used $\bar\partial$. The proof of the (probably) non-optimal upper bound $F_D \leq 4$ is much more elementary!

SCV Suita conjecture – continued

For domain D the function

$$F_D(w) = \left(\lambda(I_D^A(w))K_D(w)\right)^{1/n}$$

is a biholomorphically invariant function satisfying for convex D the inequalities $1 \leq F_D \leq 4$.

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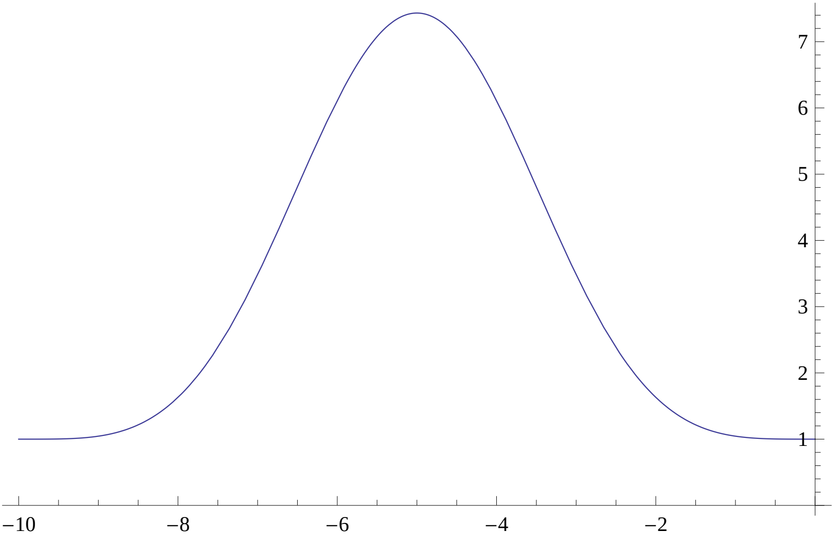
Graph of F_D - example



$\dfrac{\pi K_D}{A_D^2}$ for $D = \{e^{-5} < |z| < 1\}$ as a function of $2 \log |w|$

SCV Suita conjecture: convex case – discussion

For convex domains it was not so simple to find examples such that $F_D \not\equiv 1$.

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As we shall see the examples we could find had the possible values differing very little from 1.

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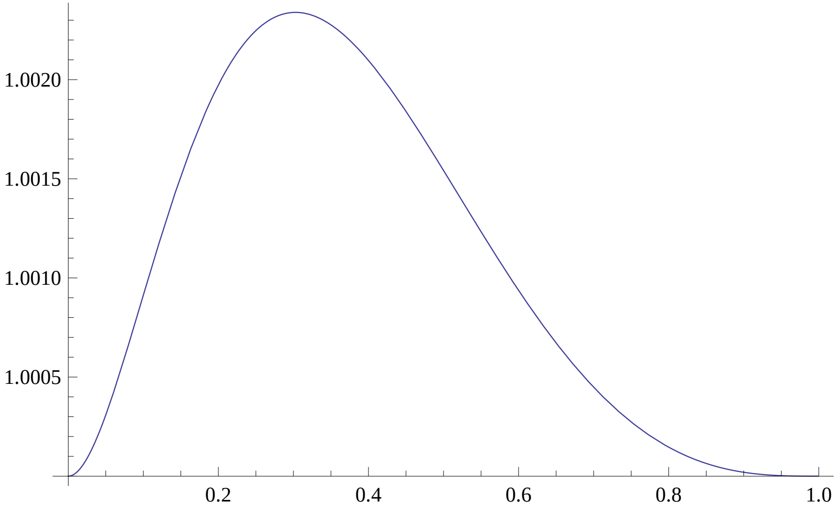
Theorem For $D = \{|z_1| + |z_2|^2 < 1\}$ and $b \in [0,1)$ one has

$$\lambda(I_D((b,0))) = \frac{\pi^2}{3}(1-b)^3(1+3b+3b^2-b^3)$$

and

$$\lambda(I_D((b,0)))K_D((b,0)) = 1 + \frac{(1-b)^3 b^2}{3(1+b)^3}.$$

$F_D \neq 1$ - first example, graph



$F_D((b,0))$ for $D = \{|z_1| + |z_2|^2 < 1\}$

Further examples - ellipsoids

Although the Kobayashi function of $\mathcal{E}(m, 1)$ is given by implicit formulas, it turns out that the volume of the Kobayashi indicatrix can be computed explicitly:

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$$= \pi^2 \Big[-\frac{m-1}{2m(3m-2)(3m-1)} b^{6m+2} - \frac{3(m-1)}{2m(m-2)(m+1)} b^{2m+2} + \frac{m}{2(m-2)(3m-2)} b^6 + \frac{3m}{3m-1} b^4 - \frac{4m-1}{2m} b^2 + \frac{m}{m+1} \Big].$$

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For $m = 2/3$

$$\lambda(I_D((b,0))) = \frac{\pi^2}{80}\Big(-65b^6 + 40b^6 \log b + 160b^4 - 27b^{10/3} - 100b^2 + 32$$

and $m = 2$

$$\lambda(I_D((b,0))) = \frac{\pi^2}{240}\big(-3b^{14} - 25b^6 - 120b^6 \log b + 288b^4 - 420b^2 + 160$$

Formula for F_D – sketch of the proof

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is a parametrization of an S^1-invariant portion of ∂I then the volume of the corresponding part of I is given by

$$\frac{\pi}{2} \int_U |H(z)| d\lambda(z), \tag{10}$$

where

$$H = |f|^2(|g_{\bar{z}}|^2 - |g_z|^2) + |g|^2(|f_{\bar{z}}|^2 - |f_z|^2) + 2\Re\big(f\bar{g}(\overline{f_z}g_z - \overline{f_{\bar{z}}}g_{\bar{z}})\big).$$

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$$\mathbb{C} \supset U \ni z \longmapsto (f(z), g(z)) \in \partial I$$

is a parametrization of an S^1-invariant portion of ∂I then the volume of the corresponding part of I is given by

$$\frac{\pi}{2} \int_U |H(z)| d\lambda(z), \qquad (10)$$

where

$$H = |f|^2(|g_{\bar{z}}|^2 - |g_z|^2) + |g|^2(|f_{\bar{z}}|^2 - |f_z|^2) + 2\Re\big(f\bar{g}(\overline{f_z}g_z - \overline{f_{\bar{z}}}g_{\bar{z}})\big).$$

Both H and the integral (10) are computed with the help of *Mathematica*.

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Both H and the integral (10) are computed with the help of *Mathematica*.
The same method is used for computations in other ellipsoids.

Proof – continued

For $D = \{|z_1|^{2m} + |z_2|^2 < 1\}$ the formula for the Bergman kernel is well known:

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so that

$$K_D((b,0)) = \frac{m+1+(1-m)b^2}{\pi^2 m(1-b^2)^3}.$$

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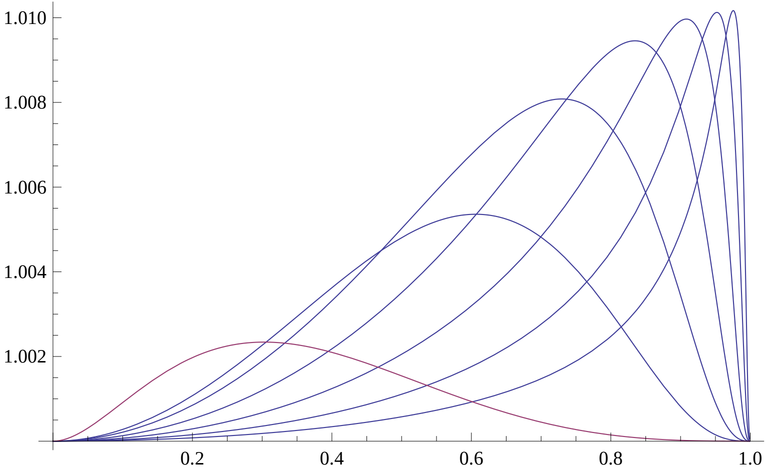
Since for $t \in \mathbb{R}$ and $a \in \Delta$ the mapping

$$D \ni z \longmapsto \left(e^{it}\frac{(1-|a|^2)^{1/2m}}{(1-\bar{a}z_2)^{1/m}}z_1, \frac{z_2-a}{1-\bar{a}z_2}\right)$$

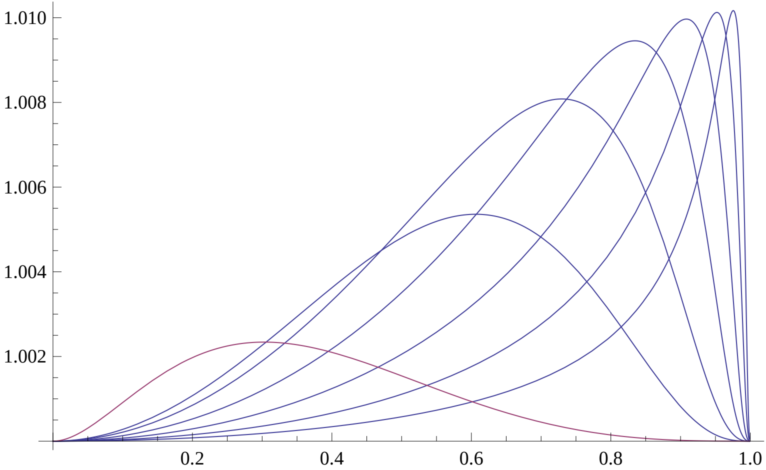
is a holomorphic automorphism of D, $F_D((b,0))$ for $b \in [0,1)$ attains all values of F_D in D.

Formulas - graphs



$F_D((b,0))$ in $D = \{|z_1|^{2m} + |z_2|^2 < 1\}$ for $m = 1/2, 4, 8, 16, 32, 64, 128$

Formulas - graphs



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$$\sup_{0<b<1} F_D((b,0)) \to 1.010182\ldots \text{ as } m \to \infty$$

(highest value of F_D obtained so far in arbitrary dimension)

Analyticity of F_D – counterexample

Theorem For $D = \{|z_1| + |z_2| < 1\}$ and $b \in [0, 1)$ one has

$$\lambda(I_D((b,0))) = \frac{\pi^2}{6}(1-b)^4\big((1-b)^4 + 8b\big)$$

and

$$\lambda(I_D((b,0)))K_D((b,0)) = 1 + b^2\frac{(1-b)^4}{(1+b)^4}.$$

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The Bergman kernel for this ellipsoid was found by Hahn-Pflug (1988):

$$K_D(w) = \frac{2}{\pi^2} \cdot \frac{3(1-|w|^2)^2(1+|w|^2) + 4|w_1|^2|w_2|^2(5-3|w|^2)}{\big((1-|w|^2)^2 - 4|w_1|^2|w_2|^2\big)^3},$$

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In all examples so far the function $w \mapsto \lambda(I_D(w))$ is analytic. Is it true in general?

Analyticity of F_D - counterexample, continued

Theorem For $D = \{|z_1| + |z_2| < 1\}$ and $b \in [0, 1/4]$ one has

$$\lambda(I_D((b,b))) = \frac{\pi^2}{6}\big(30b^8 - 64b^7 + 80b^6 - 80b^5 + 76b^4 - 16b^3 - 8b^2 + 1\big).$$

Analyticity of F_D - counterexample, continued

Since $K_D((b,b)) = \dfrac{2(3-6b^2+8b^4)}{\pi^2(1-4b^2)^3}$, we get the following picture:

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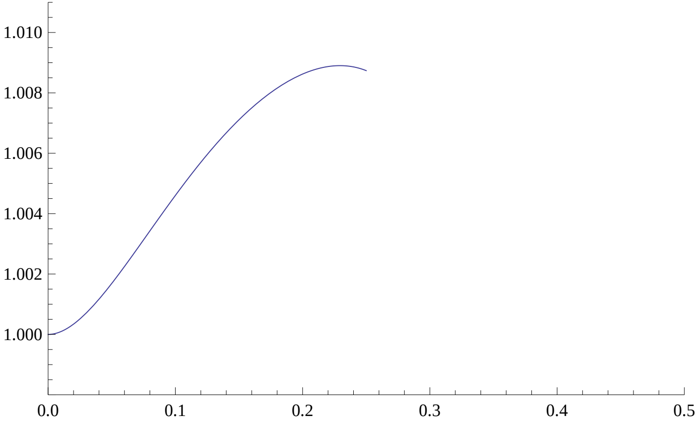

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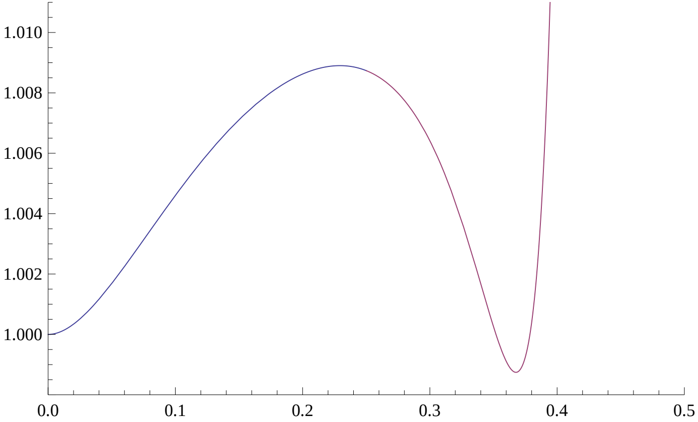

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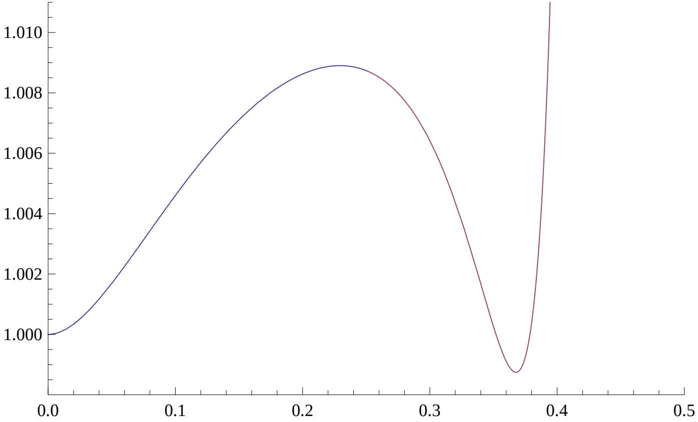

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For $b \in [1/4, 1/2)$

$$\begin{aligned}
\lambda(I_D((b,b))) &= \frac{2\pi^2 b(1-2b)^3\left(-2b^3+3b^2-6b+4\right)}{3(1-b)^2} \\
&+ \frac{\pi\left(30b^{10}-124b^9+238b^8-176b^7-260b^6+424b^5-76b^4-144b^3+89b^2-18b+1\right)}{6(1-b)^2} \\
&\quad \times \arccos\left(-1+\frac{4b-1}{2b^2}\right) \\
&+ \frac{\pi(1-2b)\left(-180b^7+444b^6-554b^5+754b^4-1214b^3+922b^2-305b+37\right)}{72(1-b)}\sqrt{4b-1} \\
&+ \frac{4\pi b(1-2b)^4\left(7b^2+2b-2\right)}{3(1-b)^2}\arctan\sqrt{4b-1} \\
&+ \frac{4\pi b^2(1-2b)^4(2-b)}{(1-b)^2}\arctan\frac{1-3b}{(1-b)\sqrt{4b-1}}.
\end{aligned}$$

Analyticity of F_D - counterexample, continued

By $\chi_-(b)$, resp. $\chi_+(b)$, denote $\lambda(I_D((b,b)))$ for $b \leq 1/4$, resp. $b \geq 1/4$.

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By $\chi_-(b)$, resp. $\chi_+(b)$, denote $\lambda(I_D((b,b)))$ for $b \leq 1/4$, resp. $b \geq 1/4$. Then at $b = 1/4$

$$\chi_- = \chi_+ = \frac{15887}{196608}\pi^2,$$

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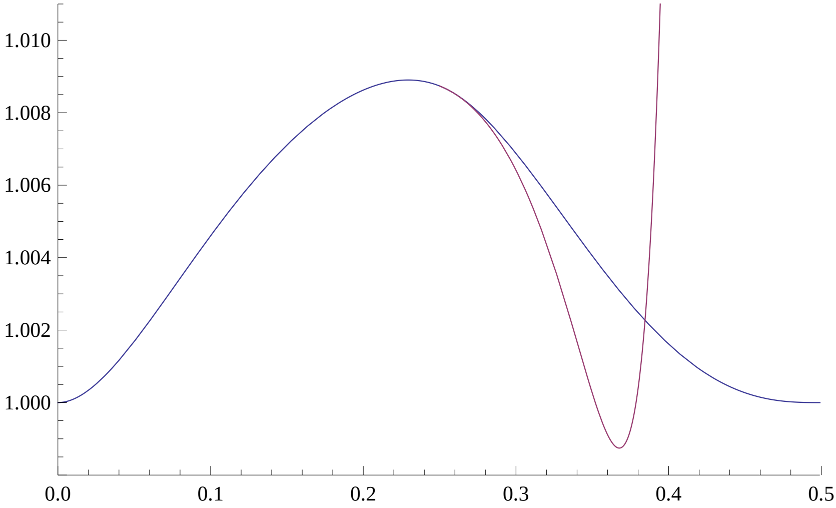
Corollary For $D = \{|z_1| + |z_2| < 1\}$ the function $w \mapsto \lambda(I_D(w))$ is not $C^{3,1}$ at $w = (1/4, 1/4)$.

Analyticity of F_D - counterexample, continued



$F_D((b,b))$ in $D = \{|z_1| + |z_2| < 1\}$ for $b \in [0, 1/2)$

Kobayashi criterion for Bergman completeness

The following is a basic tool in the study of the Bergman completeness.

Theorem

(Kobayashi, 1960) Let D be a bounded domain in $\mathbb{C}^n$. Assume that there is a dense set $E \subset L^2_h(D)$ such that for any sequence $(z^\nu) \subset D$ without accummulation point in D the equality

$$\lim_{\nu \to \infty} \frac{|f(z^\nu)|^2}{K_D(z^\nu)} = 0, \ f \in E \tag{11}$$

holds. Then D is Bergman complete.

Recall that a bounded domain $D \subset \mathbb{C}^n$ is said to be *hyperconvex* if there is $u \in PSH^-(D) \cap C(D)$ such that $\lim_{w \to \partial D} u(w) = 0$ (such a function is called negative exhausting). In dimension one hyperconvexity means regularity (and we may take $u := g_D(p, \cdot)$)

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In 2000 Bo-Yong Chen showed that in dimension one the Bergman exhaustivity implies the completeness (the Hartogs triangle $\{z \in \mathbb{C}^2 : |z_1| < |z_2|\}$ is a counterexample for the converse implication in dimension two).

Potential theory comes into play

Let D be a bounded domain in $\mathbb{C}$, $z \in \mathbb{C}$, $k \in \mathbb{N}$. Define

$$\gamma_D^{(k)}(z) := \int_0^{1/4} \frac{d\delta}{\delta^{2k+3}(-\log c(\bar{\triangle}(z,\delta) \setminus D))}. \tag{12}$$

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It is easy to see that

$$2^{-2k-3} \sum_{j=3}^{\infty} \frac{2^{2(k+1)j}}{-\log c(A_k(z) \setminus D)} \le \gamma_D^{(k)}(z) \le 2^{2k+3} \sum_{j=3}^{\infty} \frac{2^{2(k+1)j}}{-\log c(A_k(z) \setminus D)} \tag{13}$$

Potential theory comes into play – continued

We also need the following function. For a non-polar compact $K \subset \mathbb{C}$ define

$$f_K(z) := \int_K \frac{d_{\mu_K}(\lambda)}{\lambda - z}, \; z \in \mathbb{C} \setminus K. \tag{14}$$

We also put $f_K \equiv 0$ when K is polar.

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Before we formulate the main tools in the study of the Bergman functions in the complex plane let us generalize the notions used in the definition of the Bergman kernel and the Bergman metric.

Potential theory comes into play – continued

For a domain $D \subset \mathbb{C}$, $z \in D$ and $k \in \mathbb{N}$ consider the following linear functionals

$$L_h^2(D) \ni f \to f^{(k)}(z) \in \mathbb{C}. \tag{16}$$

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And let us denote the operator norm of the above operator by $K_D^{(k)}(z)$.
Note that $K_D(z) = K_D^{(0)}(z)$, $M_D(z) \leq K_D^{(1)}(z)$.

Estimates of the operator norms in terms of the potential-theoretic objects

The main result on the norms is the following.

Theorem

(Pflug-Zwonek, 2003) Fix $k \in \mathbb{N}$, $d > 1$. Then there is a $C > 0$ such that

- *for any bounded domain D with $\operatorname{diam}(D) < d$ the inequality $C\gamma_D^{(k)}(z) \leq K_D^{(k)}(z)$, $z \in D$, holds;*
- *for any bounded domain D with $1/d < \operatorname{diam}(D) < d$ the inequality $K_D^{(k)}(z) \leq C \max\{1, \gamma_D^{(k)}(z)(\log \gamma_D^{(k)}(z))^2\}$, $z \in D$, holds*

Direct consequences of estimates of the norms on Bergman functions

We get directly the following.

Corollary

Let D be a bounded domain in $\mathbb{C}$, $z_0 \in \partial D$. Then the following are equivalent

- $\lim_{D \ni z \to z_0} \gamma_D(z) = \infty$,
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Zalcman-type domains

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The domain $D := \mathbb{D} \setminus \left(\bigcup_{j=1}^{\infty} \bar{\triangle}(x_k, r_k) \cup \{0\} \right)$, where $x_j > x_{j+1} > 0$, $x_j \to 0$, $\bar{\triangle}(x_j, r_j) \subset \mathbb{D}$, $\bar{\triangle}(x_j, r_j) \cap \bar{\triangle}(x_k, r_k) = \emptyset$, $j \neq k$, is called a Zalcman-type domain.

Theorem

(Jucha, 2004) Let D be a Zalcman-type domain as above such that additionally there is a $\theta \in (0,1)$ *that* $\frac{x_{j+1}}{x_j} \leq \theta$. *Then*

Zalcman-type domains – results

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The above theorem allows us easily to construct examples of the Zalcman-type domains of the following types:

- Bergman complete that are not hyperconvex,
- Bergman complete that are not Bergman exhaustive.

More examples

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