Estimates of the Bergman distance on Dini-smooth bounded planar domains

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$$\beta_D(z;X) = \frac{M_D(z;X)}{K_D(z)}, \quad z \in D, \ X \in \mathbb{C}^n,$$

where

$$M_D(z;X) = \sup\left\{|f'(z)X| : f \in L^2_h(D), \ \|f\|_D = 1, \ f(z) = 0\right\}$$

and

$$K_D(z) = \sup\left\{|f(z)| : f \in L^2_h(D), \ \|f\|_D = 1\right\}$$

is the square root of the Bergman kernel on the diagonal.

Bergman distance

$$b_D(z,w) = \inf\Big\{ \int_{[0,1]} \beta_D(\gamma(t);\gamma'(t))dt \,|\, \gamma : [0,1] \to D \text{ smooth}, \\ \gamma(0)=z,\, \gamma(1)=w \Big\},$$

We shall use also c_D and l_D the Carathéodory distance and the Lempert function of D, respectively:

$$c_D(z, w) = \sup \Big\{ \operatorname{tgh}^{-1} |f(w)| : f \in \mathcal{O}(D, \mathbb{D}), \text{ with } f(z) = 0 \Big\},$$

$$l_D(z, w) = \inf \Big\{ \operatorname{tgh}^{-1} |\alpha| : \exists \varphi \in \mathcal{O}(\mathbb{D}, D) \text{ with } \varphi(0) = z, \varphi(\alpha) = w \Big\},$$

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$\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$.
The Kobayashi distance k_D is the largest pseudodistance not exceeding l_D. For any planar domain D we have equality of k_D and l_D.

In general, k_D is the integrted form of Kobayashi metric κ_D

$$\kappa_D(z;X) = \inf\Big\{|\alpha| : \exists\varphi \in \mathcal{O}(\mathbb{D}, D) \text{ with } \varphi(0) = z,\ \alpha\varphi'(0) = X\Big\}.$$

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$$= \inf\{\int \kappa_D(\gamma(t), \gamma'(t))dt : \gamma \text{ is a piecewise } \mathcal{C}^1 \text{ curve joining } z,\ w\}.$$

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From now $n = 1$.

The function ϑ is called *Dini-continuous* if

$$\int_0^\delta \frac{\omega(t)}{t}\,dt < \infty, \quad \text{for some } \delta > 0,$$

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A planar domain D is *Dini smooth* at a ($\in \partial D$) if there exist a neighborhood U of a and a Dini-smooth Jordan arc γ such that $\partial D \cap U = \gamma^*$.

Theorem (Warschawski Theorem, 1932)

Let F maps $\mathbb{D}$ conformally $\mathbb{D}$ onto the inner domain of the Dini-smooth Jordan curve J. Then F' extends continuously to $\overline{\mathbb{D}}$ and

$$\lim_{z\to w}\frac{F(z)-F(w)}{z-w}=F'(w)\neq 0,\ z,\ w\in\overline{\mathbb{D}}.$$

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Warschawski Theorem implies that

$$d_{\mathbb{D}}(z)\sim d_D(F(z))$$

for $z\in D$ sufficiently close to a if D is Dini-smooth at a,

Proposition (Nikolov, 2013)

Let D be a Dini-smooth bounded planar domain. Then there exists a constant $c_1 > 1$ such that

$$\log\left(1+\frac{|z-w|}{c_1\sqrt{d_D(z)d_D(w)}}+\frac{|z-w|^2}{c_1 d_D(z)d_D(w)}\right)\leqslant s_D(z,w)$$
$$\leqslant \log\left(1+\frac{c_1|z-w|}{\sqrt{d_D(z)d_D(w)}}+\frac{c_1|z-w|^2}{d_D(z)d_D(w)}\right),\quad z,\ w\in D,$$

where $s_D(z,w) = 2c_D(z,w)$ or $s_D(z,w) = 2k_D(z,w)$.

Proposition (Nikolov - T., 2015)

Led D be a Dini-smooth bounded planar domain. Then there exists a constant $c > 1$ such that

$$\sqrt{2}\log\left(1+\frac{|z-w|}{c\sqrt{d_D(z)d_D(w)}}\right) \leqslant b_D(z,w) \leqslant \sqrt{2}\log\left(1+\frac{c|z-w|}{\sqrt{d_D(z)d_D(w)}}\right), \quad z, w \in D.$$

Corollary

If D is a Dini-smooth bounded planar domain, then the differences $b_D - \sqrt{2}c_D$ and $b_D - \sqrt{2}k_D$ are bounded.

If the regularity condition is missing, then there is no constant such that the upper bound holds. Indeed, let $D \subset \mathbb{C}$ be the image of $\mathbb{D}$ under the map $z \to 2z + (1-z)\log(1-z)$. Then D is a $\mathcal{C}^1$-smooth bounded domain and

$$\lim_{\mathbb{R} \ni w \to 2^-} \frac{1 - \operatorname{tgh} l_D(0,w)}{d_D(w)} = 0$$

Proposition (Nikolov, 2013)

Assume that D is a Dini-smooth domain. For every point $p \in \partial D$ and any compact subset K of D, there exist a neighborhood V of p, and a constant $c > 0$ such that

$$|2s_D(z,w) + \log d_D(w)| \leqslant c, \quad z \in K, \ w \in D \cap V,$$

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Corollary (Nikolov, 2013)

Assume that D is a Dini-smooth, bounded domain. Let $p,\ q$ be different boundary points of D. If $s_D = k_D$ or $s_D = b_D/\sqrt{2}$, then the function

$$2s_D(z,w) + \log d_D(z) + \log d_D(w)$$

is bounded for z near q and w near p.

Theorem (Forstneric - Rossay, 1987)

Let Ω *be a bounded domain in* $\mathbb{C}^n$*. Assume that* Ω *is strictly pseudoconvex at a point* z_0 $(\in \partial\Omega)$*. Let* $\Omega_0 \subset \Omega$ *be a domain such that* $z_0 \in \operatorname{int}_{\partial\Omega} \partial\Omega_0$*. Then there exists a neighbborhood* U *of* z_0 *and a constant* $c > 0$ *such that for any point* $z \in \Omega_0 \cap U$*, any vector* $X \in \mathbb{C}^n$ *the following relation between* κ_Ω *and* κ_{Ω_0} *holds*

$$\kappa_\Omega(z; X) \geqslant \Big(1 - c\, d_\Omega(z)\Big)\kappa_{\Omega_0}(z; X). \tag{1}$$

Recall

$$\gamma_{\mathbb{B}_n}(z;X) = \sqrt[2]{\frac{\|X\|^2}{1-\|z\|^2} + \frac{|\langle z;X\rangle|^2}{(1-\|z\|^2)^2}}, \quad z \in \mathbb{B}_n,\ X \in \mathbb{C}^n.$$

Theorem (Balough - Bonk, 2000)

Let F_j, $j=1,2$ be metrics on $\mathbb{B}_2$ such that

$$(1 - C_j d_{\mathbb{B}_2}(z)^\delta)\Big(\frac{|\langle z;X\rangle|^2}{(1-\|z\|^2)^2} + 1/C_j \frac{\|X\|^2}{1-\|z\|^2}\Big)^{1/2} \leqslant F_j(z;X)$$
$$\leqslant (1 + C_j d_{\mathbb{B}_2}(z)^\delta)\Big(\frac{|\langle z;X\rangle|^2}{(1-\|z\|^2)^2} + C_j \frac{\|X\|^2}{1-\|z\|^2}\Big)^{1/2},$$

$z \in \mathbb{B}_2$, $X \in \mathbb{C}^2$, *where* $C_j > 1$, $\delta > 0$. *Then, the difference of the distance functions associated with* F_j, $j = 1, 2$ *is bounded.*

It is enough to find a constant $c > 1$ such that the respective estimates hold for $b_D(z_n, w_n)$ for every sequences (z_n), $(w_n) \subset D$ such that $z_n \to 1$ and $w_n \to 1$ for any n.
For a planar domain Ω set $\beta_\Omega(z) := \beta_\Omega(z;1)$, $M_\Omega(z) := M_\Omega(z;1)$ and $\kappa_\Omega(z) := \kappa_\Omega(z;1)$ for a point $z \in \Omega$.

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Then, we might write the following

$$\sqrt{2}\frac{\kappa_{\mathbb{D}}^2(z)}{\kappa_{E_r}(z)} = \frac{M_{\mathbb{D}}(z)}{\sqrt{K_{E_r}(z)}} \leqslant \beta_D(z) \leqslant \frac{M_{E_r}(z)}{\sqrt{K_{\mathbb{D}}(z)}} = \sqrt{2}\frac{\kappa_{E_r}^2(z)}{\kappa_{\mathbb{D}}(z)}, \quad z \in E_r \tag{2}$$

(the both equalities hold because E_r is a simply connected domain, here the smoothness of D is not required).
Fix an $r_1 \in (0, r_0)$. The localization of the Kobayashi metric implies that

$$\kappa_{\mathbb{D}}(z) \geqslant (1 - c_2\, d_{\mathbb{D}}(z))\kappa_{E_r}(z), \quad z \in E_{r_1}, \tag{3}$$

for some constant $c_2 > 0$.

Then (2) and (3) imply that there exists a $r_2 \in (0, r_1]$ with $3c_2 r_2 \leqslant 1$ such that

$$\sqrt{2}(1 - c_2\, d_{\mathbb{D}}(z))\kappa_{\mathbb{D}}(z) \leqslant \beta_D(z) \leqslant \sqrt{2}(1 + \frac{5}{2} c_2 d_{\mathbb{D}}(z))\kappa_{\mathbb{D}}(z), \quad z \in E_{r_2}$$

Since $\kappa_{\mathbb{D}}(z) = \frac{\beta_{\mathbb{D}}(z)}{\sqrt{2}} = \frac{1}{1-|z|^2}$, it follows for $c_3 = \frac{\sqrt{2}}{2} c_2$ that

$$\frac{\beta_{\mathbb{D}}(z)}{3} < \beta_{\mathbb{D}}(z) - 2c_3 < \beta_D(z) < \beta_{\mathbb{D}}(z) + 5c_3, \quad z \in E_{r_2}. \quad (4)$$

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We may assume that $z_n, w_n \in E_{r_3}$, where $r_3 \in (0, r_2)$ is such that if α_n is the shorter arc with endpoints z_n and w_n of the circle through z_n and w_n which is orthogonal to the unit circle, then $\alpha_n \subset E_{r_2}$. Hence

$$b_D(z_n, w_n) < \int_{\alpha_n} \left(\frac{\sqrt{2}}{1 - |z|^2} + 5c_3 \right) dl$$

$$= b_{\mathbb{D}}(z_n, w_n) + 5c_3\, L_{|\ |}(\alpha_n) < b_{\mathbb{D}}(z_n, w_n) + 10c_3 |z_n - w_n|$$

for any n.

The above equality follows from the description of the shortest curves with respect to the Bergman distance on $\mathbb{D}$. To get the second inequality we applied an elementary inequality $1 \leqslant \frac{x}{\sin x} < 2$ for $x \in (0, \frac{\pi}{2})$.
Now, using Lemma and the inequality

$$d_{\mathbb{D}}(z) \geqslant d_D(z), \quad z \in D,$$

it is easy to find a constant $c > 1$ such that the upper estimate for $b_D(z_n, w_n)$ holds for any n.

It is left to manage with the lower estimate. Shrinking r_3 (if necessary), we may assume that

$$d_{\mathbb{D}}(z) = d_D(z), \quad z \in E_{r_3}. \tag{5}$$

Consider the set A of all n for which there exists a smooth curve $\gamma_n : [0,1] \to D$ such that $\gamma_n(0) = z_n$, $\gamma_n(1) = w_n$, $\gamma_n((0,1)) \not\subset E_{r_2}$ and

$$b_D(z_n, w_n) + |z_n - w_n| > \int_0^1 \beta_D(\gamma_n(t); \gamma_n'(t))dt.$$

For any $n \in A$ we may find a number $t_n \in (0,1)$ such that $|u_n - 1| = r_2$, where $u_n = \gamma(t_n)$. By (4), there exists a constant $c_4 > 0$, which does not depend on $n \in A$, such that

$$b_D(z_n, w_n) + |z_n - w_n| > b_D(z_n, u_n) + b_D(u_n, w_n)$$

$$> -\frac{\log d_D(z_n)}{\sqrt{2}} - \frac{\log d_D(w_n)}{\sqrt{2}} - c_4.$$

This inequality easily provides a constant $c > 1$ for which the lower estimate for $b_D(z_n, w_n)$ holds for any $n \in A$.
Let now $n \notin A$. Then, using (4) and the formula for the Kobayashi metric for the unit ball we get that

$$b_D(z_n, w_n) + |z_n - w_n| \geqslant \sqrt{2}\hat{k}_{\mathbb{B}_2}((z_n, 0), (w_n, 0)),$$

where $\hat{k}_{\mathbb{B}_2}$ is the pseudodistance arising from the Finsler pseudometric $\hat{\kappa}_{\mathbb{B}_2}(w; Y) = (\kappa_{\mathbb{B}_2}(w; Y) - 2c_2||Y||)^+$ (i.e. its integrated form). By Balough-Bonk Theorem $\kappa_{\mathbb{B}_2}$ and $\hat{\kappa}_{\mathbb{B}_2}$, we may find a constant $c_5 > 0$ such that $0 < k_{\mathbb{B}_2} - \hat{k}_{\mathbb{B}_2} < c_5$.

It follows from here and $\beta_{\mathbb{D}} = \sqrt{2} k_{\mathbb{B}_2}|_{\mathbb{D}\times\{0\}}$ that

$$b_D(z_n, w_n) + |z_n - w_n| > b_{\mathbb{D}}(z_n, w_n) - \sqrt{2} c_5$$

which, together with Lemma and (5), easily implies the lower estimate if $|z_n - w_n|^2 > d_D(z_n) d_D(w_n)$.
To prove the lower estimate in Proposition 1.5.8 when $n \notin A$ and $|z_n - w_n|^2 \leqslant d_D(z_n) d_D(w_n)$, it suffices to observe that (4) leads to $3 b_D(z_n, w_n) \geqslant b_{\mathbb{D}}(z_n, w_n)$ and then to apply Lemma and (5).

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Thank you for your atention!