A survey on the Bernstein Markov Property I

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Budapest, August 18th 23rd 2015

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joint work with
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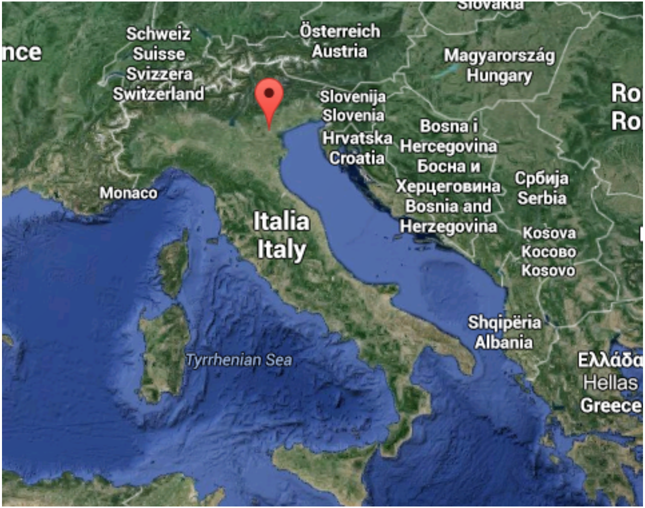
Outline

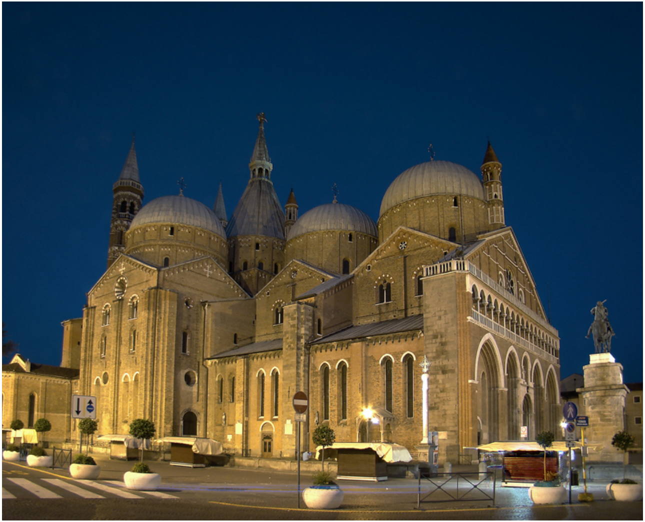
Introducing myself. . .

I am a PhD candidate at Padova



Small old town. . .

but with some interesting monuments.



with medieval down-town

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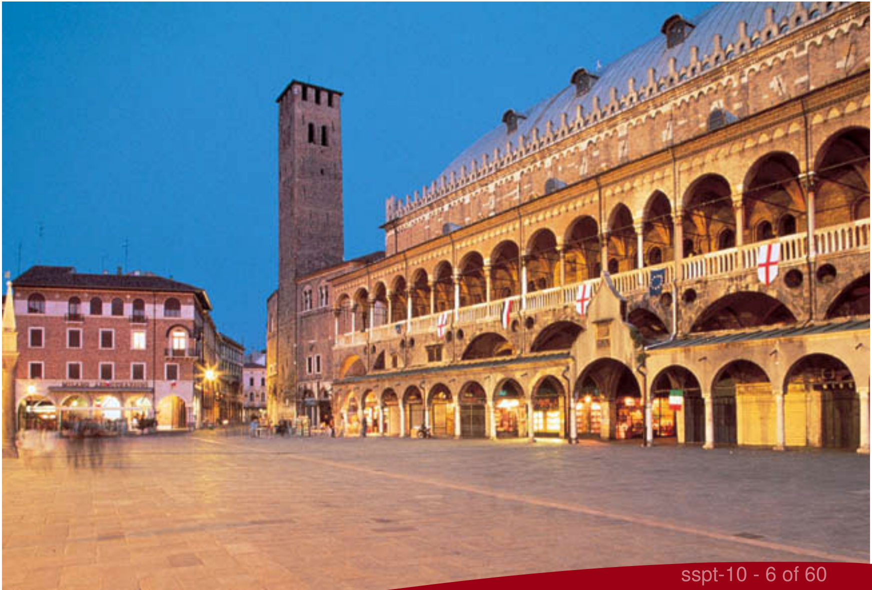


but the largest square in Europe!

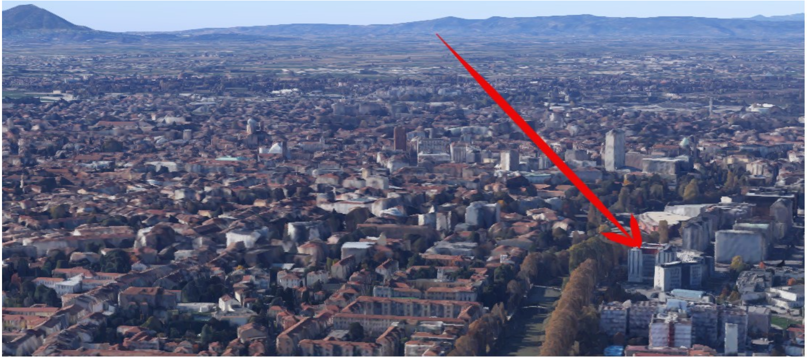
and...

the 8th oldest university in the world (founded in 1222!)

Galileo lectured there!

The Departments of Mathematics

Università degli Studi di Padova



My office!

My group

CAA Padova Verona

http://www.math.unipd.it/∼ marcov/CAA.html

M. Vianello, L. Bos, M. Calliari, A. Sommariva, S. De Marchi, G. Santin
We work on Constructive Approximation and Applications in the broad sense, with a special interest in the study and implementation of effective approximation algorithms, and in the production of reliable numerical software.

my web-page: http://www.math.unipd.it/∼fpiazzon/

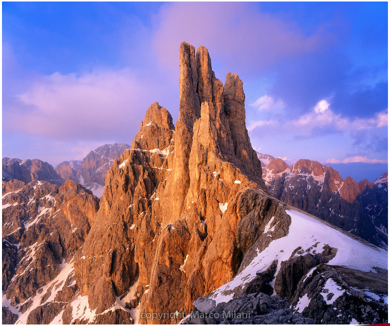
Let’s make some more AD...

Dolomites Research Week on Approximation (DRWA15)

Alba di Canazei (Italy), September 5-8, 2015

- topic: Approximation Theory and applications.
- Workshop/small conference, each year.
- Larger conference each 4 or 5 years (next in 2016!)

info at: http://events.math.unipd.it/drwa15/,or contact me by email!



DRNA

Attached to the conference there is the DRNA journal

Dolomites Research Notes on Approximation

- Peer-reviewed journal on Approximation Theory (in a broad sense) and Numerical Analysis.
- We managed to keep the journal online and, more important, free!
- The core of the journal consists of research papers, few surveys appear as well.

First Definitions and Examples

Asymptotic growth assumption on ratios of uniform and L^p_μ norms.

BMP definition

Let $E \subset \mathbb{C}$ be a compact set and μ be a Borel finite measure such that $\operatorname{supp}\mu \subseteq E$, assume that

$$\overline{\lim}_k \left(\frac{\|p_k\|_E}{\|p_k\|_{L^2_\mu}} \right)^{1/\deg(p_k)} \leq 1,$$

for any sequence of non zero polynomials $\{p_k\}$. Then we say that (E, μ) has the **Bernstein Markov property**, BMP for short, or equivalently μ is a Bernstein Markov measure on E.

Other instances of BMP

Instead of polynomials one can consider

- sequences of weighted polynomials, $e^{-\deg p_k Q} p_k$ for admissible lsc Q.
- rational functions p_k/q_k, $\max\{\deg p_k, \deg q_k\} \leq k$ with restricted poles,e.g., $\overline{\cup_k Z(q_k)} \subseteq P$, where $P \cap E = \emptyset$.

we refer to such properties as weighted BMP and rational BMP respectively.

Terminology

- it is due to Siciak, Berman and Boucksom, this name is mostly used in scv context.
- The name has been chosen (probably) because one can provide examples by using classical polynomial inequalities.
- The definition is very close to the class of measure with regular asymptotic behaviour of Stahl and Totik. For E regular w.r.t. the Dirichlet problem the two classes coincide.
- Historically it is a *very Hungarian* topic: Erdős, Szegő, Totik...

Let $\mathscr{P}_\mu^k$ be the Reproducing Kernel Hilbert Space of of polynomials of degree at most k endowed with the scalar product of L^2_μ. Let $\{q_j\}$ be its orthonormal basis (ordered by increasing degree j) then the reproducing kernel is

$$K_k^\mu(z,\zeta) := \sum_{j=0}^k q_j(z)\bar{q}_j(\zeta)\ , \quad \text{notice that } p(z) = \langle K_k^\mu(z,\cdot), p(\cdot)\rangle_{L^2_\mu}.$$

The Bergman Function of $\mathscr{P}_\mu^k$ is

$$B_k^\mu(z) := K_k^\mu(z,z) = \sum_{j=0}^k |q_j(z)|^2.$$

The Bergman Function II

Bergman Function and BMP

By Parseval Inequality we have

$$B_k^\mu(z) = \sup_{p \in \mathscr{P}^k} \frac{|p(z)|^2}{\|p\|_{L^2_\mu}^2}.$$

Hence (E, μ) has the BMP iff

$$\overline{\lim}_k \|B_k^\mu\|_E^{1/2k} = 1.$$

Example in the disk

Let $E = \{z : |z| \leq 1\}$ and μ the normalized arclength measure on ∂E. Then $q_j(z) = z^j$, $j = 1, 2, \ldots, k$. We can compute the Bergman function explicitly.

$$B_k^\mu(z) = \sum_{j=0}^{k} |z|^{2j} = \begin{cases} \frac{1-|z|^{2k-2}}{1-|z|^2}, & |z| \neq 1 \\ k+1, & |z| = 1 \end{cases}.$$

We have

$$\overline{\lim}_k \|B_k^\mu\|_E^{1/2k} = \overline{\lim}_k (k+1)^{1/2k} = 1,$$

thus

$$\boxed{(E, \mu) \text{ has the BMP.}}$$

Proposition

Let $E \subset \mathbb{C}$ be any compact set, then there exists a measure μ such that

1. $\operatorname{supp} \mu \subseteq E$.
2. μ has a countable carrier.
3. (E, μ) has the BMP.

Sketch of the Proof

Pick any sequence of Fekete arrays $\{(z_0^{(k)}, \dots, z_k^{(k)})\}_{k \in \mathbb{N}}$ for E and set

$$\mu_k := \frac{1}{\dim \mathscr{P}^k(E)} \sum_{j=0}^{k} \delta_{z_j^{(k)}}, \quad \mu := \sum_{k=1}^{\infty} \frac{\mu_k}{k^2}.$$

Conclude by interpolation at Fekete points...

Motivations and Properties

from different fields/points of view

the study of BMP is motivated by

1. Approximation Theory (Bernstein Walsh type theorems).
2. (pluri-) Potential Theory (recovering of quantities by L^2 methods).
3. Statistics and probability applications (random polynomials/matrices, large deviation principles).

Motivations from Approximation Theory

Behaviour of Least squares projection

Upper bound on diagonal of *reproducing kernel* of $\left(\mathscr{P}^k, \langle\cdot,\cdot\rangle_{L^2_\mu}\right)$ gives good behaviour of uniform polynomial approximation by L^2_μ projection

$$\mathcal{C}(E) \subset L^2_\mu \ \ni \ f \to \mathcal{L}^\mu_k[f] := \sum_{j=0}^{k} \langle f, q_j\rangle q_j(z) \ \in \ \mathscr{P}^k.$$

For bounded f we have

$$\|\mathcal{L}^\mu_k[f]\|_E \leq \left(\sum_{j=0}^{k} |\langle f, q_j\rangle|^2\right)^{1/2} \left\|\left(\sum_{j=0}^{k} |q_j(z)|^2\right)^{1/2}\right\|_E$$

$$\leq \|f\|_{L^2_\mu} \sqrt{\|B^\mu_k(z)\|_E} \leq \|f\|_E \sqrt{\mu(E)\|B^\mu_k(z)\|_E}.$$

thus (taking p_k the best unif. norm approx)

$$\|f - \mathcal{L}^\mu_k[f]\|_E \leq \|f - p_k - \mathcal{L}^\mu_k[f - p_k]\|_E \leq d_k(f, E)\left(1 + \sqrt{\mu(E)\|B^\mu_k\|_E}\right).$$

classical Bernstein Walsh Lemma

Recall that (not by definition)

$$g_E(z) = \overline{\lim}_{\zeta \to z} \left(\left\{ \frac{1}{\deg p} \log^+ |p(\zeta)|, \|p\|_E \leq 1 \right\} \right).$$

Bernstein Walsh results

Let E be a compact non polar set, then we have

$$|p(z)| \leq \|p\|_E \exp(\deg p \, g_E(z)) \quad \forall p \in \mathscr{P}(\mathbb{C}).$$

(Bernstein Walsh Ineq.)

If $f : E \to \mathbb{C}$ is any continuous function and E is polynomially convex, we set $d_k(f, E) := \inf\{\|f - p\|_E : p \in \mathscr{P}^k\}$, then for any real number $R > 1$ the following are equivalent

1. $\lim_k d_k(f, E)^{1/k} < 1/R$
2. f is the restriction to E of $\tilde{f} \in hol(D_R)$, where $D_R := \{g_E < \log R\}$.

L^2-Bernstein Walsh type lemma

If (E, μ) has the BMP, then

$$d_{k,\mu}(f)^{1/k} := \inf_{p \in \mathscr{P}^k} \|f - p\|_{L^2_\mu}^{1/k}$$

has the same asymptotic of $d_k(f, E)^{1/k}$, therefore

L^2 Bernstein Walsh Lemma

Let E be a regular compact polynomially convex subset of $\mathbb{C}$, μ a positive finite Borel measure such that $\operatorname{supp} \mu = E$ and $f \in \mathcal{C}(E)$. Then for any $R > 1$ the following are equivalent.

1. f is the restriction to K of $\tilde{f} \in hol(D_R)$.
2. $\overline{\lim}_k d_{k,\mu}(f)^{1/k} \leq 1/R$.
3. $\overline{\lim}_k \|f - \mathcal{L}_k^\mu[f]\|_E^{1/k} \leq 1/R$.

Theorem

Let K be a compact regular subset of $\mathbb{C}$, let $f \in \mathscr{C}(K)$ and let $r > 1$. The following are equivalent.

i) There exists $\tilde{f} \in \mathscr{M}_n(D_r)$ such that $\tilde{f}|_K \equiv f$.

ii) $\overline{\lim}_k d_{k,n}^{1/k}(f, K) \leq 1/r$.

iii) For any finite Borel measure μ such that $\operatorname{supp}\mu = K$ and (K, μ, P) has the rational Bernstein Markov property for any compact set P such that $P \cap K = \emptyset$, denoting by $r_{k,n}^{\mu}$ a best L^2_μ approximation to f in $\mathcal{R}_{k,n}$, one has

$$\overline{\lim}_k \left(\|f - r_{k,n}^{\mu}\|_K\right)^{1/k} \leq 1/r,$$

provided that $\overline{\{\text{Poles}(r_{k,n})\}_k} \cap K = \emptyset$.

iv) With the same hypothesis and notations as in *iii)* we have

$$\overline{\lim}_k \left(\|f - r_{k,n}^{\mu}\|_{L^2_\mu}\right)^{1/k} \leq 1/r,$$

Motivations from Potential Theory

We already got a tasty bite of applications in Potential Theory in the Danka's talk on the asymptotic of orthogonal polynomials...

let's see other possible applications/motivations.

Back to the disk example I

Let $E = \{z : |z| \leq 1\}$ and μ the arclength measure on ∂E. Then $q_j(z) = z^j$, $j = 1, 2, \dots, k$. We can compute the Bergman function explicitly.

$$B_k^\mu(z) = \sum_{j=0}^{k} |z|^{2j} = \begin{cases} \frac{1-|z|^{2k-2}}{1-|z|^2}, & |z| \neq 1 \\ k+1, & |z| = 1 \end{cases}.$$

Hence we have

$$\overline{\lim}_k \|B_k^\mu\|_E^{1/2k} = \overline{\lim}_k (k+1)^{1/2k} = 1,$$

$$\lim_k \log(B_k^\mu(z))^{1/2k} = \begin{cases} \lim_k \log\left(\frac{1-|z|^{2k-2}}{1-|z|^2}\right)^{1/2k}, & |z| \neq 1 \\ \lim_k \log(k+1)^{1/2k}, & |z| = 1 \end{cases} = \log^+ |z|.$$

We can notice that

Back to the disk example II

1. (E, μ) has the BMP.
2. $\log^+ |z|$ is the Green function for $\mathbb{C} \setminus E$ with log pole at ∞

Furthermore, we have that

3. $\frac{B_k^\mu(z)}{\dim \mathscr{P}^k}$ is bounded by one on E for any k.
4. Precisely, $B_k^\mu(z) = k + 1 = \dim \mathscr{P}^k$ μ-almost everywhere.
5. We can compute the Gram determinant $G_k^\mu(\mathscr{B}_k)$ w.r.t. the standard monomial basis $\mathscr{B}_k$, we get $G_k^\mu(\mathscr{B}_k) = 1$ and in particular

$$\lim_k G_k^\mu(\mathscr{B}_k)^{\frac{1}{2 \dim \mathscr{P}^k}} = 1 = \delta(E).$$

Here $\delta(E)$ is the transfinite diameter of E.

We are somehow cheating: μ is a very special choice, it is the equilibrium measure of E.
Can we recover results of this type for just a BM measure?

Theorem (k-th root Bergman asymptotic)

Let E be a regular compact subset of $\mathbb{C}$ and μ a positive finite Borel measure supported on it such that (E,μ) has the Bernstein Markov property. Then

$$\lim_k \frac{1}{2k} \log B_k^\mu(z) = g_K(z) \text{ locally uniformly.}$$

Therefore

$$\Delta\left(\frac{1}{2k} \log B_k^\mu(z)\right) \to^* \mu_E.$$

Sketch of the proof.

The second statement follows by the first. For the first, one can prove that

$$(\Phi_{E,k}(z))^2 \leq B_k^{\mu}(z) \leq \|B_k^{\mu}\|_E^2 (\Phi_{E,k}(z))^2,$$

where $\Phi_{E,k}(z) := \sup\{|p(z)| : \|p\|_E \leq 1, p \in \mathscr{P}^k\}$, using the reproducing property of K_k^{μ} and the extremal property of B_k^{μ}. Then one uses the asymptotic property of the Siciak function $\Phi_{E,k}(z)^{1/k} \to e^{g_E(z)}$ and that by the BMP property $\|B_k^{\mu}\|_E^{1/2k} \to 1$.

Bergman asymptotic

Theorem (Bergman asymptotic)

Let E be a compact regular subset of $\mathbb{C}$ and μ a positive finite Borel measure with $\operatorname{supp}\mu = E$ such that (E, μ) has the BMP. Then one has

$$\frac{B_k^\mu}{k+1}\mu \to^* \mu_E$$

In general much more is true

Theorem (weighted Bergman asymptotic)

Let E be a compact regular subset of $\mathbb{C}$ and μ a positive finite Borel measure with $\operatorname{supp}\mu = E$ such that for a given admissible weight Q the triple (E, μ, Q) has the weighted BMP. Then one has

$$\frac{B_k^{\mu,Q}}{k+1}\mu \to^* \mu_{E,Q}, \text{ where } B_k^{\mu,Q} := \sum_{j=0}^{k} |q_j(z)|^2 e^{-2Q(z)}.$$

Sketch of the proof I

Let us introduce/recall the quantities.

$$G_k^\mu := \det[\langle z^i \bar{z}^j \rangle_{L^2_\mu}]_{i,j=0,\dots,k},$$
$$Z_k^\mu := \int_{E^{k+1}} |\,\mathrm{VDM}(z_0,\dots,z_k)|^2 d\mu(z_0)\dots d\mu(z_k).$$

- By Gram Shmidt orthogonalization one can prove that

$$Z_k^\mu = (k+1)!\, G_k^\mu,$$
$$B_k^\mu(z) = \frac{k+1}{Z_k^\mu} \int_{E^k} |\,\mathrm{VDM}(z,\dots,z_k)|^2 d\mu(z_1)\dots d\mu(z_k).$$

Sketch of the proof II

- Using ($k+1$ times) the BMP one can compare $(Z_k^\mu)^{1/(k(k+1))}$ with the k-th order diameter of E and get

$$\lim_k (Z_k^\mu)^{1/(k(k+1))} = \delta(E).$$

Measures having such a property are termed asymptotically Fekete measures.

- Now introduce a probability on E^{k+1} setting

$$\operatorname{Prob}_k(A) := \frac{1}{Z_k^\mu}\int_A |\operatorname{VDM}(z_0,\dots,z_k)|^2 d\mu(z_0)\dots d\mu(z_k).$$

Then extend it to a probability Prob on sequences in E by taking the product space.

Sketch of the proof III

- One has a Johansson-type result. Let

  $A_{k,\eta} := \{(z_0,\dots,z_k) \in E^{k+1} : |\,\mathrm{VDM}(z_0,\dots,z_k)|^2 \geq (\delta(E) - \eta)^{(k+1)^2}\}$,

  then $\forall \eta > 0$ there exists k_η such that $\forall k > k_\eta$ we have

$$\underset{k}{\mathrm{Prob}}(E^{k+1} \setminus A_{k,\eta}) \leq \left(1 - \frac{\eta}{2\delta((E)}\right)^{(k+1)^2}$$

Sketch of the proof IV

- To prove weak* convergence fix a continuous function φ; it follows by the above equations and by symmetry that

$$
\begin{aligned}
&\int_E \varphi(z)\frac{B_k^\mu(z)}{k+1}d\mu(z)\\
=&\frac{1}{Z_k^\mu}\int_{E^{k+1}}\varphi(z)|\operatorname{VDM}(z,\dots,z_k)|^2 d\mu(z_1)\dots d\mu(z_k)d\mu(z)\\
=&\sum_{j=0}^{k}\frac{1}{k+1}\int_{E^{k+1}}\varphi(z_j)\frac{|\operatorname{VDM}(z,\dots,z_k)|^2}{Z_k^\mu}d\mu(z_0)\dots d\mu(z_k)\\
=&\int_{E^{k+1}}\sum_{j=0}^{k}\frac{\varphi(z_j)}{k+1}\operatorname*{Prob}_k(z_0,\dots,z_k).
\end{aligned}
$$

- the above integral can be divided in two parts: one over A_{k_j,η_j} and the other on $E^{k+1}\setminus A_{k_j,\eta_j}$ for suitable choice of $\eta_j \to 0$ and $k_j > k_{\eta_j}$.

Sketch of the proof V

- Finally one combines the Johansson type result with the fact that asymptotically Fekete points tends weak* to μ_E : notice that sequence of arrays lying in A_{k_j,η_j} are asymptotically Fekete.

Remark

We used just

1. Asymptotically Fekete points converge to μ_E.
2. Free-energy asymptotic $\lim_k (Z_k^\mu)^{1/(k(k+1))} = \delta(E)$.

We will compare this with its several variables counterpart in lecture #2. . .

Open problem

We saw that

$$\boxed{\text{BMP} \Rightarrow Z_k^\mu\text{-asymptotic} \Rightarrow \text{Bergman asymptotic,}}$$

what about the converse implications?

Bloom proved that the first arrow can be replaced by iff, in one complex variable and in the un-weighted case.

Applications Motivations from Probability and statistics

Example (Kac) I

We consider families of random polynomials of the form

$$p_a(z) := \sum_{j=0}^{k} a_j z^j,$$

where for each k the random coefficients are normal $(0,1)$ complex variables

$$a := (a_0, \ldots, a_k) \sim e^{-\sum_{j=0}^{k} |a_j|^2} dm(a_0) \ldots dm(a_k)$$

We define the random measure $Z_a := \frac{1}{k+1} \sum_{\zeta \text{ zero of } p_a} \delta_\zeta$ and for sequences $\{a^{(k)}\}$, $a^{(k)} = (a_0^{(k)}, \ldots, a_k^{(k)})$, we introduce

$$\langle \mathbb{E}(Z_{a^{(k)}}), \varphi \rangle := \int \int \varphi(z) dZ_{a^{(k)}} e^{-\sum_{j=0}^{k} |a_j|^2} dm(a_0) \ldots dm(a_k) \quad \forall \varphi \in \mathcal{C}_c(\mathbb{C}).$$

Example (Kac) II

which is the asymptotic of

1. $\mathbb{E}(Z_{a^{(k)}})$ and
2. $\frac{1}{k+1} \log |p_{a^{(k)}}|$?

It turns out that

1. $\mathbb{E}(Z_{a^{(k)}}) \to^* \mu_{\mathbb{S}^1}$ and
2. $\frac{1}{k+1} \log |p_{a^{(k)}}| \to g_{\mathbb{S}^1}$ a.s. with respect to the probability induced by $e^{-|a|^2} dm(a)$ on the space of sequences.

Why $\mathbb{S}^1$?

- The monomial basis z^j is the orthonormal basis w.r.t. ds,
- $(\mathbb{S}^1, ds)$ has the BMP

(more) General situation

Theorem (random polynomial asymptotic)

Let $\{a^{(k)}\}$ be a sequence of i.i.d. $(0,1)$ Normal random variables, μ a finite positive Borel measure having regular compact support $E \subset \mathbb{C}$ and such that (E,μ) has the BMP. Let $\{q_j\}_{j=0,\dots,k}$ be the orthonormal basis for $\mathscr{P}_\mu^k$ and set $p_{a^{(k)}}(z) := \sum_{j=0}^k a_j^{(k)} q_j(z)$. Then the following holds.

1. $\mathbb{E}(Z_{a^{(k)}}) \to^* \mu_E$ and
2. $\left(\frac{1}{k+1}\log|p_{a^{(k)}}|\right)^* \to g_E$ in $L^1_{\text{loc}}(\mathbb{C})$, a.s. with respect to the probability induced by $e^{-|a|^2}dm(a)$ on the space of sequences.

Further generalizations are possible (e.g., more general probabilities), see Bloom-Levenberg and Shiffman.

- Use $Z_{a^{(k)}} = \Delta(\frac{1}{k+1}\log|p_{a^{(k)}}|)$ and integrate by parts.
- Normalize the basis by a factor $\sqrt{B^k_\mu(z)}$ using $\|(q_0,\dots,q_k)(z)\|^2 = B^k_\mu(z)$,
- exchange integration order by Fubini to take the integration "over sequences" in the inner integral defining $\langle \mathbb{E}(Z_{a^{(k)}}), \varphi\rangle$

The integration is cut in two pieces because

$$\frac{1}{k+1}\log|p_{a^{(k)}}| =$$
$$= \frac{1}{2(k+1)}\log B^\mu_k(z) + \frac{1}{k+1}\log\left|\left\langle \frac{(1, q_j(z), \dots, q_k(z))}{(B^\mu_k(z))^{1/2}}, (a^{(k)}_0, \dots, a^{(k)}_k)\right\rangle\right|$$

Idea of the proof of 1 II

- on one piece we use the BMP property of μ to compare $\frac{1}{k+1}\log B_\mu^k$ with $\Phi_{k,E}$ and get the convergence to g_E, then we unwind the argument to use $\Delta(\frac{1}{k+1}\log B_\mu^k) \to \Delta g_E = \mu_E$.
- The other piece is treated using the i.i.d. assumption on the normal variables and rotational invariance of the product space to show that

$$\int \log \left| \left\langle \frac{(1, q_j(z), \dots, q_k(z))}{(B_k^\mu(z))^{1/2}}, (a_0^{(k)}, \dots, a_k^{(k)}) \right\rangle \right| \mathrm{Prob}_k = const$$

The proof of 2 involves

- Borel Cantelli Lemma
- Hartog's Lemma on subharmonic functions
- Dominated Convergence Theorem.

Prob. re-interpretation of Bergman As.

- $(z_0^{(k)},\ldots,z_k^{(k)}) \rightsquigarrow \frac{1}{k+1}\sum_{j=0}^{k}\delta_{z_j^{(k)}} =: \mu_{\boldsymbol{z}^{(k)}}$.
- $\mathrm{Prob}_k(A) := \int_A |\mathrm{VDM}(z_0^{(k)},\ldots,z_k^{(k)})|^2 d\mu(z_0^{(k)})\ldots d\mu(z_k^{(k)})$ and extend it to a probability Prob on sequences of arrays in E by taking the product measure space.
- By Johansson estimate Prob-a.e. sequence is asymptotically Fekete, thus Prob-a.e. $\mu_k \to^* \mu_E$.
- Given any test function φ define $f_k(\boldsymbol{z}^{(k)}) := \int \varphi d\mu_{\boldsymbol{z}^{(k)}}$ and extend it to a (uniformly bounded) sequence of functions F_k on sequences of arrays.
- Use the a.s. convergence $\mu_{\boldsymbol{z}^{(k)}} \to^* \mu_E$ and Dominated Convergence Theorem to get $\mathbb{E}(\mu_{\boldsymbol{z}^{(k)}}) \to^* \mu_E$.
- we saw that this is equivalent to Bergman asymptotic.

Sufficient conditions

Classical mass-density

The most powerful tool for proving BMP is the following theorem due to Stahl and Totik; Λ^* condition.

Mass density sufficient condition for BMP

Let $K \subset \mathbb{C}$ be a regular compact set and μ a finite Borel measure with $\operatorname{supp}\mu = K$, the following condition is sufficient for (μ, K) to have the BM property. There exists $t > 0$ such that

$$\mathrm{Cap}(K) = \lim_{r \to 0^+} \mathrm{Cap}\left(\{z \in K : \mu(B(z,r)) \geq r^t\}\right).$$

Sketch of the proof

The main elements/tools are

- If $\mathrm{Cap}(K_j) \to \mathrm{Cap}(K)$ for $K_j \subset K$, then $g_{K_j} \to g_K$ locally uniformly.
- Bernstein Walsh inequality gives an upper bound for $|p(z)|$ in term of $\|p\|_{K_j}$ and $g_{K_j}(z)$.
- Cauchy integral formula to give a lower bound to $|p(\zeta)|$ for $\zeta \in B(z, r)$, where $\|p\|_{K_j} = |p(z)|$.
- mass density to compare L^2_μ norms with $|p(z)|$.

Extensions

The mass-density condition has been used to prove other instances of the BMP, for instance

1. weighted polynomials
2. rational functions with restricted poles
3. Müntz polynomials
4. multivariate polynomials (in $\mathbb{C}^n$); see the next lecture (if you do not fed up with this)

Rational BMP

Definition

Let P be a compact set, $P \cap E = \emptyset$ and set

$$\mathcal{R}(P) := \Big\{ \{p_k/q_k\} \,:\, p_k, q_k \in \mathscr{P}^k, Z(q_k) \subseteq P \;\forall k \in \mathbb{N} \Big\}.$$

The triple (E, μ, P), where μ is a positive finite Borel measure μ supported on E has the rational BMP if

$$\overline{\lim}_k \left(\frac{\|r_k\|_E}{\|r_k\|_{L^2_\mu}} \right)^{1/k} \leq 1 \;\; \forall \{r_k\} \in \mathcal{R}(P),$$

Mass density for rational BMP

Theorem (Mass-density I)

Let $K \subset \mathbb{C}$ be a compact regular set and $P \subset \Omega_E$ be compact. Let $\mu \in \mathcal{M}^+(E)$, $\operatorname{supp}\mu = E$ and suppose that there exists $t > 0$ such that

$$\lim_{r \to 0^+} \operatorname{Cap}\left(\{z \in E : \mu(B(z,r)) \geq r^t\}\right) = \operatorname{Cap}(E). \tag{1}$$

Then (E, μ, P) has the rational Bernstein Markov Property.

Sketch of the proof

- If $\operatorname{Cap}(E_j) \to \operatorname{Cap}(E)$ then $g_{E_j}(z,a) \to g_E(z,a)$ locally uniformly in $z \in \mathbb{C}$ and uniformly in $a \in P$.
- Bernstein Walsh Lemma for rational functions,
- the same overall technique of the polynomial case.

The case $P \cap \hat{E} \neq \emptyset$

If $P \cap \hat{E} \neq \emptyset$ we can not use this theorem, but we can build a suitable conformal mapping...

Lemma

Let $E, P \subset \mathbb{C}$ be compact sets, where $E \cap \hat{P} = \emptyset$. Then there exist $w_1, w_2, \ldots, w_m \in \mathbb{C} \setminus (E \cup \hat{P})$ and $R_2 > R_1 > 0$ such that denoting by f the function $z \mapsto \frac{1}{\prod_{j=1}^{m}(z-w_j)}$ we have

$$E \subset\subset \{|f| < R_1\}, \quad P \subset\subset \{R_1 < |f| < R_2\}.$$

The proof is based on properties of Fekete points.

The case $P \cap \hat{E} \neq \emptyset$

Theorem (Mass- density II)

Let $E, P \subset \mathbb{C}$ be compact disjoint sets where E is regular with respect to the Dirichlet problem and $\hat{P} \cap E = \emptyset$. Let $\mu \in \mathcal{M}^+(E)$ be such that $\operatorname{supp} \mu = E$ and suppose that there exist $t > 0$ and f as in the lemma above such that the following holds

$$\lim_{r \to 0^+} \operatorname{Cap}\Big(\{z \in f(E) : f_*\mu(B(z,r)) \geq r^t\}\Big) = \operatorname{Cap}(f(E)).$$

Then (E, μ, P) has the rational Bernstein Markov Property.

Weighted polynomials

We say that $[E, \mu, Q]$ has the weighted BMP if for any sequence of polynomilas $\{p_k\}$ one has

$$\overline{\lim}_k \left(\frac{\|p_k e^{-\deg p_k Q}\|_E}{\|p_k e^{-\deg p_k Q}\|_{L^2_\mu}} \right)^{1/\deg p_k} \leq 1.$$

Mass-density for weighted BMP

Let μ be a positive finite Borel measure and $E := \operatorname{supp} \mu$ be a compact regular set. Suppose that there exists $T > 0$ such that

$$\lim_{r \to 0^+} \operatorname{Cap}\left(\{z \in E : \mu(B(z,r)) \geq r^t\}\right) = \operatorname{Cap}(E).$$

Then $[E, \mu, Q]$ has the weighted BMP for any continuous weight Q.

The proof is a modification of the un-weighted one.

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Thank You!