# On the limit zero distribution of type I Hermite–Padé polynomials

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# Introduction

We are interested in studying type I Hermite-Padé approximants.
Since they are a generalization of classical Padé approximants,
we start from the basic definition of Padé approximants.

Let $f$ be a power series,

$$f(z) = \sum_{i=0}^{\infty} c_i z^i.$$

Let $P, Q$ be polynomials of degrees at most $n, m$, respectively. Then the Padé approximant of $f$ of order $(n, m)$ is given by [Perron 1957]

$$Qf - P = \mathcal{O}(z^{n+m+1}).$$

The polynomials $P, Q$ are not unique, but their ratio $P/Q$ is uniquely determined. We say that $[n/m]_f := P/Q$ is the Padé approximant of $f$.

# General concept of Hermite-Padé

Let $f_1, f_2, \dots, f_k$ be $k$ different functions, given by power series at the point $z = 0$. Let $A_1, A_2, \dots, A_k$ be polynomials of degrees at most $\nu_1, \nu_2, \dots, \nu_k$. The type I Hermite-Padé approximant of $f_1, f_2, \dots, f_k$ is given by [Baker, Graves-Morris 1996]

$$A_1 f_1 + A_2 f_2 + \ldots + A_k f_k = \mathcal{O}(z^{\nu_1+\nu_2+\ldots+\nu_k+k-1}).$$

Apparently, the Padé approximant of $f$ is a Hermite-Padé approximant of $f_1 = -1$, $f_2 = f$ of order $\nu_1 = n$, $\nu_2 = m$, and $k = 2$.

There are other types of Hermite-Padé approximants, like type II, two-point, and others, but we discuss only type I.

# Padé approximant at the infinity point

Let $f$ be a convergent power series, i.e. a germ (single-valued branch of a multivalued analytic function given at the point $z = \infty$),

$$f = \sum_{k=0}^{\infty} \frac{c_k}{z^{k+1}}.$$

Now let $P, Q$ be the Padé polynomials at the infinity point of $f$ of degree at most $n$ such that [Nuttall 1984]

$$Qf - P = \mathcal{O}\left(\frac{1}{z^{n+1}}\right), \quad z \to \infty.$$

The ratio $[n/n]_f := P/Q$ is the diagonal Padé approximant of $f$ of order $n$ at the infinity point.

# Markov-type functions

Let $\Delta = [-1, 1]$ and $D = \overline{\mathbb{C}} \setminus \Delta$.
Let $\rho(x)$ be a weight function, holomorphic on $\Delta$ and without zeros on $\Delta$.
Define the Markov function $\widehat{\rho}(z)$ [Markov 1895], [Bernstein 1937], [GonSu 2004],

$$\widehat{\rho}(z) = \frac{1}{\pi} \int_{\Delta} \frac{\rho(x)}{z - x} \frac{dx}{\sqrt{1 - x^2}}.$$

Also, let $r(z)$ be a complex rational function, with poles in $D$ and $r(\infty) = 0$. Define the class of functions of Markov-type:

$$f = \widehat{\rho} + r.$$

# Convergence of Padé approximants

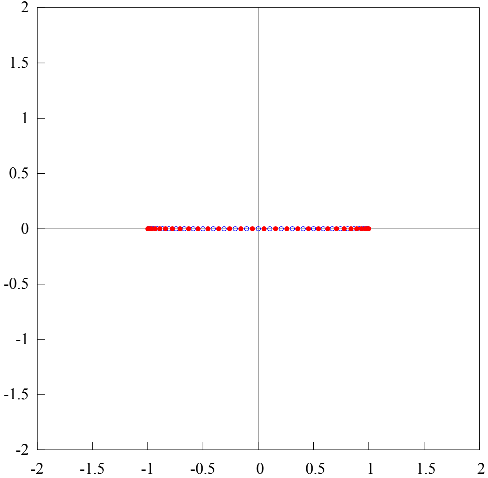
## Theorem (Gonchar, Suetin [GonSu 2004])

*Let $f = \widehat{\rho} + r$ be a Markov-type function, where $\rho(x)$ is holomorphic on $\Delta$ and without zeros on $\Delta$, all poles of $r$ are in $D$ and $r(\infty) = 0$. Then the sequence of diagonal Padé approximants $[n/n]_f$ converges uniformly in the spherical metric to $f$ inside $D$.*

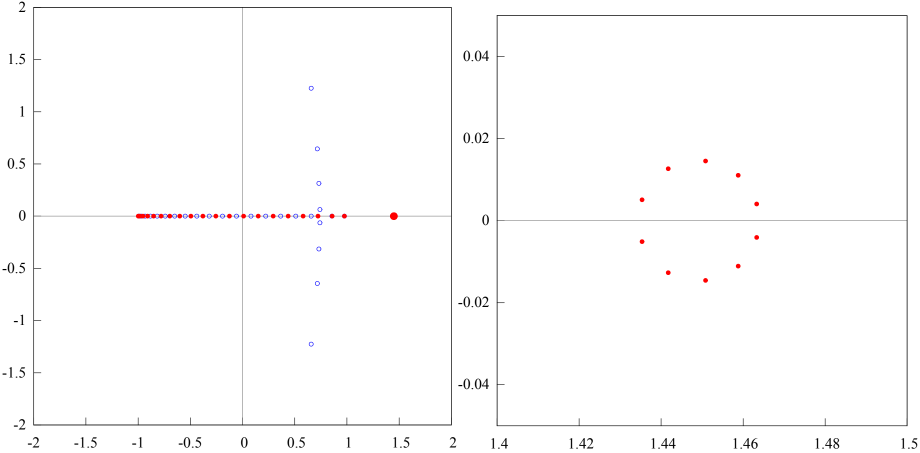
From the theorem follows that each pole of the function $f$ attracts as many poles of $[n/n]_f$ as its multiplicity. In addition all poles of $[n/n]_f$ are simple ones (this result was obtained numerically, and then mathematically proved).

The theorem was extended for a larger class of functions, meromorphic, Dini-Lipschitz, and others.

The article by Gonchar and Suetin [GonSu 2004] is the basis for our research.




Zeros (blue points) and poles (red points) of the diagonal Padé approximant $[30/30]_f$ of the function $f(z) = 1/\sqrt{z^2 - 1}$.




Zeros (blue points) and poles (red points) of the diagonal Padé approximant $[30/30]_f$ of the function $f(z) = 1/\sqrt{z^2-1} + z^{10}/(z-1.45)^{10}$.

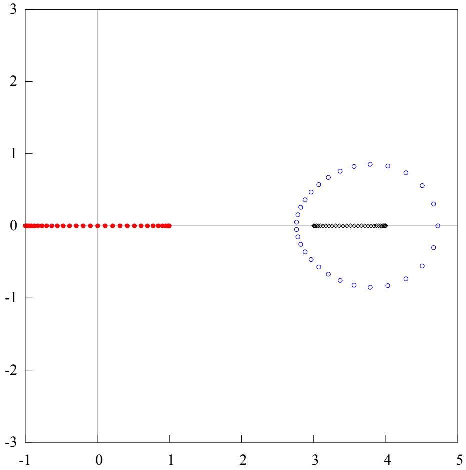
# Hermite-Padé approximant at the infinity point

We are interested in the general problem of the limit zero distribution of type I Hermite–Padé polynomials for the collection of three functions $[1, f, g]$.

Suppose that $f$ and $g$ are two convergent power series at the infinity point. The type I Hermite-Padé polynomials $P_1, P_2, P_3$ of degree at most $n$ are given by [Baker, Graves-Morris 1996], [Aptekarev 2008]

$$P_1 + P_2 f + P_3 g = \mathcal{O}\left(\frac{1}{z^{2n+2}}\right), \quad z \to \infty.$$

Comparing to the general definition, we have Hermite-Padé approximant of $f_1 = 1$, $f_2 = f$, $f_3 = g$ of order $\nu_1 = \nu_2 = \nu_3 = n$, and $k = 3$.




Zeros of $P_1$ (blue points), $P_2$ (red points) and $P_3$ (black points) of the Hermite-Padé approximant $[1, f, g]$ with $n = 30$ of the two functions $f(z) = 1/\sqrt{z^2 - 1}$ and $g(z) = 1/\sqrt{(z-3)(z-4)}$.

# Our contributions

We present only graphics about the type I Hermite-Padé approximants for the collection of three functions $[1, f, g]$, and some ideas based on these graphics. We are still working on the general theory.

There is software for computing Padé approximants (Maple, Mathematica), but there was not present a reliable software for computing Hermite-Padé approximants.

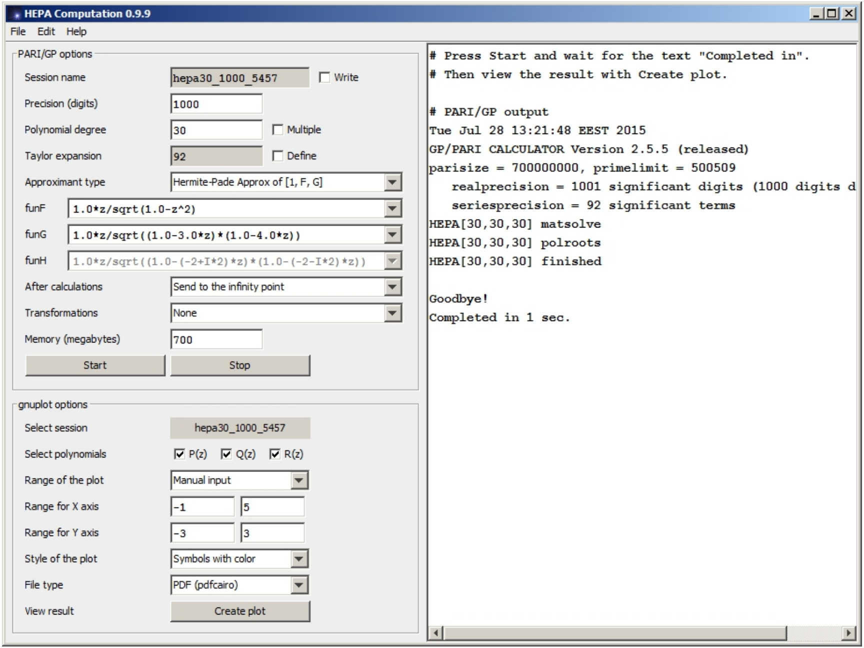
In September 2012 we started working on that task, and in December 2013 we had our own software for computing Hermite-Padé approximants for the collection of three functions $[1, f, g]$ with the resulting polynomials having the same degree $n$ (the equivalent of diagonal Padé approximant).

# Papers in arXiv

By using this software, we presented our conjectures in two papers, published in arXiv during 2015, [KovIkSu 2015a], [KovIkSu 2015b].

We are using `PARI/GP` computer algebra for the actual computations, and `gnuplot` for the plotting of the points.

This project was realised under a bilateral agreement between the Steklov Mathematical Institute (RAS, Moscow) and the Institute of Mathematics and Informatics (BAS, Sofia).


HEPA Computation 0.9.9
File Edit Help
PARI/GP options
Session name
hepa30_1000_5457
Write
Precision (digits)
1000
Polynomial degree
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Multiple
Taylor expansion
92
Define
Approximant type
Hermite-Pade Approx of [1, F, G]
funF
1.0*z/sqrt(1.0-z^2)
funG
1.0*z/sqrt((1.0-3.0*z)*(1.0-4.0*z))
funH
1.0*z/sqrt((1.0-(-2+I*2)*z)*(1.0-(-2-I*2)*z))
After calculations
Send to the infinity point
Transformations
None
Memory (megabytes)
700
Start
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gnuplot options
Select session
hepa30_1000_5457
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P(z) Q(z) R(z)
Range of the plot
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Range for X axis
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File type
PDF (pdfcairo)
View result
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# Press Start and wait for the text "Completed in".
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# PARI/GP output
Tue Jul 28 13:21:48 EEST 2015
GP/PARI CALCULATOR Version 2.5.5 (released)
parisize = 700000000, primelimit = 500509
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HEPA[30,30,30] matsolve
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N. R. Ikonomov, R. K. Kovacheva, S. P. Suetin, *On the limit zero distribution of type I Hermite–Padé polynomials*, 2015, 67 pp, arXiv:1506.08031.

# Asymptotics of Padé approximants

We present the general theory of asymptotics of classical Padé approximants.

First, we cite the general Stahl Theory about weak asymptotics of Padé approximants, second, we present a conjecture for strong asymptotics of Padé approximants.

After that, we focus on Hermite-Padé approximants.

# General results for Padé approximants

The basic general results of Padé approximants theory are of two types.

The first type is concerned with the limit zero distribution of Padé polynomials for multivalued analytic functions with a finite set of branch points on the Riemann sphere $\overline{\mathbb{C}}$. These results are referred to as "weak asymptotics of Padé approximants". The problem of first type was completely solved by H. Stahl for classical Padé approximants.

The second type of results are devoted to the "strong asymptotics of Padé approximants", which is a generalization of the classical Bernstein–Szegő asymptotic theory for polynomials, orthogonal on the unit segment $\Delta := [-1, 1]$ and related to the general case of non-Hermitian orthogonal polynomials. This problem is still open, and we present a conjecture on it.

# Stahl Theory

The Stahl Theorem gives a complete answer to the problem of limit zero-pole distribution of the classical Padé appoximants to $f$. Stahl Theorem is quite general, it admits multivalued functions with a finite number of branch points on the Riemann sphere, and also multivalued analytic functions with a singular set of zero (logarithmic) capacity.

The most important part of Stahl theory [Stahl 1985ab]–[Stahl 1986ab] is the existence of a unique (up to a compact set of zero capacity) maximal domain for the given multivalued function $f$ (at the point $z = \infty$). This "maximal domain" of holomorphy of $f$ is a domain $D = D(f) \ni \infty$, such that the given germ $f$ can be continued as holomorphic function from a neighborhood of the infinity point $z = \infty$ to $D$ (the function $f$ can be continued analytically along each path that belongs in to $D$).

# Maximal domain

Maximal domain in the sense that $\partial D$ is of "minimal capacity" among all compact sets $\partial G$, such that $G$ is a domain, $\infty \in G$ and $f$ admits a holomorphic continuation into $G$, $f \in \mathscr{H}(G)$. Such maximal domain $D$ is unique (up to a compact set of zero capacity).

The compact set $S = S(f) := \partial D$ is now called "Stahl's compact set", and $D$ is called "Stahl's domain".

For example, if we have a set of two branch points, $\{-1, 1\}$, then Stahl's compact set (with respect to the infinity point) is $S = [-1, 1]$.

# Stahl's compact set

Properties of the compact set $S$: the complement of $D = \overline{\mathbb{C}} \setminus S$ is a domain, $S$ consists of a finite number of analytic arcs, and, $S$ possesses the property of "symmetry", that is,

$$\frac{\partial g_D(z,\infty)}{\partial n^+} = \frac{\partial g_D(z,\infty)}{\partial n^-}, \quad z \in S^0,$$

where $g_D(z,\infty)$ is Green's function for the domain $D$, with the logarithmic singularity at the point $z = \infty$, $S^0$ is the union of all open arcs of $S$ (which closures constitute $S$, that is $S \setminus S^0$ is a finite set), and $\partial n^+$ and $\partial n^-$ are the inner (with respect to $D$) normal derivatives of $g_D(z,\infty)$ at a point $z \in S^0$ from the opposite sides of $S^0$.

# Set of branch points

Let $\Sigma \subset \overline{\mathbb{C}}$, $\#\Sigma < \infty$ be a finite set, and let $\mathscr{A}(\overline{\mathbb{C}} \setminus \Sigma)$ be the set of all functions analytic in the domain $\overline{\mathbb{C}} \setminus \Sigma$.

Let $\mathscr{A}^0(\overline{\mathbb{C}} \setminus \Sigma) := \mathscr{A}(\overline{\mathbb{C}} \setminus \Sigma) \setminus \mathscr{H}(\overline{\mathbb{C}} \setminus \Sigma)$, that is, a function $f$ is from this set if it is a multivalued analytic function in the domain $\overline{\mathbb{C}} \setminus \Sigma$, but not a holomorphic function in $\overline{\mathbb{C}} \setminus \Sigma$ ($f$ is analytic, but not single-valued).

Let $\Sigma = \{a_1, \dots, a_p\}$, $\#\Sigma = p < \infty$, be the set of all branch points of $f$, i.e. $f \in \mathscr{A}^0(\overline{\mathbb{C}} \setminus \Sigma)$. Clearly, if $\Sigma = \{-1, 1\}$, then Stahl's compact set (with respect to the infinity point) is $S = [-1, 1]$.

# Potential theory

We recall some basic facts from the potential theory.

For a positive Borel measure $\mu$ with a compact support $\operatorname{supp}(\mu) \Subset \overline{\mathbb{C}}$, denote by $V^{\mu}(z)$ the logarithmic potential of $\mu$, that is:

$$V^{\mu}(z) := \int_{\operatorname{supp}\mu} \log \frac{1}{|z-\zeta|}\, d\mu(\zeta).$$

Given an arbitrary polynomial $Q \in \mathbb{C}(z)$, $Q \not\equiv 0$, denote by

$$\chi(Q) := \sum_{\zeta : Q(\zeta)=0} \delta_{\zeta}$$

the associated zero counting measure of $Q$ ($\delta_{\zeta}$ denotes the Dirac measure concentrated at the point $\zeta \in \overline{\mathbb{C}}$).

# Equilibrium measure

Denote with $\lambda$ the unique probability equilibrium measure of the compact set $S$, that is

$$V^{\lambda}(z) \equiv \gamma, \quad z \in S,$$

where

$$V^{\lambda}(z) = \int \log \frac{1}{|z-\zeta|}\, d\lambda(\zeta)$$

is the logarithmic potential of the measure $\lambda$, $\gamma$ is the Robin constant for $S$.

It is known, that

$$g_D(z, \infty) \equiv \gamma - V^{\lambda}(z).$$

# Convergence

We use the notation "$\xrightarrow{*}$" for convergence of measures in the weak-star topology. We say that a sequence $\{\mu_n\}_{n=1}^{\infty}$ of Borel measures converges weakly to a measure $\mu$, if for every continuous function $g(x) \in A$, where $A := \text{supp}(\mu_n)$, we have [Landkoff 1966, Theorem 0.4]

$$\lim_{n\to\infty} \int_A g(x) d\mu_n(x) = \int_A g(x) d\mu(x), \quad \mu \in A.$$

We use the notation "$\xrightarrow{\text{cap}}$" for convergence in capacity inside (on compact subsets of) a domain. Let $\Omega$ be a domain in $\mathbb{C}$ and $\varphi$ be a continuous function defined in $\Omega$ with values in $\overline{\mathbb{C}}$. A sequence of functions $\{\varphi_n\}$, meromorphic in $\Omega$, is said to converge in "capacity inside" to $\varphi$ inside $\Omega$ if for any compact set $K \subset \Omega$ and any $\varepsilon > 0$ we have [Gonchar 1975, §2.3]

$$\text{cap}\left(\{z \in K : |\varphi - \varphi_n| \geqslant \varepsilon\}\right) \to 0, \text{ as } n \to \infty.$$

# Padé polynomials

We use the following definition for the Padé approximant.

Let $P_{n,0}, P_{n,1}$, $P_{n,1} \not\equiv 0$, be the Padé polynomials of degree at most $n$ (at the infinity point) of the function $f$, for which the following holds true:

$$(P_{n,0} + P_{n,1} f)(z) = O\left(\frac{1}{z^{n+1}}\right), \quad z \to \infty.$$

The polynomials $P_{n,0}, P_{n,1}$ are not unique, but their ratio $P_{n,0}/P_{n,1}$ is uniquely determined. The rational function $[n/n]_f := -P_{n,0}/P_{n,1}$ is called the diagonal Padé approximant of order $n$ of the function $f$ (at the infinity point).

## Stahl Theorem (H. Stahl [Stahl 1986ab], [Stahl 1987b])

*Let $f \in \mathscr{H}(\infty)$, $f \in \mathscr{A}^0(\overline{\mathbb{C}} \setminus \Sigma)$, $\#\Sigma < \infty$. Let $D = D(f)$ be Stahl's "maximal" domain for $f$, $S = \partial D$ – Stahl's compact set, $[n/n]_f = -P_{n,0}/P_{n,1}$ – the n-diagonal Padé approximant to the function $f$. Then the following statements are valid:*

1. *there exists LZD of Padé polynomials $P_{n,j}$, $j = 0, 1$, namely,*

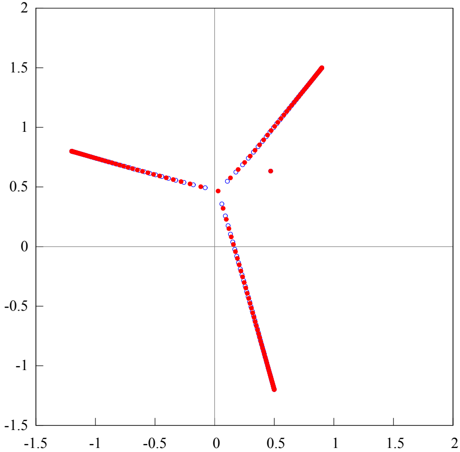
$$\frac{1}{n}\chi(P_{n,j}) \xrightarrow{*} \lambda, \quad \text{as} \quad n \to \infty, \quad j = 0, 1;$$

2. *the n-diagonal Padé approximants converge in capacity to the function $f$ inside the domain $D$,*

$$[n/n]_f(z) \xrightarrow{\text{cap}} f(z), \quad n \to \infty, \quad z \in D;$$

3. *the rate of the convergence in 2) is completely characterized by the relation*

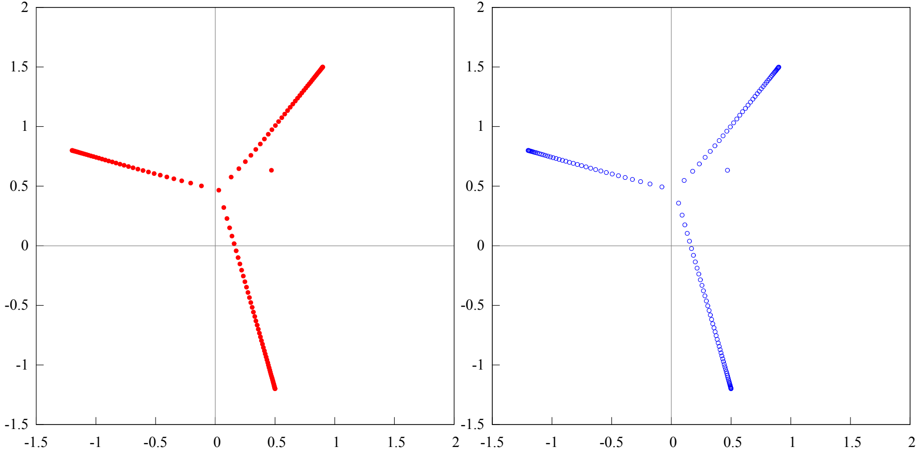
$$\left|(f - [n/n]_f)(z)\right|^{1/n} \xrightarrow{\text{cap}} e^{-2g_D(z,\infty)}, \quad n \to \infty, \quad z \in D.$$




Zeros (blue points) and poles (red points) of the diagonal Padé approximant $[130/130]_f$ of the function
$f(z) = 1/((z - (-1.2 + 0.8i))(z - (0.9 + 1.5i))(z - (0.5 - 1.2i)))^{1/3}$.

There is a Froissart doublet (spurious zero-pole pair) when $n = 130$.
It was proved that the genus of the Riemann surface is 1, there might be at most one Froissart doublet.

The Froissart doublet “attracts” the Stahl $S$-compact $S_{130}$
in full compliance with the electrostatical model by E. A. Rakhmanov [Rakhmanov 2012].




Poles (red points, left) and zeros (blue points, right) of the diagonal Padé approximant $[130/130]_f$ of the function
$f(z) = 1/((z-(-1.2+0.8i))(z-(0.9+1.5i))(z-(0.5-1.2i)))^{1/3}$.

The poles of the Padé approximant $[130/130]_f$ approximate a Chebotarev point $v_{130}$ for the $S$-compact $S_{130}$ (see [Rakhmanov 2012]). When $n \to \infty$ we have that $v_n \to v$ is a classical Chebotarev point (left picture).

The Chebotarev point should not be approximated by zeros of the Padé approximant $[130/130]_f$ (changing $f$ to $1/f$ reverses that).

There is one spurious pole of the Padé approximant $[130/130]_f$, it is accompanied by a spurious zero of the Padé approximant $[130/130]_f$.

See articles from Rakhmanov, Nuttall, Suetin, and others about Froissart doublets.

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# Strong asymptotics

We turn our attention to strong asymptotics of Padé polynomials, we first cite a theorem again by Gonchar and Suetin [GonSu 2004] (which generalizes Nuttall's results) for a special case, and then we present our conjecture for the generic case.

We layout the approach of the proof for the special case.

Let $\mathfrak{R} : \omega^2 = z^2 - 1$ be a Riemann surface.
Denote the two sheets by $D^{(1)}$ and $D^{(2)}$.
Let $\omega = +\sqrt{z^2 - 1}$ on the first sheet $D^{(1)}$ and $\omega = -\sqrt{z^2 - 1}$ on $D^{(2)}$.

For points on $\mathfrak{R}$ we use the notation $\underline{\mathbf{z}} = (z, \omega)$, where $\omega = \pm\sqrt{z^2 - 1}$,

$$\underline{\mathbf{z}} = \begin{cases} z^{(1)} := (z, \sqrt{z^2 - 1}), & \underline{\mathbf{z}} \in D^{(1)} \\ z^{(2)} := (z, -\sqrt{z^2 - 1}), & \underline{\mathbf{z}} \in D^{(2)}. \end{cases}$$

# Nuttall's psi-function

Let $\Phi(z) = z + \sqrt{z^2 - 1}$, $z = z^{(1)} \in D^{(1)}$.
We extend $\Phi(\underline{z}) \in \mathbb{C}(z, \omega)$ on $\mathfrak{R}$ as $\Phi(\underline{z}) = z + \omega$, with

$$\Phi(\underline{z}) = \begin{cases} z + \sqrt{z^2 - 1}, & \underline{\mathbf{z}} \in D^{(1)} \\ z - \sqrt{z^2 - 1}, & \underline{\mathbf{z}} \in D^{(2)}. \end{cases}$$

We introduce the Nuttall's psi-function $\Psi = \Psi_n$

$$\Psi(z) = \Phi(z)^n e^{S(z)}, \quad z = z^{(1)} \in D^{(1)},$$

where $S(z)$ is the Szegö function

$$S(z) = \frac{\sqrt{z^2 - 1}}{2\pi} \int_\Delta \frac{\log \rho(x)}{z - x} \frac{dx}{\sqrt{1 - x^2}}$$

(the classical Szegö function is actually $1/S(z)$).

# Remainder function

Recall that $\Delta = [-1, 1]$ and $D = \overline{\mathbb{C}} \setminus \Delta$.
Let $f$ be a Markov function, $f = \widehat{\rho}$, with

$$\widehat{\rho}(z) = \frac{1}{\pi} \int_{\Delta} \frac{\rho(x)}{z - x} \frac{dx}{\sqrt{1 - x^2}}.$$

We remind that

$$R_n(z) := (P_{n,0} + P_{n,1} f)(z) = O\left(\frac{1}{z^{n+1}}\right), \quad z \to \infty.$$

where $R_n(z)$ is the remainder function of the Padé approximant.

# Limits

We define the following on different sheets, $\underline{\mathbf{x}} \in \Gamma = \mathfrak{R} \setminus (D^{(1)} \sqcup D^{(2)})$, from the upper half-plane of the first sheet to $\underline{\mathbf{x}}$,

$$\lim_{z^{(1)} \to \underline{\mathbf{x}}} \Psi^{(1)}(z^{(1)}) = \Psi^{+}(\underline{\mathbf{x}}),$$

from the lower half-plane of the second sheet to $\underline{\mathbf{x}}$,

$$\lim_{z^{(2)} \to \underline{\mathbf{x}}} \Psi^{(2)}(z^{(2)}) = \Psi^{-}(\underline{\mathbf{x}}),$$

# Problem for Markov function

The function $\Psi$ is unique (up to a normalization) and is the solution of a Riemann boundary problem:

## Riemann boundary problem

*Given a $n \in \mathbb{N}$, find a function $\Psi = \Psi_n$ such that*

1. $\Psi$ *is partially meromorphic on* $\mathfrak{R} \setminus \Gamma$,
2. $\operatorname{div}(\Psi) = n\infty^{(2)} - n\infty^{(1)}$,
3. $\rho(x)\Psi^{+}(\underline{x}) = \Psi^{-}(\underline{x})$, *as* $\underline{x} \in \Gamma$,

*where* $\Gamma$ *is the cutting countour on* $\mathfrak{R}$.

# Theorem for Markov function

Under the above conditions for the psi-function $\Psi$, the asymptotic behavior of the denominators of the Pade approximant $P_{n,1}$ solve a Riemann boundary problem:

**Theorem (Gonchar, Suetin [GonSu 2004])**

*Let $f = \widehat{\rho}$ be a Markov function, where $\rho(x)$ is holomorphic on $\Delta$ and with no zeros on $\Delta$. Let $\Psi$ be Nuttall's psi-function, associated with the two sheeted Riemann surface $\mathfrak{R} : \omega^2 = z^2 - 1$.*
*Then, under suitable normalization of $P_{n,1}(z)$, we have*

1. $P_{n,1}(z) = \Psi(z)(1 + o(1))$, $n \to \infty$, *uniformly inside* $D'$,
2. $P_{n,1}(x) = \Psi^+(x) + \Psi^-(x) + o(1)$, $n \to \infty$, *uniformly on* $\Delta$,

*where* $o(1) = o(\delta_n)$, $\delta < 1$.

The function $\Psi$ is unique (up to a normalization) and is the solution of a Riemann boundary problem:

## Riemann boundary problem

*Given a* $n \in \mathbb{N}$, *find a function* $\Psi = \Psi_n$ *such that*

1. $\Psi$ *is partially meromorphic on* $\mathfrak{R} \setminus \Gamma$,
2. $\operatorname{div}(\Psi) = n\infty^{(2)} + m_1 a_1^{(1)} + \ldots + m_l a_l^{(1)}$ $- m_1 a_1^{(2)} - \ldots - m_l a_l^{(2)} - n\infty^{(1)}$,
3. $\rho(x)\Psi^+(\underline{x}) = \Psi^-(\underline{x})$, *as* $\underline{x} \in \Gamma$.

*where* $\Gamma$ *is the cutting countour on* $\mathfrak{R}$.

# Theorem for Markov-type function

Under the above conditions for the psi-function $\Psi$, the asymptotic behavior of the denominators of the Pade approximant $P_{n,1}$ solve a Riemann boundary problem:

## Theorem (Gonchar, Suetin [GonSu 2004])

*Let $f = \widehat{\rho} + r$ be a Markov-type function, where $\rho(x)$ is holomorphic on $\Delta$ and with no zeros on $\Delta$, all poles of $r$ are in $D$ and $r(\infty) = 0$.*
*Let $a_1, \ldots, a_l$ be the poles of $f$ in $D$, $m_1, \ldots, m_l$ – their multiplicities, $\Psi$ – Nuttall's psi-function, associated with the two sheeted Riemann surface $\mathfrak{R} : \omega^2 = z^2 - 1$.*
*Then, under suitable normalization of $P_{n,1}(z)$, we have*

1. $P_{n,1}(z) = \Psi(z)(1 + o(1))$, $n \to \infty$, *uniformly inside* $D'$,
2. $P_{n,1}(x) = \Psi^+(x) + \Psi^-(x) + o(1)$, $n \to \infty$, *uniformly on* $\Delta$,

*where* $D' = D \setminus \{a_1, \ldots, a_l\}$, *and* $o(1) = o(\delta_n)$, $\delta < 1$.

# Generic case

We present a conjecture on strong asymptotics for Padé polynomials in the generic case under the conditions of Stahl Theorem. It is worth noting that in general (except for the case of genus zero) such a representation is not unique (see [DeKrMc 1999], [Pastur 2006], [KomSu 2014]).

The reason is that there might exist spurious zeros of the Padé polynomials, with behavior, as $n \to \infty$, that can be described in many ways.

# Problem for the generic case

Let $\mathfrak{R}_2$ be the canonical hyperelliptic two-sheeted Riemann surface associated with the Stahl compact set $S$. The compact set $S$ is the projection of $S$ onto $\mathfrak{R}_2$, denote by $\Gamma_S$.
Let $g = g(\mathfrak{R}_2)$ be the genus of the Riemann surface.

The function $\Psi$ is unique (up to a normalization) and is the solution of a Riemann boundary problem:

## Riemann boundary problem

*Given a $n \in \mathbb{N}$, find a function $\Psi = \Psi_n$ such that*

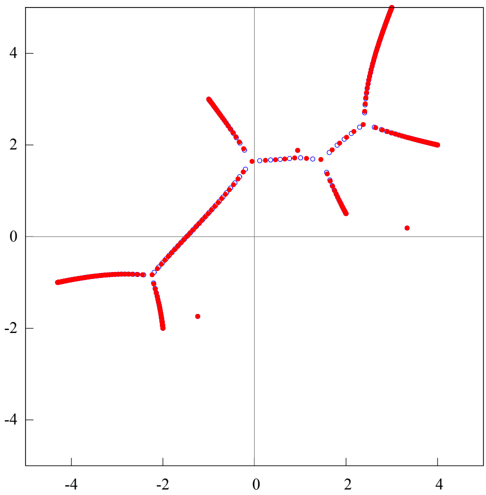
1. $\Psi$ *is partially meromorphic on* $\mathfrak{R}_2 \setminus \Gamma_S$,
2. $\operatorname{div}(\Psi) = (n-g)\infty^{(2)} + \underline{\mathbf{z}}_1(n) + \ldots + \underline{\mathbf{z}}_g(n) - n\infty^{(1)}$,
3. $\rho(\zeta)\Psi^{+}(\underline{\zeta}) = \Psi^{-}(\underline{\zeta})$, *when* $\underline{\zeta} \in \Gamma_S$.

## Conjecture (Kovacheva, Suetin [KovSu 2014])

*Let $f \in \mathscr{H}(\infty)$ and $f \in \mathscr{A}^0(\overline{\mathbb{C}} \setminus \Sigma)$ for some finite set $\Sigma \subset \overline{\mathbb{C}}$. Let $D = D(f)$ be Stahl's maximal domain associated with $f$, $S = S(f) = \partial D$ – the corresponding Stahl's compact set for $f$, $\mathfrak{R}_2$ – the canonical hyperelliptic two-sheeted Riemann surface associated with the compact set $S$, $\underline{z} = z^{(1,2)} = (z, \pm) \in \mathfrak{R}_2$ – an arbitrary point on the two-sheeted $\mathfrak{R}_2$, $\Psi_n(\underline{z}) = \Psi_n(\underline{z}; f)$ – Nuttall's psi-function associated with $f$ and $\mathfrak{R}_2$. Then, after a suitable normalization of the Padé polynomials $P_{n,j}(z) = P_{n,j}(z; f)$, $j = 0, 1$, and the remainder function $R_n$, the following relations take place in capacity inside the domain $D$:*

$$P_{n,j}(z) \overset{\text{cap}}{=} \frac{(-1)^j}{f^j(z)} \Psi_n(z^{(1)})\big(1 + o(1)\big), \qquad n \to \infty,$$

$$R_n(z) \overset{\text{cap}}{=} \frac{\Psi_n(z^{(2)})}{\omega(z^{(2)})}\big(1 + o(1)\big), \qquad n \to \infty.$$

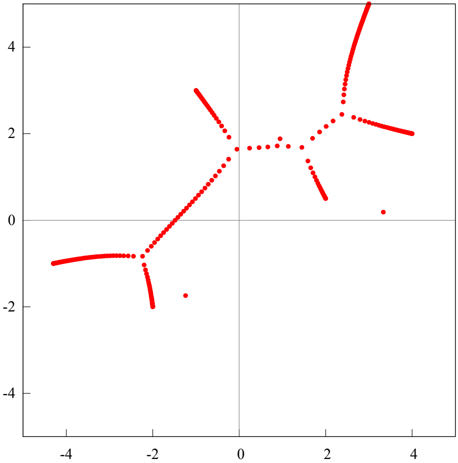



Zeros and poles of the diagonal Padé approximant $[266/266]_f$ of the function $f(z) = 1/((z-(-4.3-1.0i))(z-(2.0+0.5i))(z-(-2.0-2.0i))(z-(-1.0+3.0i))(z-(4.0+2.0i))(z-(3.0+5.0i)))^{1/6}$.

These zeros and poles are distributed in a plane, under fixed $n = 266$, accordingly to the electrostatical model by Rakhmanov [Rakhmanov 2012].

Since the genus of the Riemann surface is 4, for each $n$ there might be no more than 4 Froissart doublets. Here are observed 3 Froissart doublets.

In full compliance with the Rakhmanov model, the Froissart doublets "attract" the Stahl $S$-compact $S_{266}$. In general, the zeros and poles of the diagonal Padé approximants $[n/n]_f$ are distributed as $n \to \infty$ accordingly to Stahl Theorem [Stahl 1987b].



4
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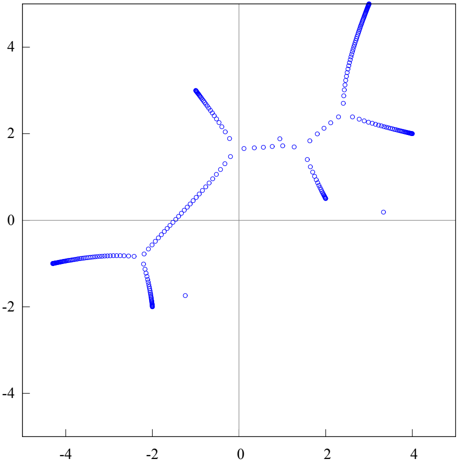
2
4

4
2
0
-2
-4
-4
-2
0
2
4

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# Numerical results

We present our numerical results about the asymptotics of type I Hermite-Padé polynomials.

First, we recall the Angelesco and Nikishin systems of functions, and then we introduce a new type of system.

Further on, we shall only be considering functions of Markov-type $f = \widehat{\rho} + r$, where

$$\widehat{\rho}(z) = \frac{1}{\pi} \int_{\Delta} \frac{\rho(x)}{z - x} \frac{dx}{\sqrt{1 - x^2}}.$$

# Angelesco system

(1) We consider the type I Hermite-Padé approximant for the collection of three functions $[1, f_1, f_2]$. Let the functions $f_1$ and $f_2$ have the following form:

$$f_1(z) = \frac{z}{\sqrt{(z-a_1)(z-a_2)}}, \qquad f_2(z) = \frac{z}{\sqrt{(z-b_1)(z-b_2)}},$$

where $a_1, a_2, b_1, b_2 \in \mathbb{C}$, $a_1 \neq a_2$, $b_1 \neq b_2$, and $[a_1, a_2] \cap [b_1, b_2] = \varnothing$. Therefore, the pair of functions $f_1, f_2$ forms an *Angelesco system* (see [Kalyagin 1979]).

The sets of branch points $\mathcal{A}_1 = \mathcal{A}(f_1)$ and $\mathcal{A}_2 = \mathcal{A}(f_2)$ do not intersect each other.

# Nikishin system

(2) Now we consider type I Hermite-Padé approximant for the collection of three functions $[1, f, f^2]$,

$$f(z) = (z^2 - 1)^{1/4}(z - a)^{-1/2}, \quad f(\infty) = 1, \quad a \notin \mathbb{R},$$

and the pair of functions $f_1, f_2$ forms a (generalized) *Nikishin system* (see [Nikishin 1980]).

The sets of branch points $\mathcal{A}_1$ and $\mathcal{A}_2$ of the functions $f_1$ and $f_2$ are equivalent.

(3) Again, consider the type I Hermite-Padé approximant for the collection of three functions $[1, f_1, f_2]$. Let $E_1 := [a_1, a_2] = [-1, a]$, $E_2 := [b_1, b_2] = [-a, 1]$, where $a \in (0, 1)$ is a real parameter.

Both segments $E_1$ and $E_2$ are overlapping, $E_1 \cap E_2 = [-a, a] \neq \varnothing$. The sets of branch points do not intersect each other.

Therefore, the new type of system is neither an Angelesco, nor a Nikishin system.

# Case 1

Let

$$f_1(z) := \int_{-1}^{a} \frac{dx}{x-z} = \log \frac{z-a}{z+1}, \quad z \notin E_1,$$
$$f_2(z) := \int_{-a}^{1} \frac{dx}{x-z} = \log \frac{z-1}{z+a}, \quad z \notin E_2.$$

We take the main branch of the logarithmic function, in the sense that

$$\log \frac{z-a}{z+1}, \ \log \frac{z-1}{z+a} \approx \log 1 = 0, \quad \text{as} \quad z \to \infty.$$

Set $Q_{n,0}, Q_{n,1}, Q_{n,2} \in \mathbb{P}_n \setminus \{0\}$ for the collection $[1, f_1, f_2]$,

$$(Q_{n,0} \cdot 1 + Q_{n,1} f_1 + Q_{n,2} f_2)(z) = \mathcal{O}\left(\frac{1}{z^{2n+2}}\right).$$

# Case 2

Let

$$g_1(z) := \left(\frac{z-a}{z+1}\right)^{1/2} = \frac{1}{\pi}\int_{-1}^{a}\sqrt{\frac{a-x}{x+1}}\frac{dx}{x-z} + 1, \quad z \notin E_1,$$
$$g_2(z) := \left(\frac{z-1}{z+a}\right)^{1/2} = \frac{1}{\pi}\int_{-a}^{1}\sqrt{\frac{1-x}{x+a}}\frac{dx}{x-z} + 1, \quad z \notin E_2.$$

We take such a branch of the $(\cdot)^{1/2}$ function, that $g_1(z), g_2(z) \to 1$ as $z \to \infty$, and under the square root function $\sqrt{\cdot}$ we mean the "arithmetic square root function", that is $\sqrt{x^2} = x$ for $x \in \mathbb{R}_+$.

Set $P_{n,0}, P_{n,1}, P_{n,2} \in \mathbb{P}_n \setminus \{0\}$ for the collection $[1, g_1, g_2]$,

$$(P_{n,0} \cdot 1 + P_{n,1}g_1 + P_{n,2}g_2)(z) = \mathcal{O}\left(\frac{1}{z^{2n+2}}\right).$$

# Case 3

Let

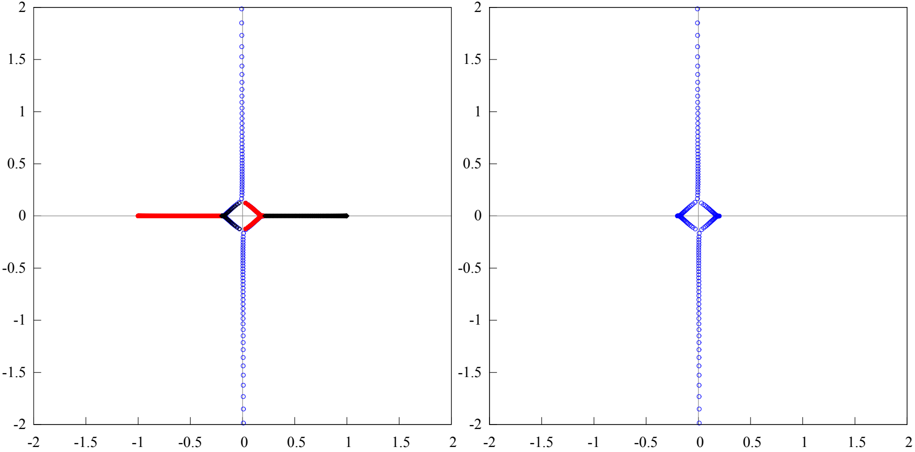
$$h_1(z) := \left(\frac{z-a}{z+1}\right)^{1/3}, \quad z \notin E_1,$$

$$h_2(z) := \left(\frac{z-1}{z+a}\right)^{1/3}, \quad z \notin E_2.$$

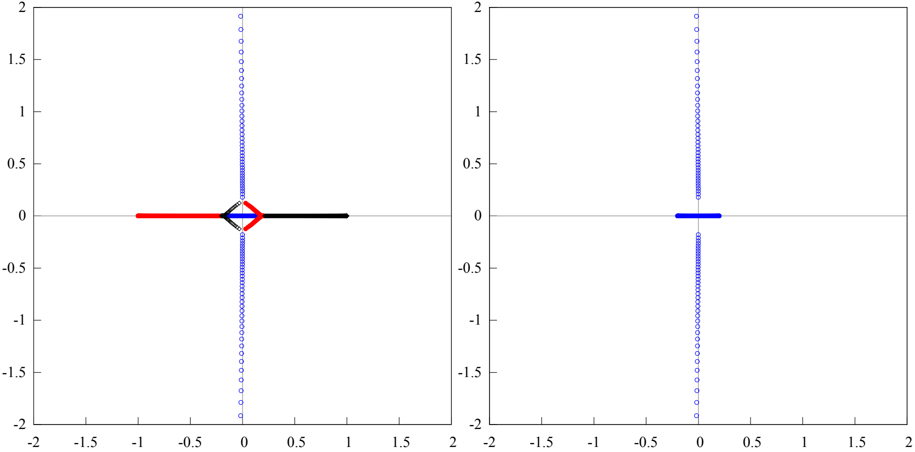
We take such a branch of the $(\cdot)^{1/3}$ function, that $h_1(z), h_2(z) \to 1$ as $z \to \infty$, and, in what follows, under the cubic root function $\sqrt[3]{\cdot}$ we mean the "arithmetic cubic root function", that is $\sqrt[3]{x^3} = x$ for $x \in \mathbb{R}_+$.

Set $U_{n,0}, U_{n,1}, U_{n,2} \in \mathbb{P}_n \setminus \{0\}$ for the collection $[1, h_1, h_2]$,

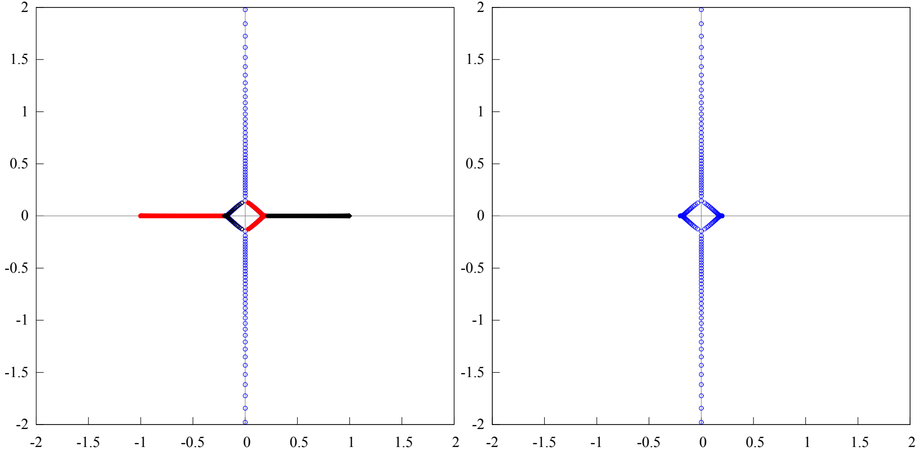
$$(U_{n,0} \cdot 1 + U_{n,1} h_1 + U_{n,2} h_2)(z) = \mathcal{O}\left(\frac{1}{z^{2n+2}}\right).$$



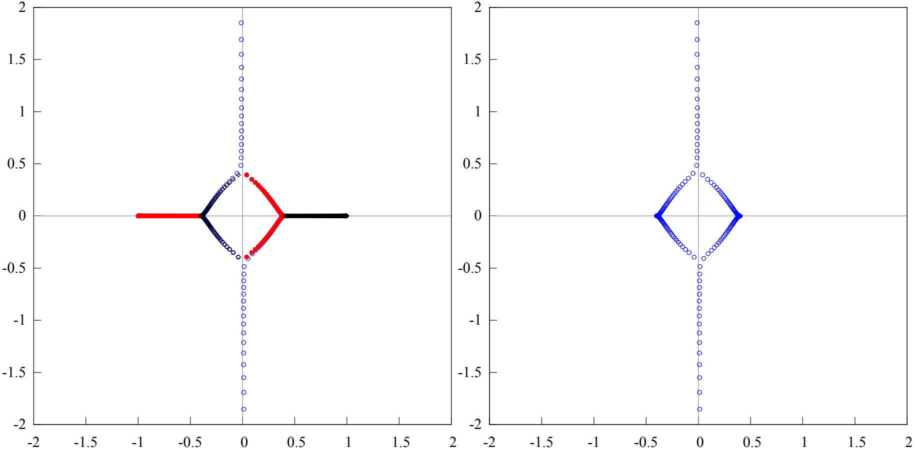

**0.2–1** Numerical distribution of zeros of type I HP polynomials $Q_{200,0}$ (blue points), $Q_{200,1}$ (red points), $Q_{200,2}$ (black points), for the collection $[1, f_1, f_2]$, where $f_1 = \log((0.2 - 1/z)/(1 + 1/z))$, $f_2 = \log((0.2 + 1/z)/(1 - 1/z))$.



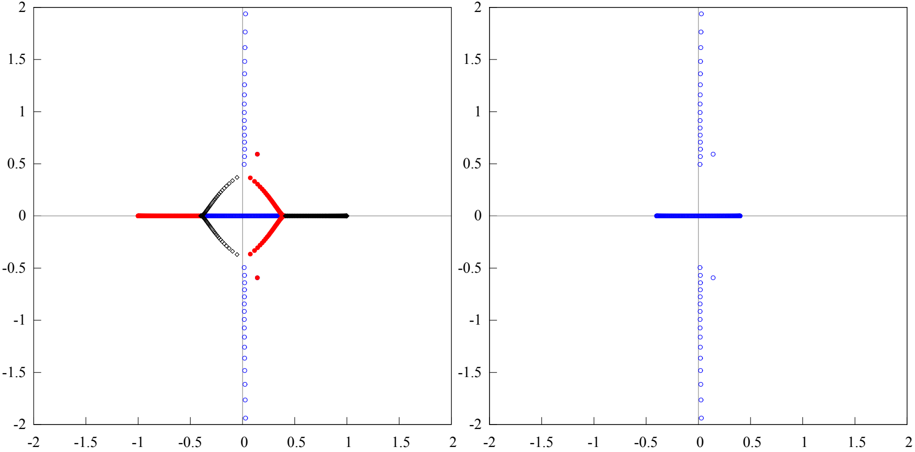

**0.2–2** Numerical distribution of zeros of type I HP polynomials $P_{200,0}$ (blue points), $P_{200,1}$ (red points), $P_{200,2}$ (black points), for the collection $[1, g_1, g_2]$, where $g_1 = ((0.2 - 1/z)/(1 + 1/z))^{1/2}$, $g_2 = ((0.2 + 1/z)/(1 - 1/z))^{1/2}$.



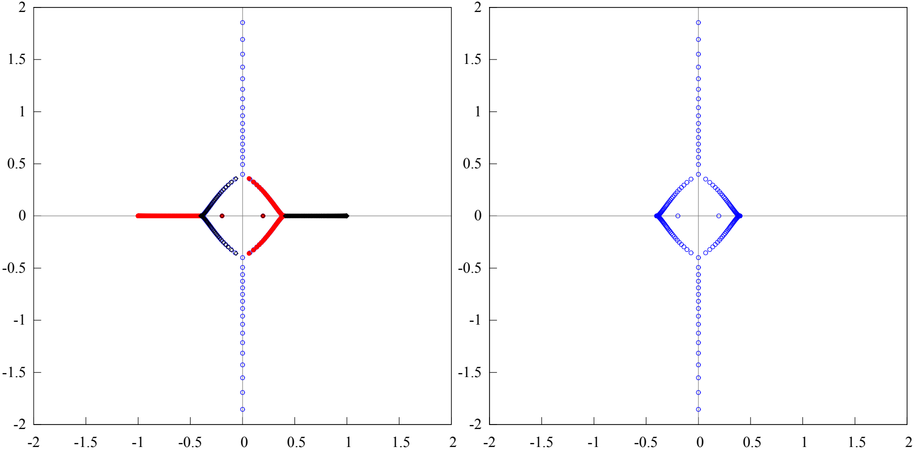

**0.2–3** Numerical distribution of zeros of type I HP polynomials $U_{200,0}$ (blue points), $U_{200,1}$ (red points), $U_{200,2}$ (black points), for the collection $[1, h_1, h_2]$, where $h_1 = ((0.2 - 1/z)/(1 + 1/z))^{1/3}$, $h_2 = ((0.2 + 1/z)/(1 - 1/z))^{1/3}$.



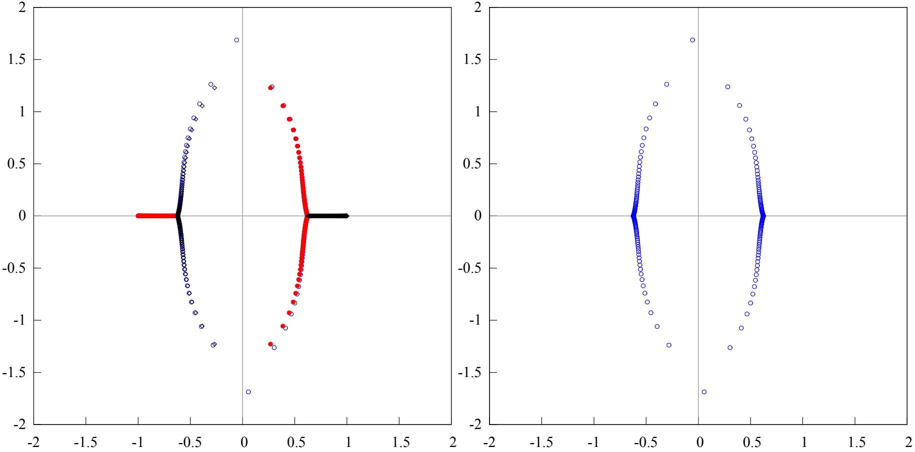

**0.4–1** Numerical distribution of zeros of type I HP polynomials $Q_{200,0}$ (blue points), $Q_{200,1}$ (red points), $Q_{200,2}$ (black points), for the collection $[1, f_1, f_2]$, where $f_1 = \log((0.4 - 1/z)/(1 + 1/z))$, $f_2 = \log((0.4 + 1/z)/(1 - 1/z))$.



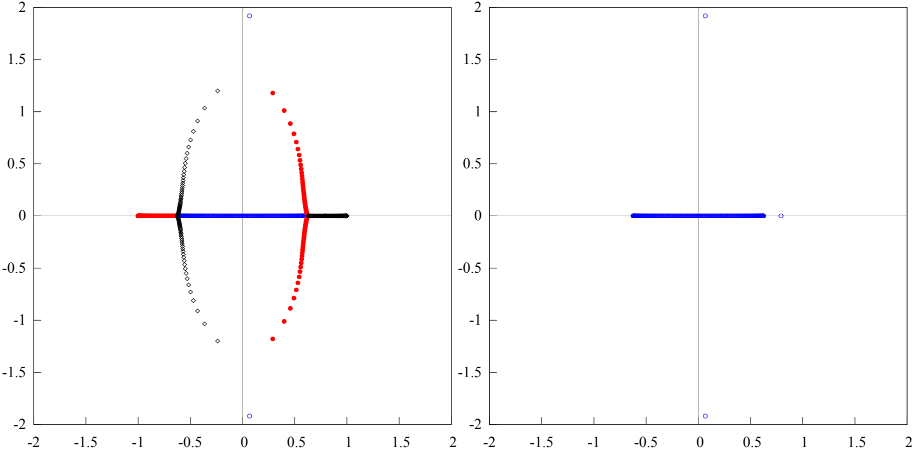

**0.4–2** Numerical distribution of zeros of type I HP polynomials $P_{200,0}$ (blue points), $P_{200,1}$ (red points), $P_{200,2}$ (black points), for the collection $[1, g_1, g_2]$, where $g_1 = ((0.4 - 1/z)/(1 + 1/z))^{1/2}$, $g_2 = ((0.4 + 1/z)/(1 - 1/z))^{1/2}$.



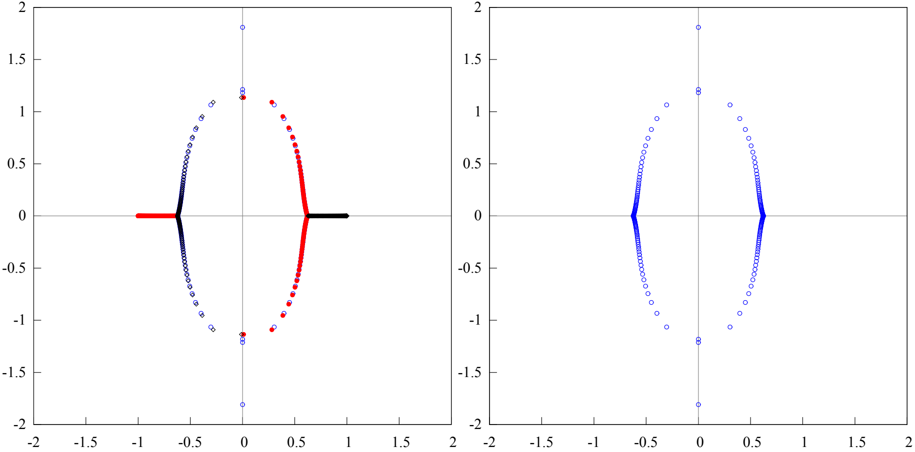

**0.4–3** Numerical distribution of zeros of type I HP polynomials $U_{200,0}$ (blue points), $U_{200,1}$ (red points), $U_{200,2}$ (black points), for the collection $[1, h_1, h_2]$, where $h_1 = ((0.4 - 1/z)/(1 + 1/z))^{1/3}$, $h_2 = ((0.4 + 1/z)/(1 - 1/z))^{1/3}$.



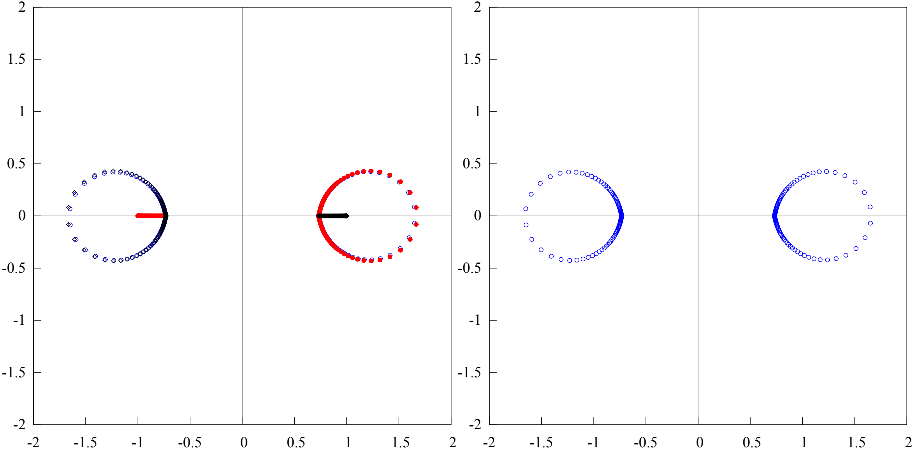

**0.625–1** Numerical distribution of zeros of type I HP polynomials $Q_{200,0}$ (blue points), $Q_{200,1}$ (red points), $Q_{200,2}$ (black points), for the collection $[1, f_1, f_2]$, where $f_1 = \log((0.625 - 1/z)/(1 + 1/z))$, $f_2 = \log((0.625 + 1/z)/(1 - 1/z))$.



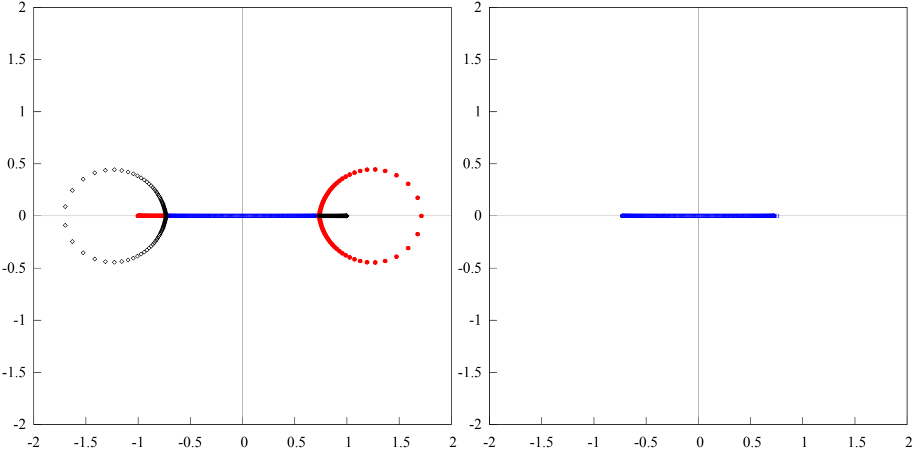

**0.625–2** Numerical distribution of zeros of type I HP polynomials $P_{200,0}$ (blue points), $P_{200,1}$ (red points), $P_{200,2}$ (black points), for the collection $[1, g_1, g_2]$, where $g_1 = ((0.625 - 1/z)/(1 + 1/z))^{1/2}$, $g_2 = ((0.625 + 1/z)/(1 - 1/z))^{1/2}$.



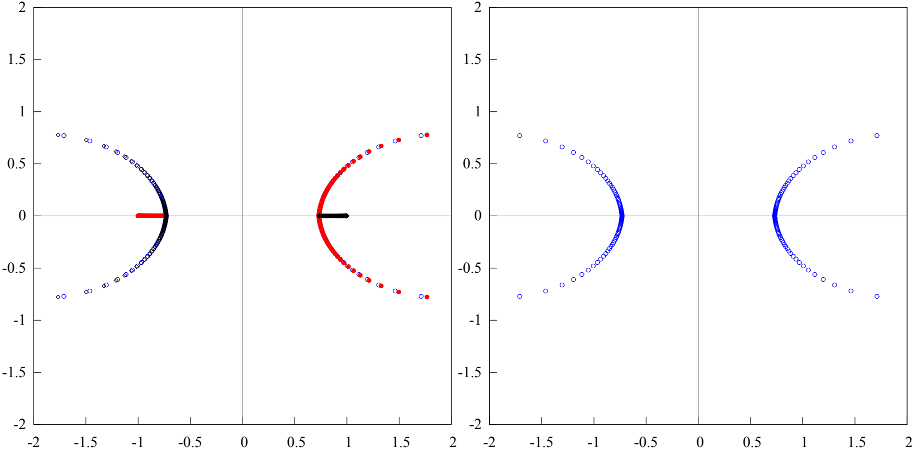

**0.625–3** Numerical distribution of zeros of type I HP polynomials $U_{200,0}$ (blue points), $U_{200,1}$ (red points), $U_{200,2}$ (black points), for the collection $[1, h_1, h_2]$, where $h_1 = ((0.625 - 1/z)/(1 + 1/z))^{1/3}$, $h_2 = ((0.625 + 1/z)/(1 - 1/z))^{1/3}$.



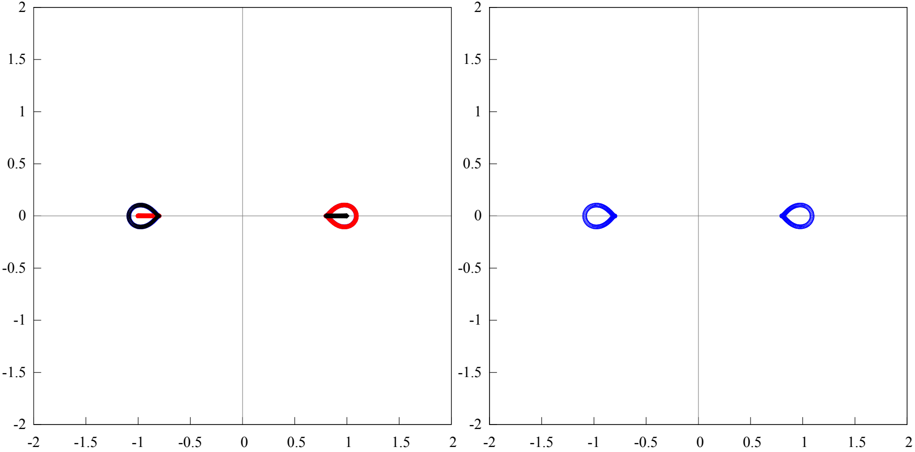

**0.73–1** Numerical distribution of zeros of type I HP polynomials $Q_{200,0}$ (blue points), $Q_{200,1}$ (red points), $Q_{200,2}$ (black points), for the collection $[1, f_1, f_2]$, where $f_1 = \log((0.73 - 1/z)/(1 + 1/z))$, $f_2 = \log((0.73 + 1/z)/(1 - 1/z))$.



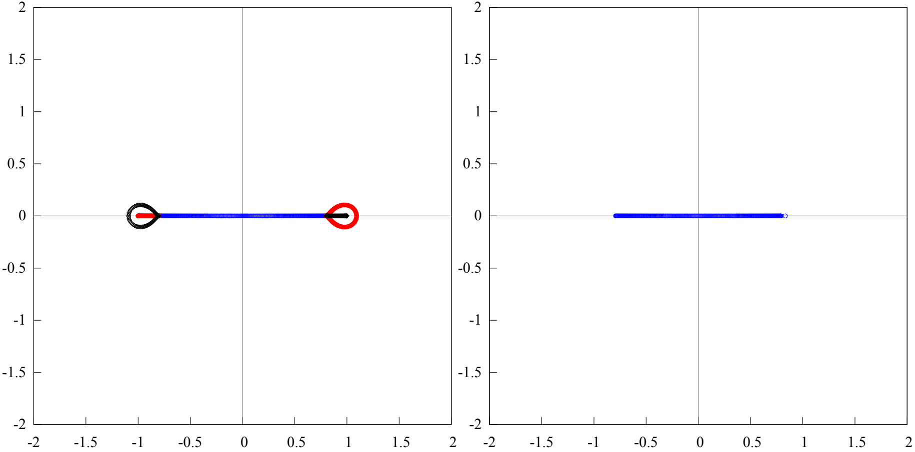

**0.73–2** Numerical distribution of zeros of type I HP polynomials $P_{200,0}$ (blue points), $P_{200,1}$ (red points), $P_{200,2}$ (black points), for the collection $[1, g_1, g_2]$, where $g_1 = ((0.73 - 1/z)/(1 + 1/z))^{1/2}$, $g_2 = ((0.73 + 1/z)/(1 - 1/z))^{1/2}$.



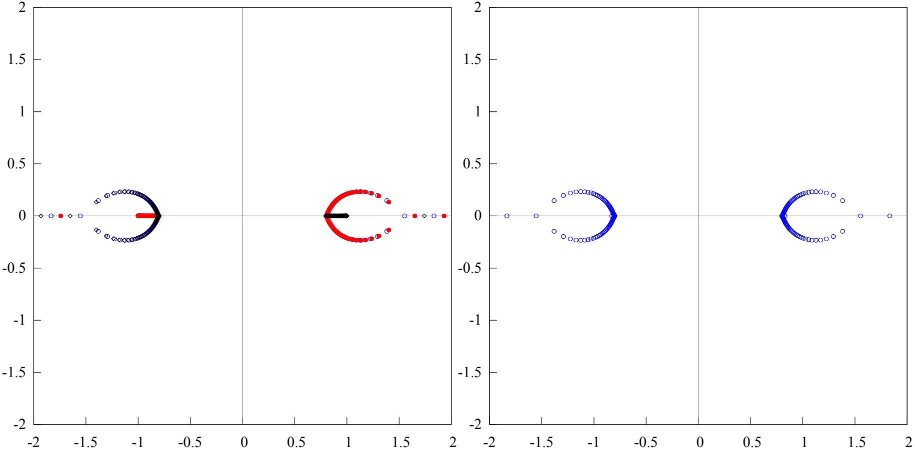

**0.73–3** Numerical distribution of zeros of type I HP polynomials $U_{200,0}$ (blue points), $U_{200,1}$ (red points), $U_{200,2}$ (black points), for the collection $[1, h_1, h_2]$, where $h_1 = ((0.73 - 1/z)/(1 + 1/z))^{1/3}$, $h_2 = ((0.73 + 1/z)/(1 - 1/z))^{1/3}$.




**0.8–1** Numerical distribution of zeros of type I HP polynomials $Q_{200,0}$ (blue points), $Q_{200,1}$ (red points), $Q_{200,2}$ (black points), for the collection $[1, f_1, f_2]$, where $f_1 = \log((0.8 - 1/z)/(1 + 1/z))$, $f_2 = \log((0.8 + 1/z)/(1 - 1/z))$.




**0.8–2** Numerical distribution of zeros of type I HP polynomials $P_{200,0}$ (blue points), $P_{200,1}$ (red points), $P_{200,2}$ (black points), for the collection $[1, g_1, g_2]$, where $g_1 = ((0.8 - 1/z)/(1 + 1/z))^{1/2}$, $g_2 = ((0.8 + 1/z)/(1 - 1/z))^{1/2}$.




**0.8–3** Numerical distribution of zeros of type I HP polynomials $U_{200,0}$ (blue points), $U_{200,1}$ (red points), $U_{200,2}$ (black points), for the collection $[1, h_1, h_2]$, where $h_1 = ((0.8 - 1/z)/(1 + 1/z))^{1/3}$, $h_2 = ((0.8 + 1/z)/(1 - 1/z))^{1/3}$.

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Thus from numerical experiments made by Kovacheva, Ikonomov, and Suetin it folows that the distribution of zeros of HP polynomials and the convergence of HP approximants itself are very sensitive to the type of branching of multivalued analytic function.

By this reason it might be very difficult to construct a general theory of limit zero distributin of HP polynomials of such type as Stahl's and Buslaev's theories are.

But in addition this sensitivity makes HP approximants very powerful to reconstruct the unknown properties of a multivalued analytic function given by a germ.

Thank You for the attention!