Cyclicity of polynomials in Dirichlet spaces in the bidisc

Łukasz Kosiński

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- If $\alpha = 0$ then we obtain the *Hardy space* H^2 comprises $f \in \mathcal{O}(\mathbb{D})$ such that
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- For $\alpha = -1$ the space D_{-1} is the *Bergman space* $\mathcal{B}$ comprises $f \in \mathcal{O}(\mathbb{D})$ such that
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- D_1 is the *Dirichlet space* $\mathcal{D}$ composed of $f \in \mathcal{O}(\mathbb{D})$ such that
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- The shift operator in ℓ^2 is a multiplication by z in H^2. To consider weighted shifts we take a multiplication by z in weighted Dirichlet spaces.

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This is equivalent to the fact that there is a sequence of polynomial p_n such that $||1 - p_n f||_\alpha \to 0$.

- if $f \in \mathcal{O}(\bar{\mathbb{D}})$ and $f \neq 0$ on $\bar{\mathbb{D}}$, then f is cyclic in D_α for any α – Taylor polynomials $p_n := T_n(1/f)$ satisfy $p_n f \to 1$ in D_α;

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- if $\alpha > 1$, then $f \in D_\alpha$ is cyclic iff it does not vanish on $\bar{\mathbb{D}}$;
- $f(z) = 1 - z$ is cyclic in D_1; note that $p_n = T_n(1/f) = \sum_{k=0}^{n-1} z^k$, satisfy $||p_n f - 1||_\alpha = ||z^n||_\alpha = (n+1)^\alpha$.

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  Generalized Riesz mean polynomials $p_n(z) = \sum_{k=0}^{n}(1 - \frac{H_k}{H_{n+1}})z^k$, where $H_n = 1 + 1/2 + \ldots + 1/n$, satisfy $||p_n f - 1||_1 \to 0$.

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- f is cyclic in H^2 iff f is outer;
- if f is cyclic in D_1, then f is outer and $c(\{f^* = 0\}) = 0$;
- (Brown-Shields conjecture): If f is an outer function such that $c(\{f^* = 0\}) = 0$, then f cyclic in D_1.

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A holomorphic function $f : \mathbb{D}^2 \to \mathbb{C}$, $f(z_1, z_2) = \sum_{k,l=0}^{\infty} a_{k,l} z_1^k z_2^l$ belongs to D_α if

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Definition

f is cyclic in D_α if and only if $span\{z_1 f, z_2 f\}$ is dense in D_α

Note that $f \in D_\alpha$ is cyclic in D_α iff there is a sequence $p_n \in \mathbb{C}[z_1, z_2]$ such that

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Let $\rho(x) = x - [x]$, $\varphi(x) = \rho(1/x)$, $x \in (0,1)$. Nyman's (1950) proved that the following is equivalent:

- All zeroes of the Riemann-ζ function are on the line $\{Rez = 1/2\}$,
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Nikolski showed that the second condition above is related to cyclicity in $H^2(\mathbb{D}_2^{\infty})$.

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$$\log|f(z_1, z_2)| = \int_{\mathbb{T}^2} \log|f^*(\zeta, \eta)| P_{(z_1,z_2)}(\zeta, \eta) |d\zeta||d\eta|.$$

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- Which polynomials are cyclic in D_α?

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- (Thomas Ransford) $f(z, w) = 2 - z - w$. Then $\{f = 0\} \cap \mathbb{T}^2 = \{(1, 1)\}$.
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- (H. Hedenmalm) If $f \in \mathcal{O}(\mathbb{D}^2) \cap \mathcal{C}(\bar{\mathbb{D}}^2)$ is such that f vanishes only in $(1, 1)$ and $f(1, \cdot)$ and $f(\cdot, 1)$ are outer functions, then f is cyclic in D_1.

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- Let $f(z_1, z_2) = g(z_1)h(z_2)$. Then $||f||_\alpha = ||g||_{D_\alpha} ||h||_{D_\alpha}$.
  Moreover, f is cyclic in D_α iff g and h are cyclic in $D_\alpha(\mathbb{D})$.
  Even more: if $f = f(z_1, z_2)$ is cyclic in D_α, then slices $f_{z_1} = f(z_1, \cdot)$ and $f_{z_2} = f(\cdot, z_2)$ are cyclic in $D_\alpha(\mathbb{D})$.

- Let $f(z,w) = 1 - zw$.
  $\{f = 0\} \cap \mathbb{T}^2 = \{(e^{i\theta}, e^{-i\theta}) : \theta \in \mathbb{R}\}$.
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  Let $p_n = \sum_{k,l} a_{k,l} z^k w^l$ be such that $p_n f \to 1$ in D_α. Let
  $\tilde{p}_n = \sum_k a_{k,k} z^k w^k = q_n(zw)$. Then
  $||p_n f - 1||_\alpha \geq ||\tilde{p}_n f - 1||_\alpha = ||(1-z)q_n - 1||_{D_{2\alpha}(\mathbb{D})}$, so $2\alpha \leq 1$.

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  $(1-zw)p_n(zw) \to 1$ in $D_{2\alpha}$.
- $f(z,w) = (1-z)(1-w)$ is cyclic if $\alpha \leq 1$.

Theorem

Suppose that $f \in \mathbb{C}[z_1, z_2]$ is not a polynomial of one variable, have no zeroes on $\mathbb{D}^2$ and $\{f = 0\}$ meets $\mathbb{T}^2$ along a curve. Then f is not cyclic in D_α for $\alpha > 1/2$.

Definition

Let $E \subset \mathbb{T}^2$ be a Borel set, μ – a probability measure supported on E. We say that μ has finite (Riesz) α-energy if

$$I_\alpha[\mu] = \int_{\mathbb{T}^2}\int_{\mathbb{T}^2} \frac{1}{|e^{i\theta_1} - e^{i\eta_1}|^{1-\alpha}} \frac{1}{|e^{i\theta_2} - e^{i\eta_2}|^{1-\alpha}} d\mu(\eta_1, \eta_2) d\mu(\theta_1, \theta_2) < \infty.$$

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Moreover,

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For $\alpha = 1$ we use kernel $\log(e/|e^{i\theta_1} - e^{i\eta_1}|) \log(e/|e^{i\theta_2} - e^{i\eta_2}|)$ in the definitions of energy and capacity.

Viewing the integral defining the energy as a convolution with a kernel of positive type it is possible to express α-energy of μ in terms of its Fourier coefficients:

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$$I_\alpha[\mu] = 1 + \sum_{k=1}^{\infty} \frac{|\hat{\mu}(k,0)|^2}{k^\alpha} + \sum_{l=1}^{\infty} \frac{|\hat{\mu}(0,l)|^2}{l^\alpha} + \frac{1}{2} \sum_{k\in\mathbb{Z}\setminus\{0\}} \sum_{l=1}^{\infty} \frac{|\hat{\mu}(k,l)|^2}{|k|^\alpha l^\alpha},$$

where $\hat{\mu}(k,l) = \int_{\mathbb{T}^2} e^{-i(k\theta_1 + l\theta_2)} d\mu(\theta_1, \theta_2)$.

Generalization of var der Corput's lemma.

S is a smooth curve in $\mathbb{T}^2 = [0, 2\pi) \times [0, 2\pi)$, $\varphi : I \to \mathbb{T}^2$ is its parametrization, $I = (-1, 1)$.

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The type of $\xi = \varphi(x) \in \varphi(I)$ is the smallest τ such that for all $\eta \in \mathbb{R}^2$, $||\eta|| = 1$ there exists $k \in \mathbb{Z}$, $k \leq \tau$, such that

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Note that $\tau \geq 2$.

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Note that $\tau \geq 2$.
Example: $\varphi(t) = (t, \psi(t))$. The curve is of type 2 at $\varphi(0)$ if for any $|\eta| = 1$ either $\eta_1 + \psi'(0)\eta_2 \neq 0$ or $\psi''(0)\eta_2 \neq 0$. This means that $\psi''(0) \neq 0$.

A curve is of type 2 is it has everywhere non-vanishing curvature.

$S \subset \mathbb{T}^2$ - a curve of finite type. Let σ be a measure on S obtained by pulling back to Lebesgue measure on the line using parametrization of D. Let $d\mu(x) = \psi(x)d\sigma(x)$, $x \in S \subset \mathbb{T}^2$, where $\psi \in \mathcal{C}_0^\infty(S)$, $\psi \geq 0$.

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Lemma (Decay of Fourier coefficients of measures on varieties)

If S of finite type $\tau \in \mathbb{N}$ and μ is as above, then there is $C > 0$ such that

$$|\hat{\mu}(k, l)| \leq C(k^2 + l^2)^{-1/(2\tau)},$$

$k, l \in \mathbb{Z}$.

Lemma

Assume that $f \in D_\alpha$ is such that $\{f^* = 0\} \cap \mathbb{T}^2$ contains a locally smooth curve S of type τ. Then f is not cyclic in D_α for any $\alpha > 1 - 1/\tau$.

It suffices to show that $c_\alpha(S) > 0$. Indeed, if $c_\alpha(S) > 0$, then $c_\alpha(\{f^* = 0\}) \cap \mathbb{T}^2) > 0$ and there is a probability measure μ on S such that $I_\alpha[\mu] < \infty$. Cauchy integral:

$$C[\mu](z_1, z_2) = \int_{\mathbb{T}^2} (1 - e^{i\theta_1} z_1)^{-1} (1 - e^{i\theta_2} z_2)^{-1} d\mu(\theta_1, \theta_2) = \int_{\mathbb{T}^2} \sum_k e^{ik\theta_1} z_1^k \sum_l e^{il\theta_2} z_2^l d\mu(\theta_1, \theta_2) = \sum_{k,l} \hat{\mu}(-k, -l) z_1^k z_2^l = \sum \bar{\hat{\mu}}(k, l) z_1^k z_2^l. \quad (1)$$

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$$||C[\mu]||_{-\alpha} = \sum_{k,l} \frac{|\hat{\mu}(k, l)|^2}{(k+1)^\alpha (l+1)^\alpha} < \infty$$

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Since μ is supported on $\{f^* = 0\}$ we get that $\langle pf, C[\mu]\rangle = 0$ for any polynomial p. In particular, f is non-cyclic.

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and $|\hat{\mu}(k,l)| \leq C(k^2 + l^2)^{-1/(2\tau)}$, $k, l \in \mathbb{Z}$.

$$\sum_{k=1} \sum_{l=1}^\infty \frac{|\hat{\mu}(k,l)|^2}{|k|^\alpha l^\alpha} \leq 2 \sum \frac{|\hat{\mu}(k,k)|^2}{(k+1)^\alpha (k+1)^\alpha} + \sum_{k=2}^\infty \sum_{l=1}^{k-1} \frac{|\hat{\mu}(k,l)|^2}{(k+1)^\alpha (l+1)^\alpha} + \sum_{l=2}^\infty \sum_{k=1}^{l-1} \frac{|\hat{\mu}(k,l)|^2}{(k+1)^\alpha (l+1)^\alpha} \leq C \sum_{k=1}^\infty \left(\frac{1}{k^{\alpha+1/\tau}} + \frac{1}{k^{2\alpha-1+2/\tau}}\right).$$

Assume that $\varphi(t) = (t, \psi(t))$ parametrizes a piece of the zero set of f on $\mathbb{T}^2$. Then $\tilde{\varphi}(t) = (\gamma(t), \psi(t))$, where $\gamma(t) = \arg m_a(e^{it})$, parametrizes a piece of the zero set of $f(m_a(z_1), z_2)$.
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$\tilde{\varphi}$ generically has type 2 at $t = 0$.
Actually, note that $\gamma'(0) > 0$ and $\gamma''(0) \neq 0$ as long as $\mathrm{Im}(a) \neq 0$.
Thus if $\psi''(0) \neq 0$, then φ has type 2. If $\psi''(0) = 0$, then equations $\eta_1\gamma'(0) + \eta_2\eta'(0) = 0$ and $\eta_2\gamma''(0) = 0$ cannot hold simultaneously, so $\tilde{\varphi}$ is of type 2 and then $f(m_a(z), w)$ is cyclic. Thus f is cyclic.

Theorem

Let $f \in \mathbb{C}[z_1, z_2]$ *have no zeros in* $\mathbb{D}^2$ *and finitely many zeroes on* $\mathbb{T}^2$*. Then* f *is cyclic in* D_α *iff* D_α *is not an algebra i.e.* $\alpha \leq 1$.

Theorem

Let $f \in \mathbb{C}[z_1, z_2]$ have no zeros in $\mathbb{D}^2$ and finitely many zeroes on $\mathbb{T}^2$. Then f is cyclic in D_α iff D_α is not an algebra i.e. $\alpha \leq 1$.

$f(z, w) = 2 - z_1 - z_2$. Note that $\{f = 0\} \cap \mathbb{T}^2 = \{(1, 1)\}$. Then $|(z_1 - 1)(z_2 - 1)| \leq 2|2 - z_1 - z_2|$, $(z_1, z_2) \in \mathbb{D}^2$. Thus for k big enough $Q(z_1, z_2) = \frac{(z_1-1)^k(z_2-1)^k}{2-z_1-z_2}$ is two times continuously differtiable on $\mathbb{T}^2$.

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$(z_1-1)^k(z_2-1)^k = Q(z_1,z_2)(2-z_1-z_2)$. But $(z_1-1)^k(z_2-1)^k$ is cyclic in D_1, so there are $p_n \in \mathbb{C}[z_1, z_2]$ such that $fQp_n \to 1$ in D_1. Since $Q \in D_1$, there are q_n such that $q_n \to Q$ in D_1. Thus $||fq_np_n - 1||_1 \to 0$.

Łojasiewicz's inequality

Let f be a nonzero real analytic function on an open set $U \subset \mathbb{R}^n$. Assume the zero set $\mathcal{Z}(f) = \{x \in U : f(x) = 0\}$ of f is nonempty. Let E be a compact subset of U. Then there are constants $C > 0$ and $q \in \mathbb{N}$, depending on E, such that

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$f \in \mathbb{C}[z_1, z_2]$ has no zeroes in $\mathbb{D}^2$ and finitely many zeroes on $\mathbb{T}^2$. Let $r(x_1, x_2) = |f(e^{ix_1}, e^{ix_2})|^2$. Set $E = [0, 2\pi]^2$. By Łojasiewicz's inequality there is $C > 0$ and q so that

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By assumption on f, $\mathcal{Z}(r) \cap E$ is finite and thus there is a constant $c > 0$ so that for $x \in E$

$$\operatorname{dist}(x, \mathcal{Z}(r))^2 \geq c \prod_{y \in \mathcal{Z}(r) \cap E} |x - y|^2.$$

But $|x-y|^2 = |x_1-y_1|^2 + |x_2-y_2|^2 \geq |e^{ix_1}-e^{iy_1}|^2 + |e^{ix_2}-e^{iy_2}|^2 \geq 2|(e^{ix_1}-e^{iy_1})(e^{ix_2}-e^{iy_2})|$.

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$$\frac{\prod_{\zeta \in \mathcal{Z}(f) \cap \mathbb{T}^2} |(z_1 - \zeta_1)(z_2 - \zeta_2)|^{q/2}}{|f(z)|^2}$$

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Thus there are

$$Q(z_1,z_2)=\frac{g(z_1)h(z_2)}{f(z_1,z_2)}$$

is 2-times continuously differentiable.

Theorem

Let $0 < \alpha \leq 1/2$. Then any polynomial that does not vanish in $\mathbb{D}^2$ is cyclic in D_α.

What may be assumed about polynomial f non-vanishing on $\mathbb{D}^2$ (Agler and McCarthy, Knese): assume that f has bidegree (n,m); put $\tilde{f}(z_1,z_2)=z_1^n z_2^m \overline{f(1/\bar{z}_1, 1/\bar{z}_2)}$.

- f does not divide $\tilde{f}$. Then f has finitely many zeros on $\mathbb{T}^2$ (zeros of f are common zeros of $\tilde{f}$ and f) - by Bezout's theorem f and $\tilde{f}$ have infinitely many zeroes iff they have a common factor;

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Additionally, f posses the following determinantal representation:

$$f(z) = c \det(I_{n+m} - U \begin{pmatrix} z_1 I_n & 0 \\ 0 & z_2 I_m \end{pmatrix}),$$

where c is a constant and U is unitary.

Equivalent norm on D_α.
Put

$$|f|_\alpha^2 = \int_{\mathbb{D}} |\partial_{z_1} f(z_1, 0)|^2 (1 - |z_1|^2)^{1-\alpha} dA(z_1) + \int_{\mathbb{D}} |\partial_{z_1} f(0, z_2)|^2 (1 - |z_2|^2)^{1-\alpha} dA(z_2) + \int_{\mathbb{D}^2} |\partial_{z_1}\partial_{z_2} f(z_1, z_2)|^2 (1 - |z_1|^2)^{1-\alpha}(1 - |z_2|^2)^{1-\alpha} dA(z_1) dA(z_2).$$

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Note that the above formula has sense for a function f holomorphic on $G\times\mathbb{D}$, where G is a domain in $\mathbb{D}$, $A(G)=A(\mathbb{D})$ and $0\in G$.
If $g\in\mathcal{O}(\mathbb{D}^2)$, then $|g(0)|+|g|_\alpha$ is a norm equivalent to Dirichlet norm in D_α.

Let $f \in \mathbb{C}[z_1, z_2]$. We may assume that f is irreducible.
We know that $g : (z_1, z_2) \mapsto 1 - z_1 z_2$ is cyclic in $\mathfrak{D}_\alpha$ iff $\alpha \leq 1/2$. This means that there exists a sequence of polynomials $(p_n)_{n \geq 1}$ such that $|p_n g - 1|_\alpha = |p_n g|_\alpha \to 0$. Factorize a general two-variable polynomial f by fixing z_1, thus obtaining

$$f(z_1, z_2) = H(z_1) \cdot (1 - h_1(z_1) z_2) \cdots (1 - h_N(z_1) z_2),$$

with H non-vanishing in $\mathbb{D}$ (so we may forget about H) and $z_1 \in \mathbb{C} \setminus A$, where points of A are isolated. Losing no generality we may assume that $0 \notin A$.
So if we choose a simply connected set D such that $\bar{D} = \bar{\mathbb{D}}$ and $D \subset \mathbb{D} \setminus A$, then $h_j \in \mathcal{O}(D)$ and

$$|1 - h_1(z_1) z_2|_\alpha$$

has sense.

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- Let p_n be a polynomial such that $q_n(z) := (1-z)p_n(z) \to 1$. Then $|(1 - h(z_1)z_2)p_n(h(z_1)z_2)|_\alpha \leq C||(1 - z_1 z_2)p_n(z_1 z_2)||_\alpha$;

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- Let p_n be a polynomial such that $q_n(z) := (1 - z)p_n(z) \to 1$. Then $|(1 - h(z_1)z_2)p_n(h(z_1)z_2)|_\alpha \leq C||(1 - z_1 z_2)p_n(z_1 z_2)||_\alpha$;
- We start by estimating the seminorm $|q_{\nu_1}(h_j z_2) q_\nu(h_i z_2)|_\alpha$, restricted to $D \times \mathbb{D}$. A computation shows that

$$\begin{aligned} &\partial_{z_1}\partial_{z_2}(q_{\nu_1}(h_j(z_1)z_2)q_\nu(h_i(z_1)z_2)) \\ &= h_j'(z_1)q_\nu(h_i(z_1)z_2)(q_{\nu_1}''(h_j(z_1)z_2)h_j(z_1)z_2 + q_{\nu_1}'(h_j(z_1)z_2)) \quad (2)\\ &+ h_j'(z_1)q_{\nu_1}'(h_j(z_1)z_2)q_\nu'(h_i(z_1)z_2)h_i(z_1)z_2 \quad (3)\\ &+ q_{\nu_1}'(h_j(z_1)z_2)h_j(z_1)q_\nu'(h_i(z_1)z_2)h_i'(z_1)z_2 \quad (4)\\ &+ q_{\nu_1}(h_j(z_1)z_2)(q_\nu''(h_i(z_1)z_2)h_i(z_1)z_2 + q_\nu'(h_i(z_1)z_2))h_i'(z_1). \quad (5) \end{aligned}$$

- Suppose $P(z_1, z_2) = z_2^n + A_1(z_1) z_2^{n-1} + \ldots + A_n(z_1)$ is holomorphic in a domain $G \times \mathbb{C} \subset \mathbb{C}^2$. If $h \in \mathcal{O}(G')$ for a domain $G' \subset\subset G$ and $P(z_1, h(z_1)) = 1$, then

$$|h'(z_1)| \leq O\left(\frac{1}{|z_1 - w|^{1-\frac{1}{n}}}\right)$$

as $z_1 \to w \in \partial G'$.

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  as $z_1 \to w \in \partial G'$.
- Let $a < 1$ and $0 \leq \beta \leq 1$. Then there is a constant C depending only on a and β such that for any $g \in \mathrm{Hol}(\mathbb{D})$,

$$\int_{\mathbb{D}} \left| \frac{g(z)}{(z-1)^a} \right|^2 (1 - |z|^2)^{1-\beta} dA(z) \leq C \left(|g(0)|^2 + \int_{\mathbb{D}} |g'(z)|^2 (1 - |z|^2)^{1-\beta} dA(z) \right). \quad (6)$$

- Applying the procedure N-times we find that $|1-(1-h_{i_1}(z)w)\dots(1-h_{i_N}(z)w)p_{\nu_1}(h_{i_1}(z)w)\dots p_{\nu_N}(h_{i_N}(z)w)|_\alpha$ may be arbitrary small for any $\{i_1,\dots,i_N\}$.
- For $\nu_1,\dots,\nu_N$ define a function $P=P_{\nu_1\dots\nu_N}$ as

$$P_{\nu_1\dots\nu_N}(z_1,z_2):=\frac{1}{N!}\sum_{\sigma\in\Sigma_N}p_{\nu_1}(h_{\sigma(1)}(z_1)z_2)\dots p_{\nu_N}(h_{\sigma(N)}(z_1)z_2),$$

  where Σ_N is the group of all permutations of the set $\{1,\dots,N\}$. Then $(1-(1-h_1(z)w)\dots(1-h_N(z)w)P(z,w)|_\alpha$ may be arbirarily small. This shows that f is cyclic.
- Finally, we observe that P extends holomorphically to a neighborhood of $\overline{\mathbb{D}^2}$, and hence it can be approximated in multiplier norm by polynomials.

Theorem

Let f be an irreducible polynomial with no zeros in $\mathbb{D}^2$.

1. *If $\alpha \leq 1/2$, then f is cyclic in D_α.*
2. *If $1/2 < \alpha \leq 1$, then f is cyclic in D_α iff $\{f = 0\} \cap \mathbb{T}^2$ is finite or empty or f does depend only on one variable.*
3. *If $\alpha > 1$ then f is cyclic in D_α iff $\{f = 0\} \cap \mathbb{T}^2$ is empty.*

The assumption about the irreducibility is harmless - any polynomial is a multiplier in D_α - for any $f \in D_\alpha$ and $p \in \mathbb{C}[z_1, z_2]$: $pf \in D_\alpha$. Thus $||pf||_\alpha \leq ||p||_{M(D_\alpha)}||f||_\alpha$ for any $f \in D_\alpha$.