Nonlinear n-term Approximation by Newtonian Potentials

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Notations

- $\mathbb{R}^d$ denotes the d-dimensional Euclidean space, $d \geq 2$.
  The unit open ball in $\mathbb{R}^d$ is given by $B^d := \{x : |x| < 1\}$.
  The unit sphere in $\mathbb{R}^d$ is $\mathbb{S}^{d-1} := \{x : |x| = 1\}$.
  Distance on the sphere is the geodesic distance, or the distance of x and y on the largest circle on $\mathbb{S}^{d-1}$ that passes through these points on the sphere, $\rho(x, y) := \arccos(x \cdot y)$.

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  Distance on the sphere is the geodesic distance, or the distance of x and y on the largest circle on $\mathbb{S}^{d-1}$ that passes through these points on the sphere, $\rho(x, y) := \arccos(x \cdot y)$.
- By $\partial_k = \frac{\partial}{\partial e_k}$ we denote differential operator in direction e_k. Then $\partial^\beta := \partial_1^{\beta_1} \ldots \partial_d^{\beta_d}$ is a differential operator of order $|\beta|$,
  $\nabla := (\partial_1, \ldots, \partial_d)$ and $\Delta := \partial_1^2 + \cdots + \partial_d^2$ stays for the Laplacian.
  By Δ_0 we denote the Laplace-Beltrami operator on $\mathbb{S}^{d-1}$.

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$$H^p = H^p(B^d) := \{U \in \mathcal{H}(B^d) : \|U\|_{H^p} := \sup_{0<r<1} \|U(r\cdot)\|_p < \infty\},$$

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- Harmonic Triebel-Lizorkin spaces $F_p^{sq} = F_p^{sq}(B^d)$

Main theorem

Let $d \geq 2$, $0 < p < \infty$, $s > 0$. There exists $c = c(d, p, s)$ such that for every $U \in B_\tau^{s\tau}$, $1/\tau = s/(d-1) + 1/p$, and every $n \in \mathbb{N}$ there exists $c_k \in \mathbb{R}$, $a_k \in \mathbb{R}^d$, $|a_k| > 1$, $k = 1, 2, \ldots, n$, such that

$$\|U - g\|_{H^p} \leq cn^{-s/(d-1)}\|U\|_{B_\tau^{s\tau}},$$

where

$$g(x) = \sum_{k=1}^{n} c_k |a_k - x|^{-d+2}, \quad d \geq 3,$$

$$g(x) = \sum_{k=1}^{n} c_k \ln 1/|a_k - x|, \quad d = 2.$$

The theorem remains true if $B_\tau^{s\tau}$ is replaced by $F_\tau^{s\tau}$.

Scheme of proof

The proof consists of the following steps:

1. Equivalence between Besov spaces $B_p^{sq}(B^d)$ of harmonic functions on the unit ball B^d and Besov spaces $\mathcal{B}_p^{sq}(\mathbb{S}^{d-1})$ of distributions on the unit spere $\mathbb{S}^{d-1}$
2. Frame theory in quasi-Banach spaces
3. Construction of new frames by "small perturbation" of "nice" existing frames in quasi-Banach spaces
4. Construction of combinations of a fixed number of Newtonian potentials which are well localized on $\mathbb{S}^{d-1}$.
5. Frame elements consisting of a fixed number of Newtonian potentials;
6. Construction of nonlinear n-term approximation of harmonic functions by Newtonian potentials

Spherical harmonics

- The restriction to $\mathbb{S}^{d-1}$ of a harmonic homogenious polynomial in $\mathbb{R}^d$ of degree k is called a spherical harmonic of order k.
- $\mathcal{H}_k = \mathcal{H}_k^d$ denotes the space of all spherical harmonics of order k on $\mathbb{S}^{d-1}$.
- If $f \in \mathcal{H}_k$ and $U(r\xi) = r^k f(\xi)$ for $r \in [0, \infty)$ and $\xi \in \mathbb{S}^{d-1}$, then $\Delta U = 0$.
- The dimension of $\mathcal{H}_k$ is $N(k,d) = \frac{2k+d-2}{k}\binom{k+d-3}{k-1} \sim k^{d-1}$.
- Let $\{Y_{kj} : j = 1, \dots, N(k,d)\}$ be any real valued orthonormal basis for $\mathcal{H}_k$. The kernel $\mathrm{P}_k(x \cdot y)$ of the orthogonal projector onto $\mathcal{H}_k$ has the representation

$$\mathrm{P}_k(x \cdot y) = \sum_{j=1}^{N(k,d)} Y_{kj}(x) Y_{kj}(y), \quad x, y \in \mathbb{S}^{d-1}.$$

Coefficients of harmonic functions

- The coefficients $c_{kj}(U)$ of $U \in \mathcal{H}(B^d)$ are defined by

$$c_{kj}(U) := \frac{1}{\rho^k} \int_{\mathbb{S}^{d-1}} U(\rho\eta) Y_{kj}(\eta) d\sigma(\eta)$$

  for some $0 < \rho < 1$.
- $c_{kj}(U)$ are independent of the choice of $\rho \in (0,1)$.
- $U \in \mathcal{H}(B^d)$ has the representation

$$U(r\xi) = \sum_{k=0}^{\infty} r^k \sum_{j=1}^{N(k,d)} c_{kj}(U) Y_{kj}(\xi), \quad 0 \le r < 1, \ \xi \in \mathbb{S}^{d-1}.$$

- We shall be interested in harmonic functions $U \in \mathcal{H}(B^d)$ such that

$$|c_{kj}(U)| \le c(k+1)^{\mu}, \quad j = 1, \dots, N(k,d), \ k = 0, 1, \dots, \qquad (1)$$

  for some constants $\mu \in \mathbb{R}$ and $c > 0$ depending on U.

For any $U \in \mathcal{H}(B^d)$, obeying (1), and $\beta \in \mathbb{R}$ we define

$$J^\beta U(r\xi) = \sum_{k=0}^{\infty} r^k (k+1)^\beta \sum_{j=1}^{N(k,d)} c_{kj}(U) Y_{kj}(\xi), \quad 0 \le r < 1, \ \xi \in \mathbb{S}^{d-1}.$$

The above series converges absolutely and uniformly on every compact subset of B^d and hence $J^\beta U$ is a well defined harmonic function on B^d. For $\beta > 0$, J^β is called the Weyl derivative of f of order β.

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Definition

Let $s \in \mathbb{R}$, $0 < p, q \le \infty$, and $\beta := s + 1$. The harmonic Besov space B_p^{sq} is defined as the set of all $U \in \mathcal{H}(B^d)$ such that

$$\|U\|_{B_p^{sq}} := \Big(\int_0^1 (1-r)^{(\beta - s)q - 1} \|J^\beta U(r\cdot)\|_{L^p(\mathbb{S}^{d-1})}^q dr \Big)^{1/q} < \infty \quad \text{if } q \ne \infty$$

and

$$\|U\|_{B_p^{s\infty}} := \sup_{0<r<1} (1-r)^{\beta - s} \|J^\beta U(r\cdot)\|_{L^p(\mathbb{S}^{d-1})} < \infty.$$

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Let $s \in \mathbb{R}$, $0 < p < \infty$, $0 < q \leq \infty$, and $\beta := s + 1$. The harmonic Triebel-Lizorkin space F_p^{sq} is defined as the set of all $U \in \mathcal{H}(B^d)$ such that

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- The quasi-norms for B_p^{sq} or F_p^{sq} are independent of the choice of $\{Y_{kj} : j = 1, \dots, N(k,d)\}$ – the real valued orthonormal basis for $\mathcal{H}_k$.
- Choosing an arbitrary $\beta > s$ in the above quantities will give equivalent quasi-norms for B_p^{sq} or F_p^{sq}.
- B_p^{sq} or F_p^{sq} are quasi-Banach spaces (Banach spaces if $p, q \geq 1$)
- The real interpolation methods preserve the system of spaces B_p^{sq} or F_p^{sq} for fixed p.

- Zonal functions on $\mathbb{S}^{d-1}$: $f(x) = F(\xi \cdot x)$ for some $\xi \in \mathbb{S}^{d-1}$ and $F : \mathbb{R} \to \mathbb{R}$.
  Similar functions in $\mathbb{R}^d$: radial or ridge functions.

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- A strange convolution on $\mathbb{S}^{d-1}$:

$$(F * g)(x) = \int_{\mathbb{S}^{d-1}} F(x \cdot y) g(y) d\sigma(y) \quad \forall x \in \mathbb{S}^{d-1}$$

  For $d \geq 3$ a convolution $f * g$ on $\mathbb{S}^{d-1}$ with the usual properties is possible only if one of f and g is zonal. Reason: the group of rotations on $\mathbb{S}^{d-1}$, i.e. in $\mathbb{R}^d$, is not comutative.

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- The class of test functions: $\mathcal{S} := C^\infty(\mathbb{S}^{d-1})$ consisting of all functions ϕ on $\mathbb{S}^{d-1}$ such that

$$\|\mathrm{P}_k * \phi\|_2 \leq c(\phi, m)(1+k)^{-m} \quad \forall k, m \geq 0.$$

  The topology in $\mathcal{S}$ is defined by the sequence of norms

$$Q_m(\phi) := \sum_{k \geq 0} (k+1)^m \|\mathrm{P}_k * \phi\|_2 = \sum_{k \geq 0} (k+1)^m \Big(\sum_{j=1}^{N(k,d)} |\langle \phi, Y_{kj} \rangle|^2 \Big)^{1/2}.$$

Distributions on $\mathbb{S}^{d-1}$

- The space $\mathcal{S}' := \mathcal{S}'(\mathbb{S}^{d-1})$ of distributions on $\mathbb{S}^{d-1}$ is defined as the space of all continuous linear functionals on $\mathcal{S}$.
- The pairing of $f \in \mathcal{S}'$ and $\phi \in \mathcal{S}$ will be denoted by $\langle f, \phi \rangle := f(\phi)$, which is consistent with the inner product $\langle f, g \rangle := \int_{\mathbb{S}^{d-1}} fg d\sigma$ on $L^2(\mathbb{S}^{d-1})$.
- More precisely, $\mathcal{S}'$ consists of all linear functionals f on $\mathcal{S}$ for which there exist constants $c > 0$ and $m \in \mathbb{N}_0$ such that

$$|\langle f, \phi \rangle| \le c Q_m(\phi) \quad \forall \phi \in \mathcal{S}.$$

- For any $f \in \mathcal{S}'$ we define $\mathrm{P}_k * f$ by

$$\mathrm{P}_k * f(x) := \langle f, \mathrm{P}_k(x \cdot \bullet) \rangle.$$

Hence $\mathrm{P}_k * f \in \mathcal{H}_k$ and for some $c > 0$ and $m \in \mathbb{N}_0$ we have

$$\|\mathrm{P}_k * f\|_2 \le c(k+1)^m \quad \forall k \ge 0,$$

$$|\langle f, Y_{kj} \rangle| \le c(k+1)^m \quad \forall k \ge 0,\ j = 1, \dots, N(k,d).$$

Theorem

(a) To any $U \in \mathcal{H}(B^d)$ with coefficients satisfying (1) there corresponds a distribution $f \in \mathcal{S}'$ (the boundary value function) defined by

$$f := \sum_{k \geq 0} \sum_{j=1}^{N(k,d)} c_{kj}(U) Y_{kj} \quad \text{(convergence in } \mathcal{S}')$$

with coefficients $\langle f, Y_{kj} \rangle = c_{kj}(U)$.

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(b) To any distribution $f \in \mathcal{S}'$ with coefficients $c_{kj}(f) := \langle f, Y_{kj}\rangle$ there corresponds a harmonic function $U \in \mathcal{H}(B^d)$ (the extension of f to B^d) defined by

$$U(x) = \sum_{k=0}^{\infty} \sum_{j=1}^{N(k,d)} c_{kj}(f)|x|^k Y_{kj}\Big(\frac{x}{|x|}\Big), \quad |x| < 1,$$

with coefficients $c_{kj}(U) = c_{kj}(f)$ obeying (1), where the series converges uniformly on every compact subset of B^d.

Littlewood-Paley decomposition of distributions

Let the cut-off function $\varphi \in C^\infty[0,\infty)$ be such that *supp* $\varphi \subset [1/2, 2]$, $\varphi(t) > 0$ for $t \in [3/5, 5/3]$, and

$$\sum_{j=1}^{\infty} \varphi(2^{-j}t) = 1 \quad \text{for } t \in [1,\infty).$$

Set

$$\Phi_0 := \mathrm{P}_0 \ \text{ and } \ \Phi_j := \sum_{k=0}^{\infty} \varphi\Big(\frac{k}{2^{j-1}}\Big)\mathrm{P}_k, \quad j = 1, 2, \dots .$$

Littlewood-Paley decomposition of distributions

$$\sum_{j \geq 0} \Phi_j * f = f \quad \text{for all } f \in \mathcal{S}' \text{ (convergence in } \mathcal{S}').$$

Let $s \in \mathbb{R}$ and $0 < q \le \infty$.

Besov spaces on $\mathbb{S}^{d-1}$

The Besov space $\mathcal{B}_p^{sq} := \mathcal{B}_p^{sq}(\mathbb{S}^{d-1})$, $0 < p \le \infty$, is defined as the set of all distributions $f \in \mathcal{S}'$ such that

$$\|f\|_{\mathcal{B}_p^{sq}} := \Big(\sum_{j=0}^{\infty} \Big(2^{sj} \|\Phi_j * f\|_{L^p(\mathbb{S}^{d-1})} \Big)^q \Big)^{1/q} < \infty,$$

where the ℓ^q-norm is replaced by the sup-norm if $q = \infty$.

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Triebel-Lizorkin spaces on $\mathbb{S}^{d-1}$

The Triebel-Lizorkin space $\mathcal{F}_p^{sq} := \mathcal{F}_p^{sq}(\mathbb{S}^{d-1})$, $0 < p < \infty$, is defined as the set of all distributions $f \in \mathcal{S}'$ such that

$$\|f\|_{\mathcal{F}_p^{sq}} := \Big\| \Big(\sum_{j=0}^{\infty} \big(2^{sj} |\Phi_j * f(\cdot)| \big)^q \Big)^{1/q} \Big\|_{L^p(\mathbb{S}^{d-1})} < \infty,$$

where the ℓ^q-norm is replaced by the sup-norm if $q = \infty$.

- All cut-off functions φ in the above quantities will give equivalent quasi-norms for $\mathcal{B}_p^{sq}$ or $\mathcal{F}_p^{sq}$.
- $\mathcal{B}_p^{sq}$ and $\mathcal{F}_p^{sq}$ are quasi-Banach spaces (Banach spaces if $p, q \geq 1$)
- $\mathcal{B}_p^{sq}$ and $\mathcal{F}_p^{sq}$ are continuously embedded in $\mathcal{S}'$, i.e. for any $s \in \mathbb{R}$ and $0 < p, q \leq \infty$ there exist constants $c > 0$ and $m \in \mathbb{N}_0$ such that

$$|\langle f, \phi \rangle| \leq c \|f\|_{\mathcal{B}_p^{sq}} Q_m(\phi), \quad \forall f \in \mathcal{B}_p^{sq} \ \forall \phi \in \mathcal{S},$$

and similarly for the Triebel-Lizorkin spaces $\mathcal{F}_p^{sq}$.

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and similarly for the Triebel-Lizorkin spaces $\mathcal{F}_p^{sq}$.

Theorem

(*a*) Let $s \in \mathbb{R}$, $0 < p, q \leq \infty$. A harmonic function $U \in B_p^{sq}$ if and only if its boundary value distribution $f = f_U$ belongs to $\mathcal{B}_p^{sq}$, moreover $\|U\|_{B_p^{sq}} \sim \|f\|_{\mathcal{B}_p^{sq}}$.
(*b*) Let $s \in \mathbb{R}$, $0 < p < \infty$, $0 < q \leq \infty$. A harmonic function $U \in F_p^{sq}$ if and only if its boundary value distribution $f = f_U$ belongs to $\mathcal{F}_p^{sq}$, moreover $\|U\|_{F_p^{sq}} \sim \|f\|_{\mathcal{F}_p^{sq}}$.

- Embeddings for Besov and Triebel-Lizorkin spaces;
- Classical spaces on $\mathbb{S}^{d-1}$:
  - $\mathcal{F}_p^{02} \approx L_p$ if $1 < p < \infty$,
  - $\mathcal{F}_p^{02} \approx H^p$ if $0 < p \leq 1$,
  - $\mathcal{F}_p^{s2} \approx W_p^s$ Sobolev spaces for integer s,
  - $\mathcal{F}_p^{s2} \approx$ Bessel potential spaces;
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Besov and Triebel-Lizorkin spaces on B^d and $\mathbb{S}^{d-1}$:

- Greenwald 1974, 1977, Besov spaces termed "Lipschitz spaces" on $\mathbb{S}^{d-1}$, $1 \leq p, q \leq \infty$;
- Oswald 1983, $d = 2$, "On Besov-Hardy-Sobolev spaces of analytic functions in the unit disc"

Quasi-Banach spaces

Denote by $\mathcal{S} := C^\infty(\mathbb{S}^{d-1})$ and let $\mathcal{S}'$ be its dual. Let $\mathfrak{B} = \mathfrak{B}(\mathbb{S}^{d-1}) \subset \mathcal{S}'$ be a quasi-Banach space of distributions on $\mathbb{S}^{d-1}$ with quasi-norm $\|\cdot\|_{\mathfrak{B}}$, which is continuously embedded in $\mathcal{S}'$. Further, we assume that $\mathcal{S}$ is a dense subset of $\mathfrak{B}$.

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We also assume that $\mathfrak{b} = \mathfrak{b}(\mathcal{X})$ with quasi-norm $\|\cdot\|_{\mathfrak{b}}$ is an associated to $\mathfrak{B}$ quasi-Banach space of real-valued sequences with domain a countable index set $\mathcal{X}$. Coupled with a frame Ψ the sequence space $\mathfrak{b}$ will be utilized for characterization of the space $\mathfrak{B}$.

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The triangle inequalities in $\mathfrak{B}$ and $\mathfrak{b}$ are $(\kappa^* \ge 1)$

$$\|f_1 + f_2\|_{\mathfrak{B}} \le \kappa^*(\|f_1\|_{\mathfrak{B}} + \|f_2\|_{\mathfrak{B}}), \ \forall f_1, f_2 \in \mathfrak{B},$$

$$\|h_1 + h_2\|_{\mathfrak{b}} \le \kappa^*(\|h_1\|_{\mathfrak{b}} + \|h_2\|_{\mathfrak{b}}), \ \forall h_1, h_2 \in \mathfrak{b}.$$

Examples: $\mathfrak{B} = \mathcal{B}_p^{sq}(\mathbb{S}^{d-1})$, $\mathfrak{b} = b_p^{sq}(\mathcal{X})$; $\mathfrak{B} = \mathcal{F}_p^{sq}(\mathbb{S}^{d-1})$, $\mathfrak{b} = f_p^{sq}(\mathcal{X})$

Old frame

$\Psi := \{\psi_\xi : \xi \in \mathcal{X}\} \subset \mathcal{S}$ is a tight normalized frame in L^2:

$$\|f\|_{L^2} = \| \langle f, \psi_\xi \rangle \|_{\ell^2} \quad \forall f \in L^2.$$

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We also assume that Ψ is a frame for $\mathfrak{B}$ in the following sense:

A1. For any $f \in \mathfrak{B}$

$$f = \sum_{\xi \in \mathcal{X}} \langle f, \psi_\xi \rangle \, \psi_\xi \quad \text{in } \mathfrak{B};$$

A2. For any $f \in \mathfrak{B}$ the sequence $\{\langle f, \psi_\xi \rangle\} \in \mathfrak{b}(\mathcal{X})$ and there exist constants $A^*, B^* > 0$ such that

$$A^* \|f\|_{\mathfrak{B}} \le \| \langle f, \psi_\xi \rangle \|_{\mathfrak{b}(\mathcal{X})} \le B^* \|f\|_{\mathfrak{B}}.$$

We call Ψ the "old frame". Examples for Ψ: needlet systems.

Frames in Hilbert spaces

Our aim is by using the idea of "small perturbation argument" to construct a new system $\Theta := \{\theta_\xi : \xi \in \mathcal{X}\} \subset \mathcal{S}$ called a "new frame" with some prescribed features and which is: (i) a frame for L^2 and (ii) a frame for $\mathfrak{B}$.

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Θ is a frame for L^2 if there exist constants $A, B > 0$ such that

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Frames: Duffin, Schaeffer 1952; Daubechies 1985–1993; Meyer 1985–1990; Mallat 1989–1992.

Definition. $\Theta := \{\theta_\xi : \xi \in \mathcal{X}\} \subset L^2$ is a frame for the quasi-Banach space $\mathfrak{B}$ with associated sequence space $\mathfrak{b}$ if:

B1. There exist constants $A_1^*, B_1^* > 0$ such that

$$A_1^* \|f\|_{\mathfrak{B}} \le \| \langle f, \theta_\xi \rangle \|_{\mathfrak{b}} \le B_1^* \|f\|_{\mathfrak{B}} \quad \forall f \in \mathfrak{B},$$

where $\langle f, \theta_\xi \rangle$ is defined by $\langle f, \theta_\xi \rangle := \sum_{\eta \in \mathcal{X}} \langle f, \psi_\eta \rangle \langle \psi_\eta, \theta_\xi \rangle$.

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B2. The frame operator $S : \mathfrak{B} \mapsto \mathfrak{B}$ defined by $Sf = \sum_{\xi \in \mathcal{X}} \langle f, \theta_\xi \rangle \theta_\xi$ is bounded and invertible on $\mathfrak{B}$; S^{-1} is also bounded on $\mathfrak{B}$ and

$$S^{-1} f = \sum_{\xi \in \mathcal{X}} \langle f, S^{-1}\theta_\xi \rangle S^{-1}\theta_\xi \quad \text{in } \mathfrak{B}.$$

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$$A_2^* \|f\|_{\mathfrak{B}} \le \| \left\langle f, S^{-1}\theta_\xi \right\rangle \|_{\mathfrak{b}} \le B_2^* \|f\|_{\mathfrak{B}} \quad \forall\, f \in \mathfrak{B},$$

where as above $\left\langle f, S^{-1}\theta_\xi \right\rangle := \sum_{\eta \in \mathcal{X}} \langle f, \psi_\eta \rangle \left\langle \psi_\eta, S^{-1}\theta_\xi \right\rangle$.

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B4. For any $f \in \mathfrak{B}$ we have

$$f = \sum_{\xi \in \mathcal{X}} \langle f, S^{-1}\theta_\xi \rangle \theta_\xi = \sum_{\xi \in \mathcal{X}} \langle f, \theta_\xi \rangle S^{-1}\theta_\xi \quad \text{in } \mathfrak{B}.$$

Remark. If $\mathfrak{B} = L^2$, then **B2**, **B3** and **B4** follow from **B1**.

Given the old frame Ψ and a new frame Θ we set

$$\begin{aligned}
\mathbf{A} &:= (a_{\xi,\eta})_{\xi,\eta\in\mathcal{X}}, \quad a_{\xi,\eta} := \langle \psi_\eta, \psi_\xi \rangle, \\
\mathbf{B} &:= (b_{\xi,\eta})_{\xi,\eta\in\mathcal{X}}, \quad b_{\xi,\eta} := \langle \theta_\eta, \psi_\xi \rangle, \\
\mathbf{C} &:= (c_{\xi,\eta})_{\xi,\eta\in\mathcal{X}}, \quad c_{\xi,\eta} := \langle \psi_\eta, \theta_\xi \rangle, \\
\mathbf{D} &:= (d_{\xi,\eta})_{\xi,\eta\in\mathcal{X}}, \quad d_{\xi,\eta} := \langle \psi_\eta, \psi_\xi - \theta_\xi \rangle, \\
\mathbf{E} &:= (e_{\xi,\eta})_{\xi,\eta\in\mathcal{X}}, \quad e_{\xi,\eta} := \langle \psi_\eta - \theta_\eta, \psi_\xi \rangle.
\end{aligned}$$

Let the operators with matrices **A**, **D**, **E** be bounded on $\ell^2(\mathcal{X})$ and on $\mathfrak{b}$:

$$\|\mathbf{A}\|_{\ell^2\to\ell^2} \le C_1, \quad \|\mathbf{D}\|_{\ell^2\to\ell^2} \le \gamma, \quad \|\mathbf{E}\|_{\ell^2\to\ell^2} \le \gamma;$$

$$\|\mathbf{A}\|_{\mathfrak{b}\to\mathfrak{b}} \le C_1^*, \quad \|\mathbf{D}\|_{\mathfrak{b}\to\mathfrak{b}} \le \gamma^*, \quad \|\mathbf{E}\|_{\mathfrak{b}\to\mathfrak{b}} \le \gamma^*.$$

In view of $\mathbf{C} = \mathbf{A} - \mathbf{D}, \mathbf{B} = \mathbf{A} - \mathbf{E}$ we get

$$\|\mathbf{B}\|_{\ell^2\to\ell^2} \le C_1 + \gamma, \quad \|\mathbf{C}\|_{\ell^2\to\ell^2} \le C_1 + \gamma;$$

$$\|\mathbf{B}\|_{\mathfrak{b}\to\mathfrak{b}} \le \kappa^*(C_1^* + \gamma^*), \quad \|\mathbf{C}\|_{\mathfrak{b}\to\mathfrak{b}} \le \kappa^*(C_1^* + \gamma^*),$$

i.e. the operators with matrices **B**, **C** are also bounded on $\ell^2(\mathcal{X})$ and on $\mathfrak{b}$.

Theorem

If

$$\gamma < 1$$

then Θ is a frame for L^2.

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Conclusion. Operators $\mathbf{D} = \{\langle \psi_\eta, \psi_\xi - \theta_\xi \rangle\}$ and $\mathbf{E} = \{\langle \psi_\eta - \theta_\eta, \psi_\xi \rangle\}$ must have small norms in L^2 and in $\mathfrak{B}$.

Hence, the name “small perturbation”.

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Hence, the name "small perturbation".

Problem. How to compute or bounded the operator norm?
Localized functions and frames.

Localized functions

Definition. The function f defined on $\mathbb{S}^{d-1}$ is localized around $x_0 \in \mathbb{S}^{d-1}$ with dilation factor $N \geq 1$ and decay rate $M > 0$ if the estimate

$$|f(x)| \leq cN^{d-1}(1 + N\rho(x_0, x))^{-M}, \quad x \in \mathbb{S}^{d-1},$$

holds for some number c depending only on d and M.

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The multiplier N^{d-1} is used as part of the decay function in order to normalize it in $L(\mathbb{S}^{d-1})$. Namely, for $M > d - 1$ we have

$$\int_{\mathbb{S}^{d-1}} |f(y)| d\sigma(y) \leq \int_{\mathbb{S}^{d-1}} \frac{N^{d-1}}{(1 + N\rho(x_0, y))^M} d\sigma(y) \leq c_0, \quad \forall x_0 \in \mathbb{S}^{d-1},$$

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where c_0 depends only on d and M.
We shall also require from the localized functions

$$\int_{\mathbb{S}^{d-1}} f(y) d\sigma(y) = 1,$$

which infers that they may have only moderate osculation.

Needlet kernels

Let $\varphi \in C^\infty[0,\infty)$ be supported in $[1/2,2]$ and

$$\varphi^2(t/2)+\varphi^2(t)=1, \quad t\in[1,2].$$

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The needlet kernel Ψ_N is defined by

$$\Psi_N(u) := \sum_{\nu=0}^{\infty} \varphi\left(\frac{\nu}{N}\right) \mathrm{P}_\nu(u) = \sum_{\nu=N/2}^{2N} \varphi\left(\frac{\nu}{N}\right) \mathrm{P}_\nu(u),$$

where P_ν is an algebraic polynomial of degree ν, such that $\mathrm{P}_\nu(x\cdot y)$ is the kernel of the orthogonal projector onto $\mathcal{H}_\nu^d$.

$$\mathrm{P}_\nu(u) = \frac{2\nu + d - 2}{(d-2)\sigma(\mathbb{S}^{d-1})} C_\nu^{(d/2-1)}(u),$$

where $\sigma(\mathbb{S}^{d-1}) = 2\pi^{d/2}/\Gamma(d/2)$ is the hypersurface area of $\mathbb{S}^{d-1}$ and $C_\nu^{(\lambda)}$ is the Gegenbauer (ultraspherical) polynomial of degree ν normalized with $C_\nu^{(\lambda)}(1) = \binom{\nu+2\lambda-1}{\nu}$.

Localization of needlet kernels

Given $\xi \in \mathbb{S}^{d-1}$ we extend $\Psi_N(\xi \cdot x)$ for $x \in \mathbb{R}^d \backslash \{0\}$ by

$$\tilde{\Psi}_N(\xi; x) = \Psi_N(\xi \cdot (x/|x|)).$$

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Theorem

For $M > 0$, $K \in \mathbb{N}_0$, multiindex β, $0 \le |\beta| \le K$, $\xi \in \mathbb{S}^{d-1}$ we have

$$\left|\partial^\beta \tilde{\Psi}_N(\xi; x)\right| \le c(d, K, M) \frac{N^{|\beta|+d-1}}{(1 + N\rho(\xi, x))^M}, \quad x \in \mathbb{S}^{d-1}.$$

Localized frames of needlets

For $j = 0, 1, 2, \dots$ let $\mathcal{X}_j$ denote a set of $O(2^{j(d-1)})$ points on $\mathbb{S}^{d-1}$, which are nodes of a cubature with positive weights of high degree of exactness. The index set is $\mathcal{X} = \cup_{j=0}^{\infty} \mathcal{X}_j$. For every $\xi \in \mathcal{X}$ set $N_\xi = 2^j$.

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Using kernels Ψ_N we define the needlet frame $\Psi = \{\psi_\xi(x) : \xi \in \mathcal{X}\} \cup \{1\}$ by

$$\psi_\xi^\diamond(x) = \Psi_{N_\xi}(\xi \cdot x), \quad \psi_\xi(x) = C_\xi^\diamond \psi_\xi^\diamond(x), \quad x \in \mathbb{S}^{d-1},\ \xi \in \mathcal{X},$$

where coefficients $C_\xi^\diamond$ satisfy

$$C_\xi^\diamond \le c(d) N_\xi^{-(d-1)/2}, \quad \xi \in \mathcal{X}.$$

$\psi_\xi^\diamond$ is normalized in $L_1(\mathbb{S}^{d-1})$.
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Needlet frames on $\mathbb{S}^{d-1}$: Narcowich, Petrushev, Wards 2006

Function and sequence Besov spaces

Let $s \in \mathbb{R}$ and $0 < q \leq \infty$.

Function Besov space $\mathcal{B}_p^{sq}(\mathbb{S}^{d-1})$, $0 < p \leq \infty$

$$\|f\|_{\mathcal{B}_p^{sq}(\mathbb{S}^{d-1})} := \Big(\sum_{j=0}^{\infty} \Big(2^{sj} \|\Phi_j * f\|_{L^p(\mathbb{S}^{d-1})} \Big)^q \Big)^{1/q} < \infty.$$

Sequence Besov space $b_p^{sq}(\mathcal{X})$, $0 < p \leq \infty$

$$\|\{h_\xi\}\|_{b_p^{sq}(\mathcal{X})} := \Big(\sum_{j=0}^{\infty} 2^{j(s+(d-1)/2-(d-1)/p)q} \Big(\sum_{\xi \in \mathcal{X}_j} |h_\xi|^p \Big)^{q/p} \Big)^{1/q} < \infty.$$

Almost diagonal matrix

Definition. The matrix $\Omega_{K,M} := \{\omega_{\xi,\eta}\}_{\xi,\eta\in\mathcal{X}}$ with entries

$$\omega_{\xi,\eta} := \left(\frac{\min\{N_\xi, N_\eta\}}{\max\{N_\xi, N_\eta\}}\right)^{K+(d-1)/2} \frac{1}{\big(1+\min\{N_\xi, N_\eta\}\rho(\xi,\eta)\big)^M}.$$

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Theorem

Suppose $s \in \mathbb{R}, 0 < q \le \infty$, $0 < p < \infty$. Set $\mathcal{J} = (d-1)/\min\{1, p, q\}$. For a fixed $\delta > 0$ assume that $K, M \in \mathbb{N}$, $K \ge \max\{s, \mathcal{J} - s - d + 1\} + \delta$ and $M \ge \mathcal{J} + \delta$. Then $\Omega_{K,M}$ is a bounded operator on f_p^{sq}.

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Sufficient condition for a new frame

Theorem

Under the above conditions if $\mathfrak{b}$ is one of the spaces f_p^{sq} or b_p^{sq} and

$$|\langle \psi_\eta, \psi_\xi - \theta_\xi \rangle| \leq \gamma_0 \omega_{\xi,\eta}, \quad \forall \xi, \eta \in \mathcal{X},$$

then Θ is a frame for $\mathfrak{B}$ provided $\gamma_0 \leq \gamma^* / \|\Omega_{K,M}\|_{\mathfrak{b}\to\mathfrak{b}}$.

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Similar theory for $\mathbb{R}^d$: the φ-transform of Frazier, Jawerth, 1985, 1990.
Theory for $\mathbb{S}^{d-1}$: Kyriazis, Petrushev, 2014.

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Reformulated sufficient condition

$$|\langle\psi_\eta^\diamond, \psi_\xi^\diamond - \theta_\xi^\diamond\rangle| \le \gamma_0 \omega_{\xi,\eta}^\diamond, \quad \forall \xi, \eta \in \mathcal{X},$$

$$\omega_{\xi,\eta}^\diamond := \left(\frac{\min\{N_\xi, N_\eta\}}{\max\{N_\xi, N_\eta\}}\right)^K \frac{\min\{N_\xi, N_\eta\}^{d-1}}{(1 + \min\{N_\xi, N_\eta\}\rho(\xi,\eta))^M}.$$

Poisson kernel

$\varepsilon > 0$, $a = 1 + \varepsilon$, $\eta \in \mathbb{S}^{d-1}$.
Newtonian potential with pole at $a\eta$:

$$F(a\eta, x) = |a\eta - x|^{-d+2}$$

Localization of F on $\mathbb{S}^{d-1}$:

$$|\varepsilon^{-1} F(a\eta, x)| \leq c(d) \frac{\varepsilon^{-d+1}}{(1 + \varepsilon^{-1}\rho(\eta, x))^{d-2}} \quad \forall x \in \mathbb{S}^{d-1}.$$

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Poisson kernel with pole at $a\eta$:

$$a^{d-1}\sigma(\mathbb{S}^{d-1}) P(a\eta, x) = \frac{a^2 - |x|^2}{|a\eta - x|^d} = \frac{2a}{d-2}(\eta \cdot \nabla) \frac{1}{|a\eta - x|^{d-2}} - \frac{1}{|a\eta - x|^{d-2}}$$

Localization of P on $\mathbb{S}^{d-1}$:

$$|P(a\eta, x)| \le c(d) \frac{\varepsilon^{-d+1}}{(1 + \varepsilon^{-1}\rho(\eta, x))^{d}} \quad \forall x \in \mathbb{S}^{d-1}.$$

Theorem

Let $d \geq 3$, $\varepsilon > 0$, $a = 1 + \varepsilon$, $\delta = (a^2 - 1)/a^2 \sim \varepsilon$, $\eta \in \mathbb{S}^{d-1}$, $K, m \in \mathbb{N}_0$. There exist coefficients $q_\ell = q_\ell(d, m, \delta) = \sum_{k=0}^{m-\ell} \alpha_{\ell,k}(d, m)\delta^k$ such that the harmonic function

$$F_m(a\eta, x) := -q_{-1}|a\eta - x|^{2-d} + \sum_{\ell=0}^{m} \frac{q_\ell \delta^\ell a^{\ell+1}}{d-2}(\eta \cdot \nabla)^{\ell+1}|a\eta - x|^{2-d}$$

satisfies

$$\int_{\mathbb{S}^{d-1}} F_m(a\eta, x)d\sigma(x) = 1,$$

$$\left|\partial^\beta \tilde{F}_m(a\eta; x)\right| \leq c(d, K, m)\frac{\varepsilon^{1-d-|\beta|}}{(1 + \varepsilon^{-1}\rho(\xi, x))^{d+2m}}, \quad \forall x \in \mathbb{S}^{d-1},$$

for every multiindex β, $0 \leq |\beta| \leq K$, where

$$\tilde{F}_m(a\eta; y) := F_m(a\eta, (y/|y|)), \quad y \in \mathbb{R}^d \setminus \{0\}.$$

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$$F_m(a\eta, x) := -q_{-1} + \sum_{\ell=0}^{m} q_\ell \delta^\ell a^{\ell+1}(\eta \cdot \nabla)^{\ell+1} \ln 1/|a\eta - x|, \text{ for } d = 2.$$

$d = 2$, $\varepsilon > 0$, $a = e^{\varepsilon}$, $\eta \in \mathbb{S}^{d-1}$, $m \in \mathbb{N}_0$.

$$F_m(a\eta, x) = -1 + \sum_{\ell=0}^{m} \left(\sum_{k=\ell}^{m} \beta(m,k)\alpha(k,\ell)\varepsilon^k \right) \frac{2a^{\ell+1}}{\ell!} (\eta \cdot \nabla)^{\ell+1} \ln 1/|a\eta - x|,$$

$$\beta(m,k) := \frac{2^k (2m-k)! m!}{k!(m-k)!(2m)!}, \quad \alpha(k,\ell) := \sum_{\nu=\ell}^{k} (-1)^{\nu-\ell} \binom{\nu}{\ell} S(k,\nu)\nu!,$$

where $S(k,\nu)$ are Stirling numbers of the second kind.

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where $S(k,\nu)$ are Stirling numbers of the second kind.

$$F_1(a\eta, x)$$
$$= -1 + (2 - 2\varepsilon)a(\eta \cdot \nabla) \ln 1/|a\eta - x| + 2\varepsilon a^2 (\eta \cdot \nabla)^2 \ln 1/|a\eta - x|$$

$$F_2(a\eta, x) = -1 + (2 - 2\varepsilon + \frac{2}{3}\varepsilon^2) a(\eta \cdot \nabla) \ln 1/|a\eta - x|$$
$$+ (2\varepsilon - 2\varepsilon^2) a^2 (\eta \cdot \nabla)^2 \ln 1/|a\eta - x| + \frac{2}{3}\varepsilon^2 a^3 (\eta \cdot \nabla)^3 \ln 1/|a\eta - x|$$

Frame of Newtonian potentials

We start the construction of frame elements $\{\theta_\xi : \xi \in \mathcal{X}\}$ of the form

$$\theta_\xi = \sum_{\nu=1}^{n_0} \frac{c_\nu}{|x - y_\nu|^{d-2}} \text{ if } d > 2; \quad \theta_\xi = \sum_{\nu=1}^{n_0} c_\nu \ln 1/|x - y_\nu| \text{ if } d = 2.$$

Here $y_\nu \in \mathbb{R}^d$ with $|y_\nu| > 1$, $c_\nu \in \mathbb{R}$, and $\{y_\nu\}_{\nu=1}^{n_0}$ and $\{c_\nu\}_{\nu=1}^{n_0}$ vary with $\xi \in \mathcal{X}$, but n_0 is fixed.

Recall that $\{\psi_\xi : \xi \in \mathcal{X}\}$ with $\mathcal{X} = \cup_{j\geq 0}\mathcal{X}_j$ is the existing old frame.

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Sufficient condition (repeated)

If for a sufficiently small γ_0 we have

$$|\langle \psi_\eta^\diamond, \psi_\xi^\diamond - \theta_\xi^\diamond \rangle| \leq \gamma_0 \omega_{\xi,\eta}^\diamond, \quad \forall \xi, \eta \in \mathcal{X},$$

$$\omega_{\xi,\eta}^\diamond := \left(\frac{\min\{N_\xi, N_\eta\}}{\max\{N_\xi, N_\eta\}} \right)^K \frac{\min\{N_\xi, N_\eta\}^{d-1}}{(1 + \min\{N_\xi, N_\eta\}\rho(\xi, \eta))^M},$$

then Θ is a frame for F_p^{sq} or B_p^{sq}.

Technical theorem

Let $K \in \mathbb{N}$, $M > K + d - 1$, $N_1, N_2 \in \mathbb{R}$, $N_2 \geq N_1 \geq 1$, $\kappa_1, \kappa_2 > 0$. Assume $f \in L^\infty(\mathbb{S}^{d-1})$, $g \in W^K_\infty(\mathbb{S}^{d-1})$, and $\tilde{g}(x) := g(x/|x|)$ for $x \in \mathbb{R}^d \backslash \{0\}$. Furthermore, assume that for some $x_1, x_2 \in \mathbb{S}^{d-1}$

$$\left|\partial^\beta \tilde{g}(x)\right| \leq \frac{\kappa_1 N_1^{|\beta|+d-1}}{(1 + N_1\rho(x_1, x))^M}, \quad \forall x \in \mathbb{S}^{d-1},\ 0 \leq |\beta| \leq K,$$

$$|f(x)| \leq \frac{\kappa_2 N_2^{d-1}}{(1 + N_2\rho(x_2, x))^M}, \quad \forall x \in \mathbb{S}^{d-1}, \quad \text{and}$$

$$\left|\int_{\mathbb{S}^{d-1}} x^\beta f(x)\, d\sigma(x)\right| \leq \kappa_2 N_2^{-K}, \quad 0 \leq |\beta| \leq K - 1.$$

Then

$$|\langle g, f\rangle| = \left|\int_{\mathbb{S}^{d-1}} g(x) f(x)\, d\sigma(x)\right| \leq c\frac{\kappa_1\kappa_2 (N_1/N_2)^K N_1^{d-1}}{(1 + N_1\rho(x_1, x_2))^M},$$

where c depends only on d, K, and M.

Construction scheme

It will be convenient to approximate the L^1 normalized frame elements $\psi_\xi^\diamond(x) := \Psi_{N_\xi}(\xi \cdot x)$, by L^1 normalized new frame elements $\{\theta_\xi^\diamond\}$. The constructions of the new frame elements $\{\theta_\xi^\diamond\}$ will be carried out in four steps:

(a) Approximation of $\Psi_{N_\xi}(\xi \cdot x)$, $\xi \in \mathcal{X}$, by convolving with the potential F_ε for appropriate values of ε.

(b) Discretization of the convolutions by using simple cubature weights.

(c) Truncation of the resulting sums.

(d) Approximation of the truncated sums by discrete versions of the operators involved.

These approximation steps will be governed by four small parameters: γ_1, γ_2, γ_3, γ_4.

For $N = 2^j$ and $K \in 2\mathbb{N}$, we set

$$\Psi_N(u) := \sum_{k=0}^{\infty} \varphi\left(\frac{k}{N}\right) \mathrm{P}_k(u) = \sum_{k=N/2}^{2N} \varphi\left(\frac{k}{N}\right) \mathrm{P}_k(u),$$

and

$$\Phi_N(u) := (-1)^{K/2} \sum_{k=N/2}^{2N} \varphi\Big(\frac{k}{N}\Big)[k(k+d-1)]^{-K/2}\mathrm{P}_k(u),$$

where $\mathrm{P}_k(x \cdot y)$ is the kernel of the orthogonal projector onto $\mathcal{H}_k^d$. Hence

$$-\Delta_0 \mathrm{P}_k(\xi \cdot x) = k(k+d-1)\mathrm{P}_k(\xi \cdot x)$$

implying

$$\Delta_0^{K/2}\Phi_N(\xi \cdot x) = \Psi_N(\xi \cdot x), \quad \xi, x \in \mathbb{S}^{d-1}.$$

Here Δ_0 is the Laplace-Beltrami operator on $\mathbb{S}^{d-1}$.
For any $\xi \in \mathbb{S}^{d-1}$ and $M > 0$ we have

$$\left|\partial^\beta \tilde{\Phi}_N(\xi; x)\right| \le c\frac{N^{-K+|\beta|+d-1}}{(1+N\rho(x_0,x))^M}, \quad x \in \mathbb{S}^{d-1},\ 0 \le |\beta| \le K,$$

For $0 < \gamma_1 \le 1$, set $\varepsilon := \gamma_1 / N_\xi$ and define

$$g_1(\xi; x) := \int_{\mathbb{S}^{d-1}} \Phi_{N_\xi}(\xi \cdot y) F_\varepsilon(y \cdot x)\, d\sigma(y), \quad x \in \mathbb{S}^{d-1}.$$

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For $0 < \gamma_2 \le \gamma_1$, let $\mathcal{Z}_j \subset \mathbb{S}^{d-1}$ be a fixed maximal $\gamma_2 2^{-j}$-net. Applying the cubature formula with nodes $\zeta \in \mathcal{Z}_j$ and weights w_ζ we obtain

$$g_2(\xi; x) := \sum_{\zeta \in \mathcal{Z}_j} w_\zeta \Phi_{N_\xi}(\xi \cdot \zeta) F_\varepsilon(\zeta \cdot x), \quad x \in \mathbb{S}^{d-1}.$$

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For $0 < \gamma_3 \le 1$, truncating the above sum to the nodes within distance $\delta_j := (\gamma_3 N_\xi)^{-1}$ from ξ we get

$$g_3(\xi;x) := \sum_{\substack{\zeta\in\mathcal{Z}_j\\ \rho(\zeta,\xi)\le\delta_j}} w_\zeta \Phi_{N_\xi}(\xi\cdot\zeta)F_\varepsilon(\zeta\cdot x), \quad x \in \mathbb{S}^{d-1}.$$

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The functions $g_1(\xi; x)$, $g_2(\xi; x)$ and $g_3(\xi; x)$ should be viewed as consecutive approximations of $\Phi_{N_\xi}(\xi \cdot x)$.

We obtain consecutive approximations of $\Psi_{N_\xi}(\xi\cdot x)$ by applying $\Delta_0^{K/2}$ in the definitions of g_1, g_2, g_3. We set

$$h_1(\xi;x) := \Delta_0^{K/2} g_1(\xi;x) = \int_{\mathbb{S}^{d-1}} \Phi_{N_\xi}(\xi\cdot y)\Delta_0^{K/2} F_\varepsilon(y\cdot x)\,d\sigma(y)$$

$$= \int_{\mathbb{S}^{d-1}} \Psi_{N_\xi}(\xi\cdot y) F_\varepsilon(y\cdot x)\,d\sigma(y) = \int_{\mathbb{S}^{d-1}} \Psi_{N_\xi}(x\cdot y) F_\varepsilon(y\cdot \xi)\,d\sigma(y),$$

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We used that the operator Δ_0 is self-adjoined and the commutativity of the scalar product of zonal functions.

Observe that h_1 is a zonal function, while, in general, h_2 and h_3 are not zonal functions. Furthermore, $h_3(\xi;x)$ is a linear combination of finitely many (independent of ξ) terms of type $\Delta_0^{K/2}(\zeta\cdot\nabla)^\ell\,|(1+\varepsilon)\zeta - x|^{-d+2}$.

Approximation of the Laplace-Beltrami operator

The rotation $Q_{1,2,t} \in SO(d)$ is given by

$$Q_{1,2,t}\zeta = Q_{1,2,t}(\zeta_1, \zeta_2, \dots, \zeta_d)$$
$$:= (\zeta_1 \cos t + \zeta_2 \sin t, -\zeta_1 \sin t + \zeta_2 \cos t, \zeta_3, \dots, \zeta_d), \quad \zeta \in \mathbb{S}^{d-1},$$

and $Q_{i,\ell,t}\zeta$ is defined similarly for every $1 \le i < \ell \le d$. The translation operator corresponding to the rotation $Q_{i,\ell,t}$, $1 \le i < \ell \le d$, is given by

$$T(Q_{i,\ell,t})f(\zeta) := f(Q_{i,\ell,t}^{-1}\zeta) = f(Q_{i,\ell,-t}\zeta).$$

Define the operator $\mathfrak{L}_t$ by

$$\mathfrak{L}_t f(\zeta) := t^{-2} \sum_{1 \le i < \ell \le d} (T(Q_{i,\ell,t})f(\zeta) + T(Q_{i,\ell,-t})f(\zeta) - 2f(\zeta)).$$

Then $\mathfrak{L}_t$ approximates $\Delta_0 f(\zeta)$ for small t.
The powers of $\mathfrak{L}_t$ are defined iteratively by $\mathfrak{L}_t^k := \mathfrak{L}_t(\mathfrak{L}_t^{k-1})$.

Definition of a new frame element

The finite difference operator $\mathfrak{D}_t^m(\zeta) := t^{-m}\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}T(\zeta, kt)$ is defined by the translation operator (in $\mathbb{R}^d$) in direction $\zeta \in \mathbb{S}^{d-1}$ with step t given by $T(\zeta, t)f(x) = f(x + t\zeta)$ for $x \in \mathbb{R}^d$.

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Definition of a new frame element

$$\theta_\xi^\diamond(x) := \kappa \sum_{\substack{\zeta\in\mathcal{Z}_j \\ \rho(\zeta,\xi)\le\delta_j}} w_\zeta \Phi_{N_\xi}(\xi\cdot\zeta)\mathfrak{L}_t^{K/2}\sum_{\ell=0}^{m} q_\ell \mathfrak{D}_t^\ell(\zeta)|(1+\varepsilon)\zeta - x|^{-d+2}.$$

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Theorem

Let $K \in \mathbb{N}$, $M > K + d - 1$, $d \ge 2$. Then for any $\gamma_0 > 0$ there exist constants γ_1, γ_2, γ_3, $\gamma_4 > 0$ depending only on d, K, M, γ_0, such that for every $\xi \in \mathcal{X}$ there exists $t > 0$ depending only on d, K, M, γ_0, N_ξ such that the element $\theta_\xi^\diamond$ obeys

$$|\langle \psi_\eta^\diamond, \psi_\xi^\diamond - \theta_\xi^\diamond \rangle| \le \gamma_0 \omega_{\xi,\eta}^\diamond, \quad \forall \eta \in \mathcal{X}.$$

Construction

Every element θ_ξ of the new frame Θ is a linear combination of n_0 Newtonian potentials.

Denote by S the frame operator, $Sf = \sum_{\xi\in\mathcal{X}} \langle f, \theta_\xi\rangle \theta_\xi$. Then

$$f = \sum_{\xi\in\mathcal{X}} \left\langle f, S^{-1}\theta_\xi\right\rangle \theta_\xi.$$

Let $U \in B_\tau^{s\tau}(B^d)$ have the boundary distribution $f_U \in \mathcal{B}_\tau^{s\tau}(\mathbb{S}^{d-1})$. The function norm of f_U in $\mathcal{B}_\tau^{s\tau}(\mathbb{S}^{d-1})$ is equivalent to the norm of $\{\left\langle f_U, S^{-1}\theta_\xi\right\rangle\}$ in the sequence Besov space $b_\tau^{s\tau}(\mathcal{X})$ given by

$$\|\{\left\langle f_U, S^{-1}\theta_\xi\right\rangle\}\|_{b_\tau^{s\tau}(\mathcal{X})} = \left(\sum_{j=0}^{\infty}\sum_{\xi\in\mathcal{X}_j}\left(N_\xi^{(s+(d-1)/2-(d-1)/\tau)}\left|\left\langle f_U, S^{-1}\theta_\xi\right\rangle\right|\right)^\tau\right)^{1/\tau}$$

with $N_\xi = 2^j$ for $\xi \in \mathcal{X}_j$.

Construction

Denote by $\mathcal{Z}_n$ the indices ξ of the largest n/n_0 numbers among

$$\{N_\xi^{(s+(d-1)/2-(d-1)/\tau)} | \langle f_U, S^{-1}\theta_\xi \rangle |\}_{\xi\in\mathcal{X}}$$

and set

$$g = \sum_{\xi\in\mathcal{Z}_n} \langle f_U, S^{-1}\theta_\xi \rangle \theta_\xi.$$

If $1/\tau = s/(d-1) + 1/p$, then

$$\|U - g\|_{H^p(B^d)} \le cn^{-s/(d-1)} \|U\|_{B_\tau^{s\tau}(B^d)}.$$

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The poles of g are located around the points

$$(1+\gamma_1 N_\xi)\xi, \quad \xi\in\mathcal{Z}_n.$$

Thank you!