p-transfinite Diameter and p-Chebyshev Constant in Locally Compact Spaces

Ágota P. Horváth

Department of Analysis, Budapest University of Technology and Economics

1. PRELIMINARIES

$K \subset \mathbb{R}^3$ compact, μ: Radon measure supported on K.

potential and energy with respect to μ:

$$U(\mu, x) = \int_{\mathbb{R}^3} \frac{d\mu(y)}{\|x - y\|}, \quad E(\mu) = \int_{\mathbb{R}^3} U(\mu, x) d\mu(x).$$

Gauss 1839: $\exists\ \mu_0$, supp$\mu_0 \subset K$, s.t.

1) $\mu_0(K) = \mu(K)$

2) $E(\mu_0) \leq E(\mu)$

3) $U(\mu_0, x) =$ constant $= W$, $x \in K$

4) $U(\mu_0, x) \leq W$ for $x \in \mathbb{R}^3 \setminus K$.

NORMALIZATION

$\mu(K) = 1$ $\qquad\qquad$ $W = 1$

$\Downarrow$ $\qquad\qquad$ $\Downarrow$

linear/classical capacity $\qquad\qquad$ p-capacity

$C(K) = W(K)^{-1}$ $\qquad\qquad$ $C_p(K) =$

$$= \inf \left\{ \|f\|^p_{\mu,p} : f \geq 0,\ \int_{\mathbb{R}^3} \frac{f(y)d\mu(y)}{\|x-y\|} \geq 1\ \forall x \in K \right\}$$

linear/classical potential theory $\qquad\qquad$ p-potential theory

ABSTRACT THEORY

$\mathbb{R}^3 \longrightarrow$ locally compact space X

Newtonian kernel $\longrightarrow$ lower semicontinuos kernel function, $k(x,y)$: $X \times X \to \mathbb{R} \cup \{\infty\}$.

This theory is developed by G. Choquet, B. Fuglede, M. Ohtsuka, M. Yamasaki, L. Carleson, and N. S. Landkof, etc.

RELATIONS

"linear theory"

continuous | discrete

capacity (n^{th}-)Chebyshev constant (n^{th}-)transfinite diameter

electrostatics

Hausdorff dimension

(orthogonal) polynomials

interpolation, approximation

Authors

Linear theory:

General: L. Carleson, G. Choquet, B. Farkas, O. Frostman, B. Fuglede, L. L. Helms, N. S. Landkof, M. A. Maria, B. Nagy ,M. Ohtsuka, E. B. Saff, V. Totik, M. Yamasaki.

Discrete: S. V. Borodachov, M. Fekete, A. L. Garcia, D. P. Hardin, Á. P. Horváth, F. Leja, E. B. Saff, J. Siciak, G. Szegő.

p-theory: D. Adams, A. Björn, J. Björn, S. Costea, P. Hajlasz, L. I. Hedberg.

The "Adams-Hedberg definition" of p-capacity

(X,ν) - measure space, $k(x,y) : \mathbb{R}^n \times X \to \mathbb{R} \cup \{\infty\}$ - kernel function s.t. lower semicontinuous on $\mathbb{R}^n$ and measurable on X, μ - Radon measure on $\mathbb{R}^n$, f - ν-measurable nonnegative function.

$$\mathcal{E}(\mu, f) = \int_{\mathbb{R}^n} \int_X k(x,y) f(y) d\nu(y) d\mu(x).$$

By a minimax theorem: $K \subset \mathbb{R}^n$

$$C_p(K)^{-\frac{1}{p}} = \sup_{f \in L} \min_{\mu \in \mathcal{M}(K)} \mathcal{E}(\mu, f) = \min_{\mu \in \mathcal{M}(K)} \sup_{f \in L} \mathcal{E}(\mu, f),$$

where $\mathcal{M}(K) = \{\mu \text{ is Radon measure on } K : \mu(K) = 1\}$, $L = \{f \geq 0 : \|f\|_{\nu,p} \leq 1\}$.

2. TRANSFINITE DIAMETER AND CAPACITY

X - locally compact Hausdorff space, ν - regular Borel measure, $k : X \times X \longrightarrow \mathbb{R} \cup \{\infty\}$ lower semicontinuous, symmetric, nonnegative kernel function - $H \subset X$.

$\mathcal{M}(H) := \{\mu : \mu$ reg. B. meas. on H, supp$\mu \subset H$ compact, $\mu(H) = 1\}$

$$L := \{f \geq 0 : \int_X f(y)^p d\nu((y) \leq 1\}$$

DEFINITION Let H be a subset of X and $X_n = \{x_1, \ldots, x_n\} \subset H$ a system of nodes in H and $f \in L$ a nonnegative, ν-measurable function on X. Let $\lambda \in (0,1)$. We define

$$d(X_n, f) := d_{k,\lambda}(X_n, f)$$

$$= (1-\lambda) \int_X f(y) \frac{1}{n} \sum_{i=1}^{n} k(x_i, y) d\nu(y) + \lambda \frac{2}{n(n-1)} \sum_{1 \le i < j \le n} k(x_i, x_j),$$

and the n^{th}-diameter of H

$$d_n(H) := d_{n,k,p,\lambda}(H) = \sup_{f \in L} \inf_{X_n \subset H} d(X_n, f).$$

Remark. If K is a compact subset of X, then $\min_{X_n \subset K} d(X_n, f) = d(X_n^*, f)$, and $X_n^* = X_{n,k,\lambda}^*(f)$ are the Fekete points of K with respect to f.

the p-transfinite diameter of a set H:

$$d(H) := d_{k,p,\lambda}(H) = \lim_{n \to \infty} d_n(H)$$

Notations

$$\mathcal{E}(\mu, f) := \mathcal{E}_k(\mu, f) = \int_X f(y) \int_X k(x, y) d\mu(x) d\nu(y),$$

$$I(\mu) := I_k(\mu) = \int_X \int_X k(x, y) d\mu(x) d\mu(y),$$

Let $\lambda \in [0, 1]$, and let

$$E(\mu, f) = E_{k,\lambda}(\mu, f) = (1 - \lambda)\mathcal{E}(\mu, f) + \lambda I(\mu)$$

the mutual energy of f and μ.

$$\mathcal{G}\mu(y) = \int_X k(x, y) d\mu(x), \quad \frac{1}{p} + \frac{1}{p'} = 1,$$

$$\epsilon(f,H) := \epsilon_{k,\lambda}(f,H) = \inf_{\mu\in\mathcal{M}(H)} E(\mu,f),$$

$$\epsilon(\mu) := \epsilon_{k,p,\lambda}(\mu) = \sup_{f\in L} E(\mu,f) = (1-\lambda)\|\mathcal{G}\mu(y)\|_{\nu,p'} + \lambda I(\mu),$$

The p-energy of a set H:

DEFINITION

$$\tilde{W}(H) := \tilde{W}_{k,p,\lambda}(H) = \sup_{f\in L} \inf_{\mu\in\mathcal{M}(H)} E(\mu,f) = \sup_{f\in L} \epsilon(f,H),$$

$$W(H) := W_{k,p,\lambda}(H) = \inf_{\mu\in\mathcal{M}(H)} \sup_{f\in L} E(\mu,f) = \inf_{\mu\in\mathcal{M}(H)} \epsilon(\mu).$$

LEMMA Let μ be a positive measure, and $f \in L$. Then the following functions are lower semicontinuous on X or in the weak*-topology:

$$\mu \to I(\mu) \tag{1}$$

$$x \to \int_X k(x,y)d\mu(y), \quad x \to \int_X k(x,y)f(y)d\nu(y) \tag{2}$$

$$\mu \to \mathcal{E}(\mu, f) \tag{3}$$

$$\mu \to \int_X k(x,y)d\mu(y) \tag{4}$$

$$\mu \to \|\mathcal{G}\mu\|_{\nu,p'} \tag{5}$$

THEOREM If $H \subset X$ then

$$d(H) \leq \tilde{W}(H), \tag{6}$$

and for a compact set $K \subset X$

$$W(K) \leq d(K). \tag{7}$$

COROLLARY

For a compact set $K \subset X$, $\tilde{W}(K) = W(K)$.

DEFINITION For a set $H \subset X$, let

$$C(H) = C_{k,p,\lambda}(H) := W_{k,p,\lambda}^{-p}(H).$$

THEOREM (k,p)-capacity is a set function with the following properties:

(a) $C(\emptyset) = 0$.

(b) If $E_1 \subset E_2$ are measurable sets, then $C(E_1) \leq C(E_2)$.

(c) Let $\dots \supset K_i \supset K_{i+1} \supset \dots$ a decreasing sequence of compact sets. Then

$$C\left(\cap_i K_i\right) = \lim_{i\to\infty} C(K_i).$$

(d) Let $\dots \subset B_i \subset B_{i+1} \subset \dots$ an increasing sequence of measurable sets. Then

$$C\left(\cup_i B_i\right) = \lim_{i\to\infty} C(B_i).$$

(e) Let $E \subset X$ is measurable. Then

$$C(E) = \sup\{C(K) : K \subset E,\ K \text{ is compact}\}.$$

COROLLARY

For each measurable set $E \subset X$,

$$W(E) = \tilde{W}(E).$$

3. POTENTIAL FUNCTION AND CHEBYSHEV CONSTANT

The mutual energy of μ and f:

$$E(\mu, f) = \int_X \int_X \left((1-\lambda) \int_X k(x,y) f(y) d\nu(y) + \lambda k(x,y) \right) d\mu(y) d\mu(x).$$

The kernel is not symmetric!

Potential depending on f

DEFINITION Let $H \subset X$, and $\mu \in \mathcal{M}(H)$, $f \in L$.

$$U(\mu, f, x) = U_{k,\lambda}(\mu, f, x)$$

$$:= \frac{1-\lambda}{2}\left(\int_X \int_X k(x,y) f(y) d\nu(y) d\mu(x) + \int_X k(x,y) f(y) d\nu(y)\right)$$

$$+\lambda \int_X k(x,y) d\mu(y).$$

Remark

$$\int_X U(\mu, f, x) d\mu(x) = E(\mu, f).$$

Notation

$$E(\mu, \sigma, f) := \int_X U(\mu, f, x) d\sigma(x)$$

Notation

$H \subset X$. $\mu_*(H) = \sup_{\substack{K \subset H \\ K \text{ is compact}}} \mu(K)$ is the inner measure of H.

LEMMA Let $H \subset X$, $f \in L$. The following conditions are equivalent:

$$\epsilon(f, H) = \infty.$$

$$\epsilon(f, K) = \infty,\ \forall K \subset H,\ K \text{ is compact.}$$

$$\mu_*(H) = 0,\ \forall \mu \text{ positive measure on } X,\ E(\mu, f) < \infty.$$

$$\mu = 0 \text{ is the only positive measure with } \operatorname{supp}\mu \subset H,\ E(\mu, f) < \infty.$$

$$\mu = 0 \text{ is the only pos. meas., } \operatorname{supp}\mu \subset K \subset H,\ K \text{ is compact, } E(\mu, f) < \infty.$$

Notation A property P is said to fulfil f-nearly everywhere (f-n.e.) on H, if denoting by

$$N := \{x \in H : P \text{ does not fulfil in } x\}, \quad \inf_{\mu \in \mathcal{M}(N)} E(\mu, f) = \infty.$$

STATEMENT Let $f \in L$ fixed, $\mu \in \mathcal{M}(X)$, $H \subset X$, $0 \leq t \leq \infty$. Then the following conditions are equivalent

$$U(\mu, f, x) \geq t \;\; f\text{-n.e. } x \in H.$$

$$E(\mu, \sigma, f) \geq t \;\; \forall \sigma \in \mathcal{M}(H),\ E(\sigma, f) < \infty.$$

LEMMA For all $f \in L$, and $K \subset X$ compact, there is an extremal measure $\mu_f := \mu_{f,K} \in \mathcal{M}(K)$, such that $\epsilon(f, K) = E(\mu_f, f)$.

THEOREM Let $f \in L$, K be a compact set in X such that $\epsilon(f, K) < \infty$, and μ_f is the equilibrium measure on K with respect to f. Then

$$U(\mu_f, f, x) \geq \epsilon(f, K) \quad f - \text{n.e. } x \in K.$$

$$U(\mu_f, f, x) \leq \epsilon(f, K) \quad \forall x \in \text{supp}\mu_f.$$

$$U(\mu_f, f, x) = \epsilon(f, K) \quad \mu_f \text{ a.e. } x \in X.$$

Escaping from f

Recall: $\mathcal{G}\mu(y) := \int_X k(x,y)d\mu(x)$, and $\epsilon(\mu) = (1-\lambda)\|\mathcal{G}\mu\|_{\nu,p'} + \lambda I(\mu)$
Let $W(K) < \infty$.

$$\mathcal{M}_c(K) := \{\mu \in \mathcal{M}(K) : \|\mathcal{G}\mu\|_{\nu,p'} \leq c\}$$

$\exists$ c s.t.

$$\inf_{\mu\in\mathcal{M}_c(K)} \sup_{f\in L} E(\mu, f) = W(K)$$

$\tilde{W}_c(K) := \sup_{f\in L} \inf_{\mu\in\mathcal{M}_c(K)} E(\mu, f) \longrightarrow \tilde{W}_c(K) = W(K)$

$\epsilon_c(f, K) := \inf_{\mu\in\mathcal{M}_c(K)} E(\mu, f)$

Construction:

$\{f_n\} \subset L$, $\lim_{n\to\infty} \epsilon_c(f_n, K) := \tilde{W}(K)$

$f_n \xrightarrow{*} f_e = f_e(K) \in L$

$\mu^c_{f_n} \in \mathcal{M}_c(K)$: $E(\mu^c_{f_n}, f_n) = \epsilon_c(f_n, K)$

$\mu^c_{f_n} \xrightarrow{*} \mu^c_e \in \mathcal{M}_c(K) \longrightarrow U^c_e(K, x) := U(\mu^c_e, f_e, x)$

LEMMA Let K be a compact set in X such that $W(K) < \infty$. If c has property (*), then

$$\int_X U_e^c(K, x) d\mu_e^c(x) = W(K).$$

COROLLARY K is a compact set in X such that $W(K) < \infty$. If c has property (*), then

$$E(\mu_e^c, f_e) = E(\mu_{f_e}^c, f_e).$$

DEFINITION - A property P is said to fulfil nearly everywhere (n.e.) on K, if $N := \{x \in K : P \text{ does not fulfil in } x\}$, $\sup_{f \in L} \inf_{\mu \in \mathcal{M}(N)} E(\mu, f) = \tilde{W}(N) = \infty$.

- A l.s.c. kernel is normal, if $\forall\ K \subset X$ compact, $W(K) < \infty\ \exists \{f_n\} \subset L$ s.t. $\lim_{n\to\infty} \epsilon(f_n, K) = \tilde{W}(K)$, and $\{\mu_{f_n}\}$ has a subsequence $\{\mu_{f_{n_k}}\}$ s.t. $\liminf_{n\to\infty} \|\mathcal{G}\mu_{f_{n_k}}\|_{\nu,p'} < \infty$.

THEOREM Let k be normal and K be a compact set in X such that $W(K) < \infty$. Then there is a $c > 0$ such that $U_e^c(K, x)$ is an equilibrium potential and μ_e^c is an equilibrium measure, that is

$$U_e^c(K, x) \geq W(K) \text{ n.e. } x \in K. \tag{8}$$

$$U_e^c(K, x) \leq W(K)\ \ \forall x \in \operatorname{supp} \mu_e^c. \tag{9}$$

$$U_e^c(K, x) = W(K)\ \ \mu_e^c \text{ a.e. } x \in X. \tag{10}$$

p-Chebyshev constant

NOTATION Let $f \in L$, $x \in X$, $H \subset X$ and $X_n \subset H$. Then let

$$M(X_n, f, x) := M_{k,\lambda}(X_n, f, x)$$

$$= \frac{1-\lambda}{2n} \sum_{i=1}^{n} \int_X k(x_i, y) f(y) d\nu(y) + \frac{1-\lambda}{2} \int_X k(x, y) f(y) d\nu(y) + \frac{\lambda}{n} \sum_{i=1}^{n} k(x, x_i).$$

$$M_n(f, H) := M_{n,k,\lambda}(f, H) = \sup_{X_n \subset H} \inf_{x \in H} M(X_n, f, x).$$

$p(X_n, f, x)$: log-polynomial of degree n w.r.t. f

$$p(X_n, f, x) := p_{k,\lambda}(X_n, f, x) = nM(X_n, f, x)$$

$$= \frac{1-\lambda}{2} \int_X \sum_{i=1}^{n} \left(k(x_i, y) + k(x, y)\right) f(y) d\nu(y) + \lambda \sum_{i=1}^{n} k(x, x_i).$$

DEFINITION

$$M(f,H) := \lim_{n\to\infty} M_n(f,H)$$

the p-Chebyshev constant with respect to f, and

$$M(H) := \sup_{f\in L} M(f,H)$$

the p-Chebyshev constant of H.

DEFINITION Let

$$L_c := \{f \in L : f \text{ is compactly supported and continuous }\}.$$

A symmetric kernel k satisfies the (L_c-) relative domination principle (cf. Ohtsuka), if $\forall f \in L_c$ and $\forall\ \mu$ with compact support and with $I(\mu) < \infty$, if

$$\int_X k(x,y)d\mu(y) \leq c_1 - c_2 \int_X k(x,y)f(y)d\nu(y) \quad x \in \text{supp}\mu,$$

then

$$\int_X k(x,y)d\mu(y) \leq c_1 - c_2 \int_X k(x,y)f(y)d\nu(y) \quad x \in X,$$

where c_i are positive constants.

THEOREM Let $H \subset X$. then

$$d(H) \leq M(H),$$

and if k satisfies the relative domination principle, then

$$M(H) \leq \tilde{W}(H).$$

COROLLARY If $K \subset X$ is compact, and k satisfies the relative domination principle, then

$$d(K) = M(K) = W(K).$$

4. GREEDY ENERGY OR LEJA POINTS

DEFINITION Let $K \subset X$ be a compact set, $f \in L$. A sequence $\{a_n\}_{n=1}^{\infty} \subset K$ is called a greedy energy sequence with respect to k, λ and f, if it is generated in the following way:

- $a_1 \in K$ is arbitrary.

- Assuming that $A_n := \{a_1, \dots, a_n\}$ have been selected, a_{n+1} is chosen to satisfy

$$\inf_{x \in K} p(A_n, f, x) = p(A_n, f, a_{n+1}).$$

THEOREM Let us assume that k satisfies the relative domination principle. Then the following statements are satisfied:

$$\sup_{f\in L}\lim_{n\to\infty} d(A_n, f) = W(K).$$

If $\epsilon(f, K) < \infty$, then the following sequence

$$\mu_n := \frac{1}{n}\sum_{k=1}^{n}\delta_{a_k}$$

has a w^*-convergent subsequence, such that

$$\mu_{n_k} \xrightarrow{*} \mu_f$$

where μ_f is an equilibrium measure with respect to f, and δ_{a_k} is the Dirac measure concentrated at the k^{th} greedy energy point.

$$\sup_{f\in L}\lim_{n\to\infty} M(A_n, f, a_{n+1}) = W(K).$$

5. FINAL REMARKS

λ tends to 0 or 1.

$$V_p(H) := \inf_{\mu \in \mathcal{M}(H)} \sup_{f \in L} \mathcal{E}(\mu, f).$$

$$\lim_{\lambda \to 0} W_\lambda(H) \geq V_p(H).$$

$$\lim_{\lambda \to 0} W_\lambda(H) = V_p(H), \text{ if } \exists M \text{ such that } \inf_{\mu \in \mathcal{M}(H)} \|\mathcal{G}\mu\|_{\nu,p'} = \inf_{\substack{\mu \in \mathcal{M}(H) \\ I(\mu) < M}} \|\mathcal{G}\mu\|_{\nu,p'}.$$

Behavior of the n^{th} Fekete set when $\lambda \to 0$:

K compact: [A-H]

$$V_p(K) = \sup_{f \in L} \inf_{\mu \in \mathcal{M}(K)} \mathcal{E}(\mu, f) = \sup_{f \in L} \int_X k(x_0, y) f(y) d\nu(y),$$

If $\epsilon_\lambda(f, K) < \infty$ for a $\lambda \in (0, 1)$, then $V_p(f, K) := \inf_{\mu \in \mathcal{M}(K)} \mathcal{E}(\mu, f) = \int_X k(x_0, y) f(y) d\nu(y)$

$\lim_{\lambda \to 0} \epsilon_\lambda(f, K) = V_p(f, K)$, if $\exists M$ such that $\inf_{\mu \in \mathcal{M}(K)} \mathcal{E}(\mu, f) = \inf_{\substack{\mu \in \mathcal{M}(H) \\ I(\mu) < M}} \mathcal{E}(\mu, f)$.

Construction: $\lambda_m \to 0$, let n be fixed, and let us denote by $X^*_{n,m} \subset K$ a Fekete set with respect to λ_m. Let $\mu_m := \frac{1}{n} \sum_{i=1}^n \delta_{x^*_{i,m}}$. It has a w^*-convergent subsequence with limit: σ_n.

$$\int_X \int_X k(x,y) f(y) d\nu(y) d\sigma_n(x) \leq V_p(f,K),$$

so if the extremal measure is unique, then for all n, $\mu_m \xrightarrow{*} \delta_{x_0}$. (E.g. if $k(x,y)$ is continuous.)

Let $k(x,x) = \infty$, $W_\lambda(K) < \infty$ for a $\lambda \in (0,1)$.
Let $g(x) := \int_X k(x,y) f(y) d\nu(y)$. Assume: $\exists\ x_0 \in K$ s. t.

$$g(x_0) = \min_{x \in K} g(x) = m < \liminf_{\substack{x \to x_0 \\ x \in K \setminus \{x_0\}}} g(x) = M.$$

$$\lim_{\lambda \to 0} \epsilon_\lambda(f,K) > V_p(f,K).$$