Widom factors and Parreau-Widom sets

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Widom factors. Definition

Let $K \subset \mathbb{C}$ be a compact set containing an infinite number of points and $\mathrm{Cap}(K)$ stand for the logarithmic capacity of K. Given $n \in \mathbb{N}$, by $\mathcal{M}_n$ we denote the set of all monic polynomials of degree at most n.

Given probability measure μ with $\mathrm{supp}(\mu) = K$ and $1 \leq p \leq \infty$, we define the n-th Widom factor associated with μ as $W_n^p(\mu) = \frac{\inf_{Q \in \mathcal{M}_n} ||Q||_p}{(\mathrm{Cap}(K))^n}$ where $|| \cdot ||_p$ is taken in the space $L^p(\mu)$. If K is polar then let $W_n^p(\mu) := \infty$.

Clearly, $W_n^p(\mu) \leq W_n^r(\mu)$ for $1 \leq p \leq r \leq \infty$; W_n^p is invariant under dilation and translation of μ.

We omit the upper index for the case $p = \infty$. Here the values $W_n(K) = \frac{||T_{n,K}||_\infty}{(\mathrm{Cap}(K))^n}$ provide us with information about behaviour of the Chebyshev polynomials $T_{n,K}$ on K.

Another important case is $p = 2$, where $\inf_{\mathcal{M}_n} ||Q||_2$ is realized on the monic orthogonal polynomial with respect to μ.

We suggest the name *Widom factor* for $W_n^p(\mu)$ because of the fundamental paper by H. Widom (1969), where the author systematically considered the corresponding ratios for finite unions of smooth Jordan curves and arcs.

Widom factors for the Sup-norm

Let $T_{n,K}$ be the $n-$th Chebyshev polynomial, $t_n(K) := ||T_{n,K}||_\infty$.
By M. Fekete and G. Szegő, $t_n(K)^{\frac{1}{n}} \to \mathrm{Cap}(K)$ as $n \to \infty$.
By Bernstein-Walsh inequality, $t_n(K) \geq (\mathrm{Cap}(K))^n$ for all n.
Thus, $W_n(K) \geq 1$ and $(W_n(K))_{n=1}^\infty$ has subexponential growth, that is $\log W_n(K)/n \to 0$. This is a restriction on growth from both sides!
Two important cases: $W_n(\partial\mathbb{D}) = 1$ and $W_n([-1,1]) = 2$ for all $n \in \mathbb{N}$.
J. P. Thiran and C. Detaille (1991): K is a subarc of the unit circle with angle 2α. Then $W_n(K) \sim 2\cos^2(\alpha/4)$. The circle and the interval can be considered now as limit cases with $\alpha \to \pi$ and $\alpha \to 0$.
By K. Schiefermayr (2008): $W_n(K) \geq 2$ if K lies on the real line.
The behaviour of $(W_n(K))$ may be rather irregular, even for simple compact sets. N. I. Achieser (1932,33): $K = [a,b] \cup [c,d]$. If $K = P^{-1}[-1,1]$ then $(W_n(K))_{n=1}^\infty$ has a finite number of accumulation points from which the smallest is 2. Otherwise, the accumulation points of $(W_n(K))$ fill out an entire interval of which the left endpoint is 2.

Parreau-Widom sets

Let $K \subset \mathbb{R}$ be regular with respect to the Dirichlet problem. Then $g_{\mathbb{C}\setminus K}$ is continuous throughout $\mathbb{C}$. By $\mathcal{C}$ we denote the set of critical points of $g_{\mathbb{C}\setminus K}$, so $g'_{\mathbb{C}\setminus K} = 0$ on $\mathcal{C}$. Then K is called *a Parreau-Widom set* ($K \in PW$) if $PW(K) := \sum_{z\in\mathcal{C}} g_{\mathbb{C}\setminus K}(z) < \infty$.
J. S. Christiansen, B. Simon and M. Zinchenko (2015):
$K \in PW \Longrightarrow W_n(K) \leq 2\exp(PW(K))$.
V. Totik and P. Yuditskii (2015) considered $K = \cup_{j=1}^{p} K_j$, a union of p disjoint C^{2+} Jordan curves, symmetric wrt $\mathbb{R}$. Then the accumulation points of $(W_n(K))_{n=1}^{\infty}$ lie in $[1, \exp(PW(K))]$. If the values $(\mu_K(K_j))_{j=1}^{p}$ are rationally independent ($\sum_{j=1}^{n} \alpha_j x_j = 0$ with $a_j \in \mathbb{Z}$ implies that $a_j = 0$ for all j), then the limit points of $W_n(K)$ fill out the whole interval above.
V. V. Andrievskii (2015), V. Totik and T. Varga (2015) analyze behaviour of $W_n(K)$ when K is a finite union of disjoint Jordan curves or arcs (not necessarily smooth), where quasi-smoothness or Dini-smoothness is required instead of smoothness.

Unbounded Widom-Chebyshev factors

Parreau-Widom sets have positive Lebesgue measure. All finite gap sets and symmetric Cantor sets with positive length are Parreau-Widom sets. Hence, in all cases considered above the sequence of Widom factors is bounded.

Question: What is the maximal possible growth of $(W_n(K))_{n=1}^{\infty}$ for non Parreau-Widom sets?

AG and B. Hatinoğlu (2015): Any subexponential growth of $(W_n(K))_{n=1}^{\infty}$ can be achieved. There are sets with highly irregular behaviour of Widom factors. Namely,

1) For each (M_n) of subexponential growth there is K with $W_n(K) \geq M_n$ for all n.

2) Given $\sigma_n \searrow 0$ and $M_n \to \infty$ (of subexponential growth), there is K such that $W_{n_j}(K) < 2(1 + \sigma_{n_j})$ and $W_{m_j}(K) > M_{m_j}$ for some subsequences (n_j) and (m_j).

These Cantor-type sets will be discussed later.

General Orthogonal Polynomials

Let μ be a probability measure on $\mathbb{C}$ with compact support $K, |K| = \infty$. The Gram-Schmidt process in $L^2(\mu)$ defines orthonormal polynomials $p_n(z, \mu) = \kappa_n z^n + \cdots$ with $\kappa_n > 0$. Let $q_n = \kappa_n^{-1} p_n$. Then $||q_n||_2 = \kappa_n^{-1} = \inf_{Q \in \mathcal{M}_n} ||Q||_2$, where, as above, $\mathcal{M}_n = \{z^n + \cdots\}$.
If $K \subset \mathbb{R}$ then a three-term recurrence relation

$$x\, q_n(x) = q_{n+1}(x) + b_n\, q_n(x) + a_{n-1}^2\, q_{n-1}(x)$$

is valid with the Jacobi parameters $a_n = \kappa_n / \kappa_{n+1}$, $b_n = \int x\, p_n^2(x)\, d\mu(x)$.
Since $\mu(\mathbb{R}) = 1$, we have $p_0 = q_0 \equiv 1$, so $\kappa_0 = 1$ and $a_0 a_1 \cdots a_{n-1} = \kappa_n^{-1}$.
The Jacobi parameters define the matrix

$$\begin{pmatrix} b_0 & a_0 & 0 & 0 & \dots \\ a_0 & b_1 & a_1 & 0 & \dots \\ 0 & a_1 & b_2 & a_2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where μ is is the spectral measure for the unit vector δ_1 and the self-adjoint operator J on $l_2(\mathbb{N})$, which is defined by this matrix.

Weak asymptotics

For a wide class of measures the polynomials $p_n = p_n(\cdot, \mu)$ enjoy regular limit behaviour. Let K be not polar, $\Omega = \overline{\mathbb{C}} \setminus K$ and ν_{p_n} be the counting measure on the zeros of p_n. Let us consider the asymptotics:
(i) $\kappa_n^{1/n} \to Cap(K)^{-1}$
(ii) $|p_n|^{1/n} \rightrightarrows \exp g_\Omega$ (locally uniformly on $\overline{\mathbb{C}} \setminus Conv.hull(K)$)
(iii) $\limsup |p_n(z)|^{1/n} \overset{q.e.}{=} 1$ on $\partial\Omega$
(iv) $\frac{1}{n}\nu_{p_n} \overset{w^*}{\to} \mu_e$ (equilibrium measure on K).
The conditions **(i)** $-$ **(iii)** are pairwise equivalent. If, in addition, $K \subset \partial\Omega$ and $c_\mu > 0$ (minimal carrier capacity), then **(i)** $\sim$ **(iv)** (..., Faber, Szegő, Bernstein, Ulman, Erdös, Freud, Stahl, Totik).

Def. $\mu \in$ **Reg** (regular in the Stahl-Totik sense) if **(i)** is valid.
Till now there is no complete description of regularity in terms of the size of μ. We use (Erdös-Turán): $\mu \in$ **Reg** provided $d\mu/d\mu_e > 0$, $\mu_e - a.e.$
By definition, $W_n^2(\mu) = ||q_n||_2 \, Cap^{-n}(K) = \kappa_n^{-1} \, Cap^{-n}(K)$.
Thus, $\mu \in$ **Reg** if and only if $(W_n^2(\mu))_n$ has subexponential growth.

Szegő class. Strong asymptotics

Suppose $d\mu = \omega(x)dx$ on $[-1,1]$. Then, by definition, $\mu \in Sz[-1,1]$ if $I(\omega) := \int_{-1}^{1} \frac{\log \omega(x)}{\pi\sqrt{1-x^2}}\,dx = \int \log \omega(x)\, d\mu_e(x) > -\infty$, which means that it converges for it cannot be $+\infty$. For such measures (G.Szegő (1938))

$$p_n(z,\mu) = \kappa_n z^n + \cdots = (1+o(1))\,(z+\sqrt{z^2-1})^n \frac{1}{\sqrt{2\pi}} D_\mu^{-1}(z)$$

with $D_\mu(z) = \exp\{\frac{1}{2}\sqrt{z^2-1}\ \int \frac{\log[\omega(x)\sqrt{1-x^2}]}{z-x}\, d\mu_e(x)\}$ - certain outer function in the Hardy space on $\mathbb{C}\setminus[-1,1]$.

Now $z\to\infty$ implies not only that $\kappa_n^{1/n} \to 2$, so $\mu \in \mathbf{Reg}$, but also that $W_n^2(\mu) \to W := \sqrt{\pi}\exp(I(\omega)/2) > 0$ as $n\to\infty$, which is essentially stronger.

The inverse implication is also valid: if $\lim_n W_n^2(\mu)$ exists in $(0,\infty)$ then we have $\mu \in Sz[-1,1]$.

The Szegő theory was extended first to the case of measures that generate a finite gap Jacobi matrix and then for measures on $\mathbb{R}$ such that the essential support of μ is a Parreau-Widom set.

Widom's characterization of the Szegő class

Let $\{y_j\}_j$ be the set of all isolated points of the support of μ and $K = \mathrm{ess\,supp}(\mu)$, so $\mathrm{supp}(\mu) = K \cup \{y_j\}_j$. Suppose that K is a Parreau-Widom set. Let $\omega(x)\,dx$ be the absolutely continuous part of $d\mu$ in its Lebesgue decomposition. In addition, let $\sum g_{\mathbb{C}\setminus K}(y_j) < \infty$. Then, by (G. Szegő, H.Widom, ... , F.Peherstorfer, P. Yuditskii, V. Totik, B. Simon, J. Christiansen, M. Zinchenko)

$$\int \log \omega(x) d\mu_K(x) > -\infty \iff \limsup_{n\to\infty} W_n^2(\mu) > 0.$$

In addition, there is $M > 0$ such that $\frac{1}{M} < W_n^2(\mu) < M$.
We write $\mu \in Sz(K)$ if $I(\omega) = \int \log[d\mu/dx]\, d\mu_e(x) > -\infty$.
This definition can be applied only to measures with nontrivial absolutely continuous part. On the other hand, the Widom condition (on the right side) is applicable to any measure.
The Widom condition is the main candidate to characterize the Szegő class in the general case.

Widom factors in Hilbert norm. Measures on $[-1,1]$

1) Jacobi weight. For $-1<\alpha,\beta<\infty$ let $C_{\alpha,\beta}=\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}\,dx$. Then $d\mu_{\alpha,\beta}=C_{\alpha,\beta}^{-1}(1-x)^{\alpha}(1+x)^{\beta}dx$ and $W_n^2(\mu_{\alpha,\beta})\to W_{\alpha,\beta}$, where $W_{\alpha,\beta}:=\sqrt{\frac{\pi}{2^{\alpha+\beta}\,C_{\alpha,\beta}}}$. Here $W_{\alpha,\beta}\to 0$ as (α,β) approaches $\partial(-1,\infty)^2$ and attains its maximal value at $\alpha=\beta=-1/2$.

By Jensen's inequality, $I(\omega)\le I(\omega_e)$ for each $\mu\in Sz[-1,1]$:
$\int\log(\omega/\omega_e)\,d\mu_e\le\log\int\omega/\omega_e\,d\mu_e=\log\int_{-1}^{1}\omega(x)\,dx=0.$

2) Beyond the Szegő class. Pollaczek polynomials: $a,b\in\mathbb{R}, a\ge|b|$, $\omega(x)=\frac{1+a}{2\pi}\exp(-2\,t\cdot\arcsin x)\cdot|\,\Gamma(1/2+i\,t)\,|^2$ with $t=\frac{ax+b}{2\sqrt{1-x^2}}$. Then $d\mu=\omega\,dx$ is regular, as $\omega>0$ for $|x|<1$, but here $\omega\to 0$ exponentially fast near ±1, $I(\omega)$ diverges, $\mu\notin Sz[-1,1]$, $\lim_n W_n^2(\mu)\cdot n^{a/2}=\Gamma(\frac{a+1}{2})$.

3) $\mu\notin$ **Reg** : $\forall\sigma_n\searrow 0\,\exists\mu:\ W_n^2(\mu)<\sigma_n$. Here, $c_\mu\ne Cap(supp(\mu))$. Z_n - zeros of T_{3^n}, $\mu_n=3^{-n}\sum_{x\in Z_n}\delta_x$. Given positive $(a_n)_{n=1}^{\infty}$ with $\sum_{n=1}^{\infty}a_n=1$, let $\mu=\sum_{n=1}^{\infty}a_n\,\mu_n$. Then $W_m^2(\mu)\le 4\sqrt{2}\cdot\sqrt{a_n}$ for $3^{n-1}\le m<3^n$. (Technique from the book by Stahl-Totik.)

Widom factors in Hilbert norm. Other measures

4) C. Martinez (2008) Periodic Jacobi parameters (a_n) and zero (or slowly oscillating) main diagonal. Let $b > 0, a = b + 2$ and $a_{2n-1} = a, a_{2n} = b$. If $b_n = 0$ then B_0 has the spectrum $\sigma(B_0) = [-b - a, b - a] \cup [a - b, a + b]$. The same (a_n) with $b_n = \sin n^\gamma$ for $0 < \gamma < 1$ give the Jacobi matrix B with $\sigma(B) = [-b - a - 1, b - a + 1] \cup [a - b - 1, a + b + 1]$. Let μ_0 and μ be the spectral measures for B_0 and B correspondingly. Then we have $W_{2n}^2(\mu_0) = 1$ and $W_{2n-1}^2(\mu_0) = \sqrt{a/b}$. The measure μ_0 is absolutely continuous wrt the Lebesgue measure and $\mu_0 \in Sz(\sigma(B_0))$.
On the other hand, $W_{2n}^2(\mu) = (\frac{b}{b+1})^n$ and $W_{2n+1}^2(\mu) = (\frac{b}{b+1})^n \sqrt{\frac{a}{b+1}}$.
Thus, $W_n^2(\mu) \to 0$ as $n \to \infty$, $\mu \notin Sz(\sigma(B))$ and $\mu \notin$ **Reg**.

5) Julia sets generated by $T(z) = z^3 - \lambda z$ with $\lambda > 3$ (M. Barnsley et al.) Iterations $T_0 = z, T_n = T_{n-1}(T)$ define a Cantor-type Julia set, which is $supp(\mu_e)$. Let $W_k := W_k^2(\mu_e)$. Then $W_{3^n} = 1, W_{3^n-1} = a_{3^n}^{-1} \to \infty$. Also, $W_{3^n+1} \to \sqrt{2\lambda/3}$, $W_{3^n+2} =\to \sqrt{2}\,\lambda/3$, etc.