Orthogonal polynomials on Cantor-type sets

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Szegő class for the finite gap case

For a probability measure μ on $\mathbb{C}$ with compact support $S_\mu, |S_\mu| = \infty$, we consider the orthonormal polynomials $p_n(z,\mu) = \kappa_n z^n + \cdots$ with $\kappa_n > 0$. Then $q_n = \kappa_n^{-1} p_n$ minimizes $|| \cdot ||_2$ in $\mathcal{M}_n = \{z^n + \cdots\}$.
We write $\mu \in$ **Reg** (regular in the Stahl-Totik sense) if $\kappa_n^{1/n} \to Cap(S_\mu)^{-1}$. Here S_μ may be polar.
Let $K = \cup_{k=1}^N [a_k, b_k]$ and $d\mu = \omega(x)dx$ on K. By def., $\mu \in Sz(K)$ if $I(\omega) = \int \log(\omega(x))\, d\mu_K(x) > -\infty$. For $\mu \in Sz(K)$, the polynomials $p_n(\cdot, \mu)$ enjoy nice limit behaviour, stronger than for only regular μ.
Widom's characterization: $\mu \in Sz(K) \iff \limsup_n W_n^2(\mu) > 0$, where $W_n^2(\mu) := ||q_n||_2\, Cap^{-n}(K) = \kappa_n^{-1}\, Cap^{-n}(K)$ are *Widom factors*.
There are similar results about $Sz(K)$ for the case $\mu = \mu_a + \mu_p$ (here a condition of Blaschke-type is added) and when K is a finite union of disjoint smooth Jordan curves or arcs. But, in general, $\mu = \mu_a + \mu_p + \mu_{sc}$.

PROBLEM: How to characterize $Sz(S_\mu)$ if $\mu_{sc} \neq 0$, for example $\mu = \mu_{sc}$?

OP for Cantor-type sets. The known results

M. Barnsley et al (1983, 85): OP on Julia polynomial sets;
G. Mantica (1996-15): numerical computing of Jacobi parameters for IFS;
S. Heilman, P. Owrutsky, R. Strichardz (2014): OP wrt self-similar measures (numerically);
J. Christiansen (2012): Parreau-Widom sets (they have positive Lebesgue measure);
H. Kruger & B. Simon (2015): some conjectures (based on numerical computing) about OP on the Cantor ternary set;
B. Simon et al. (80th-90th): almost periodic Schrödinger operators (discrete Schrödinger's operator is a particular case of Jacobi operator. Cantor-type sets appear as the spectral measures in some cases, e.g. in the one-dimensional quasi-crystal model or for the almost Mathieu operator with an irrational parameter).
Remark: in theoretical results orthogonal polynomials are taken mainly with respect to the equilibrium measure μ_K.

Weakly equilibrium Cantor sets

AG(2014): For $\gamma = (\gamma_s)_{s=1}^{\infty}$ with $0 < \gamma_s < \frac{1}{4}$ let $r_0 = 1$ and $r_s = \gamma_s r_{s-1}^2$.
Take $P_2(x) = x(x-1)$ and $P_{2^{s+1}} = P_{2^s} \cdot (P_{2^s} + r_s)$ for $s \geq 1$. Then
$K(\gamma) := \bigcap_{s=0}^{\infty} \overline{D}_s = \bigcap_{s=0}^{\infty} E_s = \bigcap_{s=0}^{\infty} \left(\frac{2}{r_s} P_{2^s} + 1\right)^{-1}([-1,1])$, where
$D_s = \{z \in \mathbb{C} : |P_{2^s}(z) + r_s/2| < r_s/2\} \searrow$, so we can apply the Harnack Principle, and $E_s = \{x \in \mathbb{R} : |P_{2^s}(x) + r_s/2| \leq r_s/2\} = \cup_{j=1}^{2^s} I_{j,s}$ is the inverse polynomial image. Here, $P_{2^s} + r_s/2$ is the 2^s-th Chebyshev polynomial on $K(\gamma)$.

The set $K(\gamma)$ has positive Lebesgue measure if γ_s are rather closed to $\frac{1}{4}$.
Moreover, in the limit case $\gamma_s = \frac{1}{4}$ for all s we have $K(\gamma) = [0,1]$.

At least for small γ, the set $K(\gamma)$ is weakly equilibrium in the following sense. Let us distribute uniformly the mass 2^{-s} on each $I_{j,s}$ for $j \leq 2^{-s}$.
Let λ_s be the normalized in this sense Lebesgue measure on E_s, so $d\lambda_s = (2^s l_{j,s})^{-1} dt$ on $I_{j,s}$. Then $\lambda_s \xrightarrow{*} \mu_{K(\gamma)}$ provided $K(\gamma)$ is not polar.

Some results for $K(\gamma)$

The representation $g_{\mathbb{C}\setminus K(\gamma)}(z) = \lim_{s\to\infty} 2^{-s} \log|P_{2^s}(z)/r_s|$ for $z \notin K(\gamma)$ allows to construct Green's functions with diverse moduli of continuity and sets with preassigned growth of subsequence of Markov's factors.

AG 2014: $\exists\ K \subset \mathbb{R}$: the Markov inequality with the best Markov's exponent is not valid on K (a question posed by M. Baran et al.);

$\exists\ K$: the best exponents in the local and the global versions of Markov's inequalities are essentially different (a question posed by L.Frerick et al.)

G. Alpan & AG(2015): $\mu_{K(\gamma)}$ and the corresponding Hausdorff measure are mutually absolutely continuous. This is not valid for symmetric Cantor-type sets, where these measures are mutually singular. (N. Makarov & A. Volberg (1986) for the classical Cantor set).

AG and B. Hatinoğlu (2015): a set with any subexponential growth of Widom-Chebyshev factors and a set K for which $(W_n(K))_{n=1}^{\infty}$ have highly irregular behaviour.

G. Alpan & AG, "OP on WECS": OP with respect to $\mu_{K(\gamma)}$.

Orthogonal polynomials on $K(\gamma)$. Main theorem

Theorem (1)

$q_{2^s}(\cdot, \mu_{K(\gamma)})$ coincides with 2^s-th Chebyshev polynomial for all s.

Sketch of the proof: Let $\nu_s = 2^{-s}\sum_{k=1}^{2^s}\delta_{x_k}$, where $(x_k)_{k=1}^{2^s}$ are the zeros of $P_{2^s} + r_s/2$ (they are simple and real). Then for $s > m$ we can decompose all zeros $(x_k)_{k=1}^{2^s}$ into 2^{s-m-1} groups, on which the values of $P_{2^m} + r_m/2$ are controllable. This gives $\int \left(P_{2^m} + \frac{r_m}{2}\right) d\nu_s = 0$. Since $\nu_s \xrightarrow{*} \mu_{K(\gamma)}$ we have $\int \left(P_{2^m} + \frac{r_m}{2}\right) d\mu_{K(\gamma)}$. Similarly it was shown that $\int \left(P_{2^{i_1}} + \frac{r_{i_1}}{2}\right)\left(P_{2^{i_2}} + \frac{r_{i_2}}{2}\right)\dots\left(P_{2^{i_n}} + \frac{r_{i_n}}{2}\right) d\nu_s = 0$ for any indices $0 \le i_1 < \dots < i_n < s$. Each P of degree $< 2^s$ is a linear combination of polynomials of the type $\left(P_{2^{s-1}} + \frac{r_{s-1}}{2}\right)^{n_{s-1}}\dots\left(P_2 + \frac{r_1}{2}\right)^{n_1}\left(x - \frac{1}{2}\right)^{n_0}$.
Therefore, q_{2^s} coincides with $P_{2^s} + r_s/2$.
In addition, we have a simple representation $||q_{2^s}||^2 = (1 - 2\,\gamma_{s+1})\, r_s^2/4$.
Application of Newton's identities is a crucial point of the proof.

Orthogonal polynomials on $K(\gamma)$. Other values of n

In the next step, $A-$type and $B-$type polynomials were introduced. In particular, for $2^m \le n < 2^{m+1}$ with $n = i_m 2^m + \cdots + i_0$ with $i \in \{0,1\}$ we take $B_n = (q_{2^m})^{i_m}(q_{2^{m-1}})^{i_{m-1}} \dots (q_1)^{i_1}$. The polynomials $B_{(2k+1)\cdot 2^s}$ and $B_{(2j+1)\cdot 2^m}$ are orthogonal for all $j,k,m,s \in \mathbb{Z}_+$ with $s \neq m$. They can be considered as a basis in the set of polynomials: for each $n \in \mathbb{N}$ with $n = 2^s(2k+1)$, the polynomial q_n has a unique representation as a linear combination of $B_{2^s}, B_{3\cdot 2^s}, B_{5\cdot 2^s} \dots, B_{(2k-1)\cdot 2^s}, B_{(2k+1)\cdot 2^s}$. This allows to present formulas to express coefficients of each q_n.

For example, $B_{3\cdot 2^s} = q_{2^s} q_{2^{s+1}}$, so $q_{3\cdot 2^s} = q_{2^{s+1}} q_{2^s} - \frac{\|q_{2^{s+1}}\|^2}{\|q_{2^s}\|^2} q_{2^s}$.

Similarly, $B_{5\cdot 2^s} = q_{2^s} q_{2^{s+2}}$ and $q_{5\cdot 2^s} = c_0 q_{2^s} + c_1 q_{2^{s+1}} q_{2^s} + q_{2^s} q_{2^{s+2}}$ with

$$c_0 = \frac{||q_{2^{s+2}}||^2}{||q_{2^s}||^4 - ||q_{2^{s+1}}||^2}, \quad c_1 = -c_0 \, \frac{||q_{2^s}||^2}{\|q_{2^{s+1}}\|^2}.$$

All coefficient can be expressed only in terms of $(\gamma_k)_{k=1}^{\infty}$. As k gets bigger, the complexity of calculations increases.

In general, the polynomial q_n is not Chebyshev.

Orthogonal polynomials on $K(\gamma)$. Jacobi parameters

Jacobi parameters also can be calculated recursively: $a_1 = ||q_1||$ and $a_2 = ||q_2||/||q_1||$. Suppose a_i are given for $i \le n$. If $n+1 = 2^s > 2$ then

$$a_{n+1} = \frac{||q_{2^s}||}{||q_{2^{s-1}}|| \cdot a_{2^{s-1}+1} \cdot a_{2^{s-1}+2} \cdots a_{2^s-1}}.$$

Otherwise, $n+1 = 2^s(2k+1)$ for some $s \in \mathbb{Z}_+$ and $k \in \mathbb{N}$. Here,

$$a_{n+1}^2 = a_{2^s(2k+1)}^2 = \frac{\|q_{2^s}\|^2 - a_{2^{s+1}k}^2 \cdots a_{2^{s+1}k-2^s+1}^2}{a_{2^s(2k+1)-1}^2 \cdots a_{2^{s+1}k+1}^2}.$$

If $\gamma_s \le 1/6$ for all s hen $\lim_{s\to\infty} a_{j\cdot 2^s+n} = a_n$ for $j \in \mathbb{N}$ and $n \in \mathbb{Z}_+$. Here, $a_0 := 0$. In particular, $\liminf a_n = 0$.

The formulas can be applied as well for the case $\gamma_n < 1/4$ for $1 \le n \le s$ and $\gamma_n = 1/4$ for $n > s$. Then $K(\gamma) = E_s$ is a finite union of intervals. If $\gamma_n = 1/4$ for all n then $K(\gamma) = [0,1]$ and all $a_n = 1/4$, which corresponds to the case of the Chebyshev polynomials on this set.

Orthogonal polynomials on $K(\gamma)$. Widom factors

Let $W_n := W_n^2(\mu_{K(\gamma)}) = \|q_n\| \,/\, (Cap(K))^n$.
We have $W_{2^s} \geq \sqrt{2}$ for each γ. If $\gamma_n \leq 1/6$ for all s then $\liminf_{n\to\infty} W_n = \liminf_{s\to\infty} W_{2^s} \geq \sqrt{6}$ and $\limsup_{n\to\infty} W_n = \infty$.
Next examples illustrate the behaviour of Widom factors:
1) If $\gamma_n \to 0$ then $W_{2^s} \to \infty$. Therefore $W_n \to \infty$.
2) There exists $\gamma_n \not\to 0$ with $W_n \to \infty$. One can take $\gamma_{2k} = 1/6$, $\gamma_{2k-1} = 1/k$.
3) If $\gamma_n \geq c > 0$ for all n then $\liminf_{n\to\infty} W_n \leq 1/2c$.
4) There exists γ with $\inf \gamma_n = 0$ and $\liminf_{n\to\infty} W_n < \infty$. Here we can take $\gamma_n = 1/6$ for $n \neq n_k$ and $\gamma_{n_k} = 1/k$ for a sparse sequence $(n_k)_{k=1}^{\infty}$. Then $(W_{2^{n_k}})_{k=1}^{\infty}$ is bounded.
Later it was shown that $K(\gamma)$ is a Parreau-Widom set if and only if $\sum_{n=1}^{\infty} \sqrt{\frac{1}{4} - \gamma_n} < \infty$.

Orthogonal polynomials on generalized Julia sets

Let $(f_n(z))_{n=1}^{\infty}$ be a sequence of rational functions with $\deg f_n \geq 2.$ in $\overline{\mathbb{C}}$. Let us define $F_n(z) := f_n \circ F_{n-1}(z)$ recursively for $n \geq 1$ and $F_0(z) = z$. Then the Fatou set $F_{(f_n)}$ is defined as the domain of normality for $(F_n)_{n=1}^{\infty}$ in the sense of Montel, its complement $J_{(f_n)}$ is the Julia set. If $f_n = f$ for some fixed rational f for all n then we have *autonomous* $F(f)$ and $J(f)$.

M. F. Barnsley,J. S. Geronimo, A. N. Harrington (1982, 83) presented $q_{k \cdot n}(\cdot, \mu_{J(f)}), k \in \mathbb{N}$ for $f = z^n + \cdots$.

We (with G. Alpan) extend this result to $J_{(f_n)}$ for a regular in the Brück - Büger sense polynomial sequence (f_n). Also this is a generalization of the "OP on WECS": if we take $f_n(z) = \frac{1}{2\gamma_n}(z^2 - 1) + 1$ for all n, then $K_1(\gamma) := J_{(f_n)}$ is a stretched version of the set $K(\gamma)$.

Towards the Szegő class

In the finite gap case (some additional mass points fast converging to the essential support are allowed) we have

$$\int \log[d\mu/dx]\, d\mu_K(x) > -\infty \iff \limsup_{n\to\infty} W_n^2(\mu) > 0.$$

The Szegő condition on the left means that the corresponding integral converges. This condition cannot be applied to singular continuous measures, whereas the Widom condition ($W_n^2(\mu) \nrightarrow 0$) is applicable to any measure, even with a polar support. The Widom condition is the main candidate to characterize the Szegő class in the general case. Therefore the analysis of Widom factors for small sets, especially for $\mu \neq \mu_K$, is rather interesting.

Conjectures

1) If a compact set K is regular with respect to the Dirichlet problem then μ_K always belongs to the Szegő class (in its Widom's description).
Remark: $\mu_K \in \mathbf{Reg}$; in known cases $W_n^2(\mu_K) > a > 0$ for all n.
2) In the case of measures with non-polar support K, the Szegő condition should be done as $I(\mu) := \int \log(d\mu/d\mu_K)d\mu_K > -\infty$.
Arguments in favour:
μ_K is the most natural measure in the theory of general OP;
this condition coincides with the Szegő condition in known cases;
by Jensen's inequality, the value $I(\mu)$ is nonpositive and it attains its maximum 0 just in the case $\mu = \mu_K$;
$I(\mu)$ is exactly the relative entropy of μ_K with respect to μ.

Objections (based on the numerical evidence from H. Krüger & B. Simon): (a_n) were calculated for $n \leq 200.000$ in the case μ_{CL} on the classical Cantor set K_0. For these values $W_n^2(\mu_{CL})$ behave as a bounded below (by a positive number) sequence. But $\mu_{CL} \perp \mu_{K_0}$ and $I(\mu) = -\infty$.

K Ö S Z Ö N Ö M !