Chebyshev constants, energies, rendezvous numbers and the transfinite diameter

B. Farkas

joint with

B. Nagy, Sz. Gy. Révész

SSPT10, Budapest, 18-23 August 2015

Transfinite diameter: $K \subseteq \mathbb{C}$ compact set

$$D(K) := \lim_{n\to\infty} \sup_{w_1,\dots,w_n} \prod_{\substack{i,j=1\\ i\neq j}}^{n} |w_i - w_j|^{\frac{1}{n(n-1)}}$$

take $-\log$

$$-\log D(K) = \lim_{n\to\infty} \inf_{w_1,\dots,w_n} \frac{1}{n(n-1)} \sum_{\substack{i,j=1\\ i\neq j}}^{n} -\log|w_i - w_j|$$

Transfinite diameter: $K \subseteq \mathbb{C}$ compact set

$$D(K) := \lim_{n\to\infty} \sup_{w_1,\dots,w_n} \prod_{\substack{i,j=1\\ i\neq j}}^{n} |w_i - w_j|^{\frac{1}{n(n-1)}}$$ take $-\log$

$$-\log D(K) = \lim_{n\to\infty} \inf_{w_1,\dots,w_n} \frac{1}{n(n-1)} \sum_{\substack{i,j=1\\ i\neq j}}^{n} -\log|w_i - w_j|$$

Chebyshev constant: $K \subseteq \mathbb{C}$ compact set

$$M(K) := \lim_{n\to\infty} \inf_{w_1,\dots,w_n} \sup_{x\in K} \prod_{j=1}^{n} |x - w_j|^{\frac{1}{n}}$$ take $-\log$

$$-\log M(K) = \lim_{n\to\infty} \sup_{w_1,\dots,w_n} \inf_{x\in K} \frac{1}{n} \sum_{j=1}^{n} -\log|x - w_j|$$

Transfinite diameter: $K \subseteq \mathbb{C}$ compact set

$$D(K) := \lim_{n\to\infty} \sup_{w_1,\ldots,w_n} \prod_{\substack{i,j=1\\ i\neq j}}^{n} |w_i - w_j|^{\frac{1}{n(n-1)}}$$ take $-\log$

$$-\log D(K) = \lim_{n\to\infty} \inf_{w_1,\ldots,w_n} \frac{1}{n(n-1)} \sum_{\substack{i,j=1\\ i\neq j}}^{n} -\log|w_i - w_j|$$

Chebyshev constant: $K \subseteq \mathbb{C}$ compact set

$$M(K) := \lim_{n\to\infty} \inf_{w_1,\ldots,w_n} \sup_{x\in K} \prod_{j=1}^{n} |x - w_j|^{\frac{1}{n}}$$ take $-\log$

$$-\log M(K) = \lim_{n\to\infty} \sup_{w_1,\ldots,w_n} \inf_{x\in K} \frac{1}{n} \sum_{j=1}^{n} -\log|x - w_j|$$

For all $K \subseteq \mathbb{C}$ compact sets we have $D(K) = M(K)$.

Transfinite diameter: $K \subseteq \mathbb{C}$ compact set

$$D(K) := \lim_{n\to\infty} \sup_{w_1,\dots,w_n} \prod_{\substack{i,j=1\\ i\neq j}}^{n} |w_i - w_j|^{\frac{1}{n(n-1)}} \qquad \text{take } -\log$$

$$-\log D(K) = \lim_{n\to\infty} \inf_{w_1,\dots,w_n} \frac{1}{n(n-1)} \sum_{\substack{i,j=1\\ i\neq j}}^{n} -\log |w_i - w_j|$$

Chebyshev constant: $K \subseteq \mathbb{C}$ compact set

$$M(K) := \lim_{n\to\infty} \inf_{w_1,\dots,w_n} \sup_{x\in K} \prod_{j=1}^{n} |x - w_j|^{\frac{1}{n}} \qquad \text{take } -\log$$

$$-\log M(K) = \lim_{n\to\infty} \sup_{w_1,\dots,w_n} \inf_{x\in K} \frac{1}{n} \sum_{j=1}^{n} -\log |x - w_j|$$

For all $K \subseteq \mathbb{C}$ compact sets we have $D(K) = M(K)$.

Maximum principle: for every μ compactly supported probability measure

$$\sup_{x\in\mathbb{C}} \int -\log|x-y| \,\mathrm{d}\mu(y) = \sup_{x\in\operatorname{supp}\mu} \int -\log|x-y| \,\mathrm{d}\mu(y).$$

H. Gross (1967):

(X, d) a compact, connected metric space

There exists uniquely a number $r = r(X)$, such that for all $w_1, w_2, \ldots, w_n \in X$ there is an $x \in X$ with

$$\frac{1}{n}\sum_{i=1}^{n} d(w_i, x) = r$$

H. Gross (1967):

(X, d) a compact, connected metric space

There exists uniquely a number $r = r(X)$, such that for all $w_1, w_2, \ldots, w_n \in X$ there is an $x \in X$ with

$$\frac{1}{n}\sum_{i=1}^{n} d(w_i, x) = r$$

Some examples

For $[0, 1]$: $r([0, 1]) = 1/2$ (take $x_1 = 0$, $x_2 = 1$, then $x - x_1 + x_2 - x = 1$)

H. Gross (1967):

(X, d) a compact, connected metric space

There exists uniquely a number $r = r(X)$, such that for all $w_1, w_2, \ldots, w_n \in X$ there is an $x \in X$ with

$$\frac{1}{n}\sum_{i=1}^{n} d(w_i, x) = r$$

Some examples

For $[0,1]$: $r([0,1]) = 1/2$ (take $x_1 = 0$, $x_2 = 1$, then $x - x_1 + x_2 - x = 1$)

$r(\text{unit disc}) = 1$, $r(\text{unit circle}) = 4/\pi$ etc.

H. Gross (1967):

(X, d) a compact, connected metric space

There exists uniquely a number $r = r(X)$, such that for all $w_1, w_2, \ldots, w_n \in X$ there is an $x \in X$ with

$$\frac{1}{n} \sum_{i=1}^{n} d(w_i, x) = r$$

Some examples

For $[0, 1]$: $r([0, 1]) = 1/2$ (take $x_1 = 0$, $x_2 = 1$, then $x - x_1 + x_2 - x = 1$)

$r(\text{unit disc}) = 1$, $r(\text{unit circle}) = 4/\pi$ etc.

For unit spheres in $\ell_p^n(\mathbb{R})$ or $\ell_p^n(\mathbb{C})$ not known unless $p = 1, 2, \infty$.

P. K. Lin, R. Wolf, J. C. García-Vázquez, R. Villa, 1994–...

Generalisations

W. Stadje (1981):

For $k : X \times X \to \mathbb{R}_+$ continuous and symmetric (i.e., $k(x,y) = k(y,x)$) there exists a unique number $r = r(X,k)$, such that for all $w_1, w_2, \ldots, w_n \in X$ there is $x \in X$ with

$$\frac{1}{n}\sum_{i=1}^{n} k(w_i, x) = r$$

Generalisations

W. Stadje (1981):

For $k : X \times X \to \mathbb{R}_+$ continuous and symmetric (i.e., $k(x,y) = k(y,x)$) there exists a unique number $r = r(X,k)$, such that for all $w_1, w_2, \ldots, w_n \in X$ there is $x \in X$ with

$$\frac{1}{n}\sum_{i=1}^{n} k(w_i, x) = r$$

G. Elton, J. M. Cleary, S. A. Morris, D. Yost (1986):

For all $\mu \in \mathfrak{M}_1(X)$ Borel probability measures there is some $x \in X$ with

$$\int\limits_X k(x,y)\,\mathrm{d}\mu(y) = r$$

B. Fuglede, M. Ohtsuka, G. Choquet

X a (locally compact) topological space

$k : X \times X \to \mathbb{R} \cup \{+\infty\}$, positive (symmetric), lower semicontinuous kernel.

B. Fuglede, M. Ohtsuka, G. Choquet

X a (locally compact) topological space

$k : X \times X \to \mathbb{R} \cup \{+\infty\}$, positive (symmetric), lower semicontinuous kernel.

$\mu \in \mathfrak{M}_1(X)$ probability measure

$$U^\mu(x) := \int_X k(x,y)\,\mathrm{d}\mu(y)$$ **potential**

B. Fuglede, M. Ohtsuka, G. Choquet

X a (locally compact) topological space

$k : X \times X \to \mathbb{R} \cup \{+\infty\}$, positive (symmetric), lower semicontinuous kernel.

$\mu \in \mathfrak{M}_1(X)$ probability measure

$$U^\mu(x) := \int_X k(x,y)\,\mathrm{d}\mu(y) \qquad \textbf{potential}$$

$$u(H) := \inf_{\operatorname{supp}\mu \Subset H} \sup_{x \in X} U^\mu(x) \qquad \textbf{uniform energy}$$

$$v(H) := \inf_{\operatorname{supp}\mu \Subset H} \sup_{x \in \operatorname{supp}\mu} U^\mu(x) \qquad \textbf{de la'Vallée-Poussin energy}$$

$$w(H) := \inf_{\operatorname{supp}\mu \Subset H} \int_X U^\mu(x)\,\mathrm{d}\mu(x) \qquad \textbf{Wiener energy}$$

B. Fuglede, M. Ohtsuka, G. Choquet

X a (locally compact) topological space

$k : X \times X \to \mathbb{R} \cup \{+\infty\}$, positive (symmetric), lower semicontinuous kernel.

$\mu \in \mathfrak{M}_1(X)$ probability measure

$$U^\mu(x) := \int_X k(x,y)\,\mathrm{d}\mu(y)$$ **potential**

$$u(H) := \inf_{\operatorname{supp}\mu \Subset H} \sup_{x \in X} U^\mu(x)$$ **uniform energy**

$$v(H) := \inf_{\operatorname{supp}\mu \Subset H} \sup_{x \in \operatorname{supp}\mu} U^\mu(x)$$ **de la'Vallée-Poussin energy**

$$w(H) := \inf_{\operatorname{supp}\mu \Subset H} \int_X U^\mu(x)\,\mathrm{d}\mu(x)$$ **Wiener energy**

in general: $w(H) \le v(H) \le u(H)$; For symmetric kernels: $w(H) = v(H)$

B. Fuglede, M. Ohtsuka, G. Choquet

X a (locally compact) topological space

$k : X \times X \to \mathbb{R} \cup \{+\infty\}$, positive (symmetric), lower semicontinuous kernel.

$\mu \in \mathfrak{M}_1(X)$ probability measure

$$U^\mu(x) := \int_X k(x,y)\,\mathrm{d}\mu(y)$$ **potential**

$$u(H) := \inf_{\operatorname{supp}\mu \Subset H} \sup_{x\in X} U^\mu(x)$$ **uniform energy**

$$v(H) := \inf_{\operatorname{supp}\mu \Subset H} \sup_{x\in\operatorname{supp}\mu} U^\mu(x)$$ **de la'Vallée-Poussin energy**

$$w(H) := \inf_{\operatorname{supp}\mu \Subset H} \int_X U^\mu(x)\,\mathrm{d}\mu(x)$$ **Wiener energy**

in general: $w(H) \le v(H) \le u(H)$; For symmetric kernels: $w(H) = v(H)$

Frostman's maximum principle: $\sup_{\operatorname{supp}\mu} U^\mu = \sup_X U^\mu$ holds for all $\mu \in \mathfrak{M}_1$;

this trivially implies $u = v$.

$H, L \subset X$ arbitrary

(Dual) energy of a set H with respect to L

$$q(H, L) := \inf_{\operatorname{supp}\mu \Subset H} \sup_{x \in L} U^\mu(x) \qquad \underline{q}(H, L) := \sup_{\operatorname{supp}\mu \Subset H} \inf_{x \in L} U^\mu(x)$$

$q(H) := q(H, H)$

$H, L \subset X$ arbitrary

(Dual) energy of a set H with respect to L

$$q(H,L) := \inf_{\operatorname{supp}\mu \Subset H} \sup_{x \in L} U^\mu(x) \qquad \underline{q}(H,L) := \sup_{\operatorname{supp}\mu \Subset H} \inf_{x \in L} U^\mu(x)$$

$q(H) := q(H,H)$

Motivation 1

Recall: mutual energy $\mu, \nu \in \mathfrak{M}_1(H)$

$$I(\mu,\nu) = \int_X \int_X k(x,y)\,\mathrm{d}\mu(x)\,\mathrm{d}\nu(y)$$

$H, L \subset X$ arbitrary

(Dual) energy of a set H with respect to L

$$q(H,L) := \inf_{\operatorname{supp}\mu \Subset H} \sup_{x \in L} U^\mu(x) \qquad \underline{q}(H,L) := \sup_{\operatorname{supp}\mu \Subset H} \inf_{x \in L} U^\mu(x)$$

$q(H) := q(H,H)$

Motivation 1

Recall: mutual energy $\mu, \nu \in \mathfrak{M}_1(H)$

$$I(\mu,\nu) = \int_X \int_X k(x,y)\, \mathrm{d}\mu(x)\, \mathrm{d}\nu(y)$$

Motivation 2

$w(H) \leq v(H) \leq q(H) \leq u(H) = q(H,X)$.

$H, L \subset X$ arbitrary

(Dual) energy of a set H with respect to L

$$q(H,L) := \inf_{\operatorname{supp}\mu \Subset H} \sup_{x \in L} U^\mu(x) \qquad \underline{q}(H,L) := \sup_{\operatorname{supp}\mu \Subset H} \inf_{x \in L} U^\mu(x)$$

$q(H) := q(H,H)$

Motivation 1

Recall: mutual energy $\mu, \nu \in \mathfrak{M}_1(H)$

$$I(\mu,\nu) = \int_X \int_X k(x,y)\,\mathrm{d}\mu(x)\,\mathrm{d}\nu(y)$$

Motivation 2

$w(H) \le v(H) \le q(H) \le u(H) = q(H,X)$.

Motivation 3

$\inf \sup \Longrightarrow$ “saddle point”. Monotonicity only in separate variables.

Gy. Pólya, G. Szegő, L. Carleson, G. Choquet

n^{th} Chebyshev constant of L w.r.t. H

$$M_n(H,L) := \sup_{w_1,\dots,w_n\in H} \inf_{x\in L} \frac{1}{n}\left(\sum_{k=1}^{n} k(x,w_k)\right)$$

think of $k = -\log$

Gy. Pólya, G. Szegő, L. Carleson, G. Choquet

n^{th} Chebyshev constant of L w.r.t. H

$$M_n(H,L) := \sup_{w_1,\dots,w_n \in H} \inf_{x \in L} \frac{1}{n}\left(\sum_{k=1}^{n} k(x,w_k)\right)$$

think of $k = -\log$

n^{th} dual Chebyshev constant of L w.r.t. H

$$\overline{M}_n(H,L) := \inf_{w_1,\dots,w_n \in H} \sup_{x \in L} \frac{1}{n}\left(\sum_{j=1}^{n} k(x,w_j)\right)$$

Gy. Pólya, G. Szegő, L. Carleson, G. Choquet

n^{th} Chebyshev constant of L w.r.t. H

$$M_n(H,L) := \sup_{w_1,\dots,w_n \in H} \inf_{x\in L} \frac{1}{n}\left(\sum_{k=1}^{n} k(x,w_k)\right)$$

think of $k = -\log$

n^{th} dual Chebyshev constant of L w.r.t. H

$$\overline{M}_n(H,L) := \inf_{w_1,\dots,w_n \in H} \sup_{x\in L} \frac{1}{n}\left(\sum_{j=1}^{n} k(x,w_j)\right)$$

Proposition. For arbitrary $H, L \subset X$ both $\overline{M}_n(H,L)$ and $M_n(H,L)$ converge. The limits $\overline{M}(H,L)$, $M(H,L)$ are called the Chebyshev constant of L w.r.t to H.
$M(H) := M(H,H)$, $\overline{M}(H) := M(H,H)$.

Definition. Let $H \subset X$ be fixed. We define the n^{th} diameter of H as

$$D_n(H) := \inf_{w_1,\dots,w_n \in H} \frac{1}{(n-1)\,n} \Bigg(\sum_{1 \le j \ne l \le n} k(w_j, w_l) \Bigg);$$

The limit $D(H) := \lim_{n\to\infty} D_n(H)$ is the transfinite diameter of H.

Definition. Let $H \subset X$ be fixed. We define the n^{th} diameter of H as

$$D_n(H) := \inf_{w_1,\dots,w_n \in H} \frac{1}{(n-1)\,n} \Bigg(\sum_{1 \le j \ne l \le n} k(w_j, w_l) \Bigg);$$

The limit $D(H) := \lim_{n\to\infty} D_n(H)$ is the transfinite diameter of H.

Theorem. [with Béla Nagy]

Assume that the kernel k is positive, symmetric and satisfies the maximum principle. Let $K \subset X$ be any compact set. Then the transfinite diameter, the Chebyshev constant and the energy of K coincide:

$$D(K) = M(K) = w(K).$$

In fact:

- $M(K) \ge D(K)$ always
- $D(K) \ge w(K)$ always
- $w(K) \ge M(K)$ with maximum principle

with Szilárd Révész

For arbitrary subsets $H, L \subset X$ the n^{th} (weak) rendezvous set of L w.r.t. H is

$$R_n(H,L) := \bigcap_{w_1,\dots,w_n \in H} \overline{\operatorname{conv}} \left\{ \frac{1}{n} \sum_{j=1}^{n} k(x, w_j) \quad : \quad x \in L \right\} .$$

Correspondingly, one defines

$$R(H,L) := \bigcap_{n=1}^{\infty} R_n(H,L)$$

with Szilárd Révész

For arbitrary subsets $H, L \subset X$ the n^{th} (weak) rendezvous set of L w.r.t. H is

$$R_n(H,L) := \bigcap_{w_1,\dots,w_n \in H} \overline{\operatorname{conv}}\left\{\frac{1}{n}\sum_{j=1}^{n} k(x,w_j) \quad : \quad x \in L\right\}.$$

Correspondingly, one defines

$$R(H,L) := \bigcap_{n=1}^{\infty} R_n(H,L)$$

Similarly, one defines the (weak) average set of L w.r.t. H as

$$A(H,L) := \bigcap_{\mu \in \mathfrak{M}_1(H)} \overline{\operatorname{conv}}\left\{U^{\mu}(x) \quad : \quad x \in L\right\}$$

M. Baronti, E. Casini, P. L. Papini

In general $R(H, L)$ and $A(H, L)$ may be empty

If $a \in A(H, L) = \bigcap_{\mu \in \mathfrak{M}_1(H)} \overline{\mathrm{conv}} \Big\{ U^\mu(x) \quad : \quad x \in L \Big\}$ then:

For all measures $\mu \in \mathfrak{M}_1(H)$ and for all $\varepsilon > 0$ there are $x_1, x_2 \in L$

$$U^\mu(x_1) - \varepsilon < a < U^\mu(x_2) + \varepsilon$$

In general $R(H,L)$ and $A(H,L)$ may be empty

If $a \in A(H,L) = \bigcap_{\mu \in \mathfrak{M}_1(H)} \overline{\text{conv}}\Big\{U^\mu(x) \quad : \quad x \in L\Big\}$ then:

For all measures $\mu \in \mathfrak{M}_1(H)$ and for all $\varepsilon > 0$ there are $x_1, x_2 \in L$

$$U^\mu(x_1) - \varepsilon < a < U^\mu(x_2) + \varepsilon$$

If $r \in R(H,L)$ then:

For all $w_1, w_2, \ldots, w_n \in H$ and for all $\varepsilon > 0$ there are $x_1, x_2 \in L$

$$\frac{1}{n}\sum_{i=1}^{n} k(w_i, x_1) - \varepsilon < r < \frac{1}{n}\sum_{i=1}^{n} k(w_i, x_2) + \varepsilon$$

If L is compact, k is continuous

$$R(H,L) = \bigcap_{n\in\mathbb{N}, w_1,\dots,w_n\in H} \operatorname{conv}\left\{\frac{1}{n}\sum_{j=1}^{n} k(x,w_j) \quad : \quad x\in L\right\}$$

$$A(H,L) = \bigcap_{\mu\in\mathfrak{M}_1(H)} \operatorname{conv}\left\{U^{\mu}(x) \quad : \quad x\in L\right\}$$

If L is compact, k is continuous

$$R(H,L) = \bigcap_{n\in\mathbb{N},w_1,\ldots,w_n\in H} \operatorname{conv}\left\{\frac{1}{n}\sum_{j=1}^{n} k(x,w_j) \quad : \quad x\in L\right\}$$

$$A(H,L) = \bigcap_{\mu\in\mathfrak{M}_1(H)} \operatorname{conv}\left\{U^{\mu}(x) \quad : \quad x\in L\right\}$$

If $r \in R(H,L)$ then for any $w_1, w_2, \ldots, w_n \in H$ there are $x_1, x_2 \in L$

$$\frac{1}{n}\sum_{i=1}^{n} k(w_i,x_1) \le r \le \frac{1}{n}\sum_{i=1}^{n} k(w_i,x_2)$$

C. Thomassen (2000): weak rendezvous number

if L is compact and connected, k is continuous

$$R(H,L) = \bigcap_{n\in\mathbb{N}, w_1,\dots,w_n\in H} \left\{ \frac{1}{n}\sum_{j=1}^{n} k(x,w_j) \quad : \quad x \in L \right\}$$

$$A(H,L) = \bigcap_{\mu\in\mathfrak{M}_1(H)} \left\{ U^{\mu}(x) \quad : \quad x \in L \right\}$$

if L is compact and connected, k is continuous

$$R(H,L) = \bigcap_{n\in\mathbb{N},w_1,\ldots,w_n\in H} \left\{\frac{1}{n}\sum_{j=1}^{n} k(x,w_j) \quad : \quad x\in L\right\}$$

$$A(H,L) = \bigcap_{\mu\in\mathfrak{M}_1(H)} \left\{U^{\mu}(x) \quad : \quad x\in L\right\}$$

$a\in A(H,L)$ means:

For all measures $\mu\in\mathfrak{M}_1(H)$ there is $x\in L$

$$\int_H k(x,y)\,\mathrm{d}\mu(y) = U^{\mu}(x) = a$$

$$A(H,L) = \bigcap_{\mu \in \mathfrak{M}_1(H)} \operatorname{conv}\left\{ U^\mu(x) \quad : \quad x \in L \right\}$$

Proposition. $R(H,L) = [M(H,L), \overline{M}(H,L)], \quad A(H,L) = [\underline{q}(H,L), q(H,L)]$

4. Existence of rendezvous numbers

$$A(H,L) = \bigcap_{\mu \in \mathfrak{M}_1(H)} \operatorname{conv}\Big\{ U^\mu(x) \quad : \quad x \in L \Big\}$$

Proposition. $R(H,L) = [M(H,L), \overline{M}(H,L)], \quad A(H,L) = [\underline{q}(H,L), q(H,L)]$

Theorem. Let $\emptyset \neq H \subseteq L \subseteq X$ be arbitrary, and let $k \geq 0$ be any l.s.c symmetric kernel. Then the intervals $R(H,L)$ and $A(H,L)$ are nonempty.

$$A(H,L) = \bigcap_{\mu \in \mathfrak{M}_1(H)} \operatorname{conv}\Big\{ U^\mu(x) \quad : \quad x \in L \Big\}$$

Proposition. $R(H,L) = [M(H,L), \overline{M}(H,L)], \quad A(H,L) = [\underline{q}(H,L), q(H,L)]$

Theorem. Let $\emptyset \neq H \subseteq L \subseteq X$ be arbitrary, and let $k \geq 0$ be any l.s.c symmetric kernel. Then the intervals $R(H,L)$ and $A(H,L)$ are nonempty.

Proof.

It suffices $A(H,L) \neq \emptyset$. Let $\mu \in \mathfrak{M}_1(H)$, $\nu \in \mathfrak{M}_1(L)$. We have

$$\inf_{x \in L} U^\mu(x) = \inf_{x \in L} \int_H k(x,y)\,\mathrm{d}\mu(y) \leq \int_L \int_H k(x,y)\,\mathrm{d}\mu(y)\,\mathrm{d}\nu(x) =$$

$$= \int_H \int_L k(x,y)\,\mathrm{d}\nu(x)\,\mathrm{d}\mu(y) \leq \sup_{y \in H} \int_L k(x,y)\,\mathrm{d}\nu(x) = \sup_{y \in H} U^\nu(y)$$

$$A(H,L) = \bigcap_{\mu\in\mathfrak{M}_1(H)} \operatorname{conv}\Big\{U^\mu(x) \ : \ x \in L\Big\}$$

Proposition. $R(H,L) = [M(H,L), \overline{M}(H,L)], \quad A(H,L) = [\underline{q}(H,L), q(H,L)]$

Theorem. Let $\emptyset \neq H \subseteq L \subseteq X$ be arbitrary, and let $k \geq 0$ be any l.s.c symmetric kernel. Then the intervals $R(H,L)$ and $A(H,L)$ are nonempty.

Proof.

It suffices $A(H,L) \neq \emptyset$. Let $\mu \in \mathfrak{M}_1(H)$, $\nu \in \mathfrak{M}_1(L)$. We have

$$\inf_{x\in L} U^\mu(x) = \inf_{x\in L} \int_H k(x,y)\,\mathrm{d}\mu(y) \leq \int_L \int_H k(x,y)\,\mathrm{d}\mu(y)\,\mathrm{d}\nu(x) =$$

$$= \int_H \int_L k(x,y)\,\mathrm{d}\nu(x)\,\mathrm{d}\mu(y) \leq \sup_{y\in H} \int_L k(x,y)\,\mathrm{d}\nu(x) = \sup_{y\in H} U^\nu(y)$$

So $\underline{q}(H,L) \leq q(L,H)$. Notice that $q(H,L)$ is decreasing in H and increasing in L. Hence

$$\underline{q}(H,L) \leq q(L,H) \leq q(L,L) \leq q(H,L) \quad \Longrightarrow \quad A(H,L) = [\underline{q}(H,L), q(H,L)] \neq \emptyset$$

✓

Theorem. Let $K \Subset X$ be compact and $L \subset X$ be arbitrary.

$$q(K,L) = \inf_{\mu\in\mathfrak{M}_1(K)} \sup_{\nu\in\mathfrak{M}_1(L)} \int_L \int_K k(x,y)\,\mathrm{d}\mu(y)\,\mathrm{d}\nu(x) =$$

$$= \sup_{\nu\in\mathfrak{M}_1(L)} \inf_{\mu\in\mathfrak{M}_1(K)} \int_L \int_K k(x,y)\,\mathrm{d}\mu(y)\,\mathrm{d}\nu(x) = \underline{q}(L,K)$$

Theorem. Let $K \Subset X$ be compact and $L \subset X$ be arbitrary.

$$q(K,L) = \inf_{\mu\in\mathfrak{M}_1(K)} \sup_{\nu\in\mathfrak{M}_1(L)} \int\limits_L \int\limits_K k(x,y)\,\mathrm{d}\mu(y)\,\mathrm{d}\nu(x) =$$

$$= \sup_{\nu\in\mathfrak{M}_1(L)} \inf_{\mu\in\mathfrak{M}_1(K)} \int\limits_L \int\limits_K k(x,y)\,\mathrm{d}\mu(y)\,\mathrm{d}\nu(x) = \underline{q}(L,K)$$

Proof.

$$q(K,L) = \inf_{\mu\in\mathfrak{M}_1(K)} \sup_{\substack{\nu=\delta_z\\ z\in L}} \iint\limits_{L\,K} k(x,y)\,\mathrm{d}\mu(y)\,\mathrm{d}\nu(x) \le \inf_{\mu\in\mathfrak{M}_1(K)} \sup_{\nu\in\mathfrak{M}_1(L)} \iint\limits_{L\,K} k(x,y)\,\mathrm{d}\mu(y)\,\mathrm{d}\nu(x)$$

Theorem. Let $K \Subset X$ be compact and $L \subset X$ be arbitrary.

$$q(K,L) = \inf_{\mu\in\mathfrak{M}_1(K)} \sup_{\nu\in\mathfrak{M}_1(L)} \int_L \int_K k(x,y)\,\mathrm{d}\mu(y)\,\mathrm{d}\nu(x) =$$

$$= \sup_{\nu\in\mathfrak{M}_1(L)} \inf_{\mu\in\mathfrak{M}_1(K)} \int_L \int_K k(x,y)\,\mathrm{d}\mu(y)\,\mathrm{d}\nu(x) = \underline{q}(L,K)$$

Proof.

$$q(K,L) = \inf_{\mu\in\mathfrak{M}_1(K)} \sup_{\substack{\nu=\delta_z \\ z\in L}} \iint_{L\,K} k(x,y)\,\mathrm{d}\mu(y)\,\mathrm{d}\nu(x) \le \inf_{\mu\in\mathfrak{M}_1(K)} \sup_{\nu\in\mathfrak{M}_1(L)} \iint_{L\,K} k(x,y)\,\mathrm{d}\mu(y)\,\mathrm{d}\nu(x)$$

Glicksberg's Minimax Theorem:

$$q(K,L) \le \inf_{\mu\in\mathfrak{M}_1(K)} \sup_{\nu\in\mathfrak{M}_1(L)} \iint_{L\,K} k(x,y)\,\mathrm{d}\mu(y)\,\mathrm{d}\nu(x) = \sup_{\nu\in\mathfrak{M}_1(L)} \inf_{\mu\in\mathfrak{M}_1(K)} \iint_{L\,K} k(x,y)\,\mathrm{d}\mu(y)\,\mathrm{d}\nu(x)$$

$$= \sup_{\nu\in\mathfrak{M}_1(L)} \inf_{\mu\in\mathfrak{M}_1(K)} \int_L U^\mu(x)\,\mathrm{d}\nu(x) \le \sup_{\nu\in\mathfrak{M}_1(L)} \inf_{\mu\in\mathfrak{M}_1(K)} \sup_{x\in L} U^\mu(x) = q(K,L)$$

We have seen:

$$q(K,L) = \inf_{\mu\in\mathfrak{M}_1(K)} \sup_{\nu\in\mathfrak{M}_1(L)} \int_L \int_K k(x,y)\,\mathrm{d}\mu(y)\,\mathrm{d}\nu(x) =$$

$$= \sup_{\nu\in\mathfrak{M}_1(L)} \inf_{\mu\in\mathfrak{M}_1(K)} \int_L \int_K k(x,y)\,\mathrm{d}\mu(y)\,\mathrm{d}\nu(x)$$

We have seen:

$$q(K,L) = \inf_{\mu\in\mathfrak{M}_1(K)} \sup_{\nu\in\mathfrak{M}_1(L)} \int_L \int_K k(x,y)\,\mathrm{d}\mu(y)\,\mathrm{d}\nu(x) =$$
$$= \sup_{\nu\in\mathfrak{M}_1(L)} \inf_{\mu\in\mathfrak{M}_1(K)} \int_L \int_K k(x,y)\,\mathrm{d}\mu(y)\,\mathrm{d}\nu(x)$$

Then:

$$\underline{q}(L,K) \le q(K,L) = \sup_{\nu\in\mathfrak{M}_1(L)} \inf_{\mu\in\mathfrak{M}_1(K)} \int_L\int_K k(x,y)\,\mathrm{d}\mu(y)\,\mathrm{d}\nu(x) =$$
$$= \sup_{\nu\in\mathfrak{M}_1(L)} \inf_{\mu\in\mathfrak{M}_1(K)} \int_K U^\nu(y)\,\mathrm{d}\mu(y) \le \sup_{\nu\in\mathfrak{M}_1(L)} \inf_{y\in K} U^\nu(y) = \underline{q}(L,K)$$

✓

Theorem. Let $k \geq 0$ be any l.s.c., symmetric kernel and $\emptyset \neq K \Subset X$ compact. Then $A(K,K)$ is one single point. For continuous k, even $R(K,K)$ is a singleton.

Theorem. Let $k \geq 0$ be any l.s.c., symmetric kernel and $\emptyset \neq K \Subset X$ compact. Then $A(K, K)$ is one single point. For continuous k, even $R(K, K)$ is a singleton.

Theorem. Let $H, L \subset X$, then

$$M(H, L) = \underline{q}(H, L) \leq q(L, H) \leq \overline{M}(L, H)$$

If $L \subset X$ is compact, then

$$M(H, L) = \underline{q}(H, L) = q(L, H)$$

If $K \subset X$ is compact and k is continuous, then

$$q(L, K) = \overline{M}(L, K)$$

Theorem. Let $k \geq 0$ be any l.s.c., symmetric kernel and $\emptyset \neq K \Subset X$ compact. Then $A(K,K)$ is one single point. For continuous k, even $R(K,K)$ is a singleton.

Theorem. Let $H, L \subset X$, then

$$M(H,L) = \underline{q}(H,L) \leq q(L,H) \leq \overline{M}(L,H)$$

If $L \subset X$ is compact, then

$$M(H,L) = \underline{q}(H,L) = q(L,H)$$

If $K \subset X$ is compact and k is continuous, then

$$q(L,K) = \overline{M}(L,K)$$

Remark: For K compact $D(K) = w(K)$ and $M(K) = q(K)$
So with maximum principle $w(K) \leq q(K) \leq u(K)$
and $D(K) = M(K)$ follows.

S.Morris, P.Nickolas

a measure $\mu \in \mathfrak{M}_1(H)$ is k-invariant (on L), if the respective potential integral is constant:

$$U_k^\mu(x) := \int_X k(x,y)\,\mathrm{d}\mu(y) \equiv \text{const.} \qquad \text{(for all } x \in L\text{)}\ .$$

S.Morris, P.Nickolas

a measure $\mu \in \mathfrak{M}_1(H)$ is k-invariant (on L), if the respective potential integral is constant:

$$U_k^\mu(x) := \int_X k(x,y)\,\mathrm{d}\mu(y) \equiv \text{const.} \qquad \text{(for all } x \in L) \ .$$

Theorem. Assume that there exists a measure $\mu \in \mathfrak{M}_1(H)$ which is k-invariant on L (potential constant c). Then we have

$$A(H,L) = \{c\}.$$

Furthermore, if k is continuous and L is compact, then we even have

$$R(H,L) = \{c\}.$$

S.Morris, P.Nickolas

a measure $\mu \in \mathfrak{M}_1(H)$ is k-invariant (on L), if the respective potential integral is constant:

$$U_k^\mu(x) := \int_X k(x,y)\,\mathrm{d}\mu(y) \equiv \text{const.} \qquad \text{(for all } x \in L\text{)} .$$

Theorem. Assume that there exists a measure $\mu \in \mathfrak{M}_1(H)$ which is k-invariant on L (potential constant c). Then we have

$$A(H,L) = \{c\}.$$

Furthermore, if k is continuous and L is compact, then we even have

$$R(H,L) = \{c\}.$$

Corollary. (Morris–Nickolas) Let (X,d) be a compact (connected) metric space. Assume that there exists a d-invariant measure $\mu_0 \in \mathfrak{M}_1(X)$. Then we have

$$A(X) = R(X) = \{r(X)\} \qquad U^{\mu_0}(x) \equiv r(X) \quad (\forall x \in X) .$$

Basic background from potential theory:

Theorem. (Frostman, Fuglede, 1959). Let k be a positive, symmetric kernel and $K \Subset X$ with $w(K) < +\infty$. Every $\mu \in \mathfrak{M}_1(K)$ having minimal energy ($I(\mu, \mu) = w(K)$) satisfies

$$U^\mu(x) \geq w(K) \text{ for nearly every } x \in K \;,$$
$$U^\mu(x) \leq w(K) \text{ for every } x \in \operatorname{supp} \mu \;,$$
$$U^\mu(x) = w(K) \text{ for } \mu\text{-almost every } x \in X \;.$$

Basic background from potential theory:

Theorem. (Frostman, Fuglede, 1959). Let k be a positive, symmetric kernel and $K \Subset X$ with $w(K) < +\infty$. Every $\mu \in \mathfrak{M}_1(K)$ having minimal energy ($I(\mu, \mu) = w(K)$) satisfies

$$U^\mu(x) \geq w(K) \text{ for nearly every } x \in K \ ,$$
$$U^\mu(x) \leq w(K) \text{ for every } x \in \operatorname{supp} \mu \ ,$$
$$U^\mu(x) = w(K) \text{ for } \mu\text{-almost every } x \in X \ .$$

If the kernel is continuous: “nearly every” = “every”

Theorem. [R. Wolf; B.F., Sz. Révész] If k is continuous and K is compact, then $r(K) \geq w(K)$

Furthermore, if $r(K) = w(K)$, then there exists some k-invariant measure in $\mathfrak{M}_1(K)$.

Theorem. [R. Wolf; B.F., Sz. Révész] If k is continuous and K is compact, then $r(K) \geq w(K)$

Furthermore, if $r(K) = w(K)$, then there exists some k-invariant measure in $\mathfrak{M}_1(K)$.

Proof. $M(K) \geq w(K)$ is already known

Theorem. [R. Wolf; B.F., Sz. Révész] If k is continuous and K is compact, then $r(K) \geq w(K)$

Furthermore, if $r(K) = w(K)$, then there exists some k-invariant measure in $\mathfrak{M}_1(K)$.

Proof. $M(K) \geq w(K)$ is already known

continuity of k on the compact set K implies $\{M(K)\} = A(K) = R(K) = \{r(K)\}$.

Theorem. [R. Wolf; B.F., Sz. Révész] If k is continuous and K is compact, then $r(K) \geq w(K)$

Furthermore, if $r(K) = w(K)$, then there exists some k-invariant measure in $\mathfrak{M}_1(K)$.

Proof. $M(K) \geq w(K)$ is already known

continuity of k on the compact set K implies $\{M(K)\} = A(K) = R(K) = \{r(K)\}$.

Assume $r(K) = w(K)$. We have $w(K) < +\infty$

Take $\mu \in \mathfrak{M}_1(K)$ minimising $\sup_{x\in K} U^\mu(x)$, i.e., with $\sup_{x\in K} U^\mu(x) = q(K) = r(K) \Longrightarrow$

$$w(K) \leq I(\mu,\mu) = \int_K U^\mu(x)\,\mathrm{d}\mu(x) \leq \sup_{x\in K} U^\mu(x) = r(K) = w(K),$$

Theorem. [R. Wolf; B.F., Sz. Révész] If k is continuous and K is compact, then $r(K) \geq w(K)$

Furthermore, if $r(K) = w(K)$, then there exists some k-invariant measure in $\mathfrak{M}_1(K)$.

Proof. $M(K) \geq w(K)$ is already known

continuity of k on the compact set K implies $\{M(K)\} = A(K) = R(K) = \{r(K)\}$.

Assume $r(K) = w(K)$. We have $w(K) < +\infty$

Take $\mu \in \mathfrak{M}_1(K)$ minimising $\sup_{x \in K} U^\mu(x)$, i.e., with $\sup_{x \in K} U^\mu(x) = q(K) = r(K) \Longrightarrow$

$$w(K) \leq I(\mu, \mu) = \int\limits_K U^\mu(x)\, \mathrm{d}\mu(x) \leq \sup_{x \in K} U^\mu(x) = r(K) = w(K),$$

Hence μ minimises also $I(\mu, \mu)$, so Frostman's theorem is applicable

$\Longrightarrow w(K) \leq U^\mu(x) \leq= r(K) = w(K)$, so $U^\mu(x) = w(K)$ holds for all $x \in K$

Theorem. [M.Yamasaki/B. Nagy, B.F.] The kernel k is symmetric and continuous. If $M(K) = D(K)$ for all $K \subseteq X$ compact sets, then the kernel has the maximum principle.

Theorem. [M.Yamasaki/B. Nagy, B.F.] The kernel k is symmetric and continuous. If $M(K) = D(K)$ for all $K \subseteq X$ compact sets, then the kernel has the maximum principle.

Proof. $D(K) = w(K)$ for all $K \subseteq X$ compact

$\Longrightarrow$ existence of invariant measure on K

Theorem. [M.Yamasaki/B. Nagy, B.F.] The kernel k is symmetric and continuous. If $M(K) = D(K)$ for all $K \subseteq X$ compact sets, then the kernel has the maximum principle.

Proof. $D(K) = w(K)$ for all $K \subseteq X$ compact
$\Longrightarrow$ existence of invariant measure on K

First case: X finite. Induction: for $\#X = 2$, $X = \{a, b\}$.
Assume wlog: $k(a, a) \leq k(b, b)$.
The possible μ's: $\mu = \alpha\delta_a + \beta\delta_b$.
For $\mu = \delta_a$ to show $k(a, b) \leq k(a, a)$

Theorem. [M.Yamasaki/B. Nagy, B.F.] The kernel k is symmetric and continuous. If $M(K) = D(K)$ for all $K \subseteq X$ compact sets, then the kernel has the maximum principle.

Proof. $D(K) = w(K)$ for all $K \subseteq X$ compact

$\Longrightarrow$ existence of invariant measure on K

First case: X finite. Induction: for $\#X = 2$, $X = \{a, b\}$.

Assume wlog: $k(a, a) \leq k(b, b)$.

The possible μ's: $\mu = \alpha\delta_a + \beta\delta_b$.

For $\mu = \delta_a$ to show $k(a, b) \leq k(a, a)$

- $D(X) \leq k(a,a)$ holds by definition.
- For an energy minimising probability measure $\nu_p := p\delta_a + (1-p)\delta_b$ on X the potential is constant, so

$$pk(a,a) + (1-p)k(b,a) \ = \ pk(a,b) + (1-p)k(b,b) = M(X) = D(X) \leq k(a,a).$$

For $p = 1$, then $k(a,a) = k(a,b)$. If $p < 1$, then

$$(1-p)k(b,a) \leq (1-p)k(a,a), \quad \text{hence} \quad k(b,a) \leq k(a,a).$$

Let $\#X = n+1$, and μ on X we have to prove $\sup_{x\in X} U^\mu(x) = \sup_{x\in \operatorname{supp}\mu} U^\mu(x)$

- If $\operatorname{supp}\mu = X$, then there is nothing to prove.

7. The maximum principle; $n > 2$

Let $\#X = n+1$, and μ on X we have to prove $\sup_{x\in X} U^\mu(x) = \sup_{x\in \operatorname{supp}\mu} U^\mu(x)$

- If $\operatorname{supp}\mu = X$, then there is nothing to prove.
- if there are two distinct points $x_1 \neq x_2$, $x_1, x_2 \in X \setminus \operatorname{supp}\mu$, then by the induction hypothesis we have

$$\sup_{x\in X\setminus\{x_1\}} U^\mu(x) = \sup_{x\in \operatorname{supp}\mu} U^\mu(x) = \sup_{x\in X\setminus\{x_2\}} U^\mu(x).$$

Let $\#X = n + 1$, and μ on X we have to prove $\sup_{x \in X} U^\mu(x) = \sup_{x \in \operatorname{supp} \mu} U^\mu(x)$

- If $\operatorname{supp} \mu = X$, then there is nothing to prove.
- if there are two distinct points $x_1 \neq x_2$, $x_1, x_2 \in X \setminus \operatorname{supp} \mu$, then by the induction hypothesis we have

$$\sup_{x \in X \setminus \{x_1\}} U^\mu(x) = \sup_{x \in \operatorname{supp} \mu} U^\mu(x) = \sup_{x \in X \setminus \{x_2\}} U^\mu(x).$$

Let μ defy the maximum principle. We have $\# \operatorname{supp} \mu = n$, say $\operatorname{supp} \mu = X \setminus \{x_{n+1}\}$.
Set $K = \operatorname{supp} \mu$ and let μ' be an invariant measure on K.
μ' is also violating the maximum principle, because

- Consider $\mu_t := t\mu + (1 - t)\mu'$.
- $\exists\, \tau > 1$ such that μ_τ is still a probability measure and $\operatorname{supp} \mu_\tau \subsetneq \operatorname{supp} \mu$
- Induction: $U^{\mu_\tau}(x_{n+1}) \leq U^{\mu_\tau}(a)$ for some $a \in \operatorname{supp} \mu_\tau$.
- We know: $U^\mu(x_{n+1}) = U^{\mu_1}(x_{n+1}) > U^{\mu_1}(a)$
  $\Longrightarrow U^{\mu'}(x_{n+1}) = U^{\mu_0}(x_{n+1}) > U^{\mu_0}(a) = U^{\mu'}(y)$ for all $y \in K$.

Let now ν be an invariant measure on X. We have

$$\begin{aligned} M(X) &= U^{\nu}(y) = \sup_{x \in X} U^{\nu}(x) = D(X) \\ &\leq D(K) = \sup_{x \in K} U^{\mu'}(x) = U^{\mu'}(z) < U^{\mu'}(x_{n+1}) \end{aligned}$$

for all $y \in X$, $z \in K$.

Let now ν be an invariant measure on X. We have

$$\begin{aligned} M(X) &= U^\nu(y) = \sup_{x\in X} U^\nu(x) = D(X) \\ &\le D(K) = \sup_{x\in K} U^{\mu'}(x) = U^{\mu'}(z) < U^{\mu'}(x_{n+1}) \end{aligned}$$

for all $y \in X$, $z \in K$.

$\Longrightarrow U^\nu(y) \le U^{\mu'}(y)$ for all $y \in X$ and even "$<$" for $y = x_{n+1}$

Integrate w.r.t. ν

$$\int_X \int_X k \, \mathrm{d}\nu \, \mathrm{d}\nu = M(X) < \int_X \int_X k \, \mathrm{d}\mu' \, \mathrm{d}\nu = \int_X \int_X k \, \mathrm{d}\nu \, \mathrm{d}\mu' = M(X),$$

hence a contradiction, unless $\nu(\{x_{n+1}\}) = 0$

Let now ν be an invariant measure on X. We have

$$\begin{aligned} M(X) &= U^{\nu}(y) = \sup_{x \in X} U^{\nu}(x) = D(X) \\ &\leq D(K) = \sup_{x \in K} U^{\mu'}(x) = U^{\mu'}(z) < U^{\mu'}(x_{n+1}) \end{aligned}$$

for all $y \in X$, $z \in K$.
$\Longrightarrow U^{\nu}(y) \leq U^{\mu'}(y)$ for all $y \in X$ and even "$<$" for $y = x_{n+1}$

Integrate w.r.t. ν

$$\int\limits_X \int\limits_X k \, \mathrm{d}\nu \, \mathrm{d}\nu = M(X) < \int\limits_X \int\limits_X k \, \mathrm{d}\mu' \, \mathrm{d}\nu = \int\limits_X \int\limits_X k \, \mathrm{d}\nu \, \mathrm{d}\mu' = M(X),$$

hence a contradiction, unless $\nu(\{x_{n+1}\}) = 0$

If $\nu(\{x_{n+1}\}) = 0$ held, then ν would be an energy minimising measure on K:

$$M(X) = \int\limits_K \int\limits_K k \, \mathrm{d}\nu \, \mathrm{d}\mu' = \int\limits_K \int\limits_K k \, \mathrm{d}\mu' \, \mathrm{d}\nu = M(K) \quad \text{holds.}$$

A contradiction since ν satisfies the maximum principle.

Let now ν be an invariant measure on X. We have

$$\begin{aligned} M(X) &= U^{\nu}(y) = \sup_{x\in X} U^{\nu}(x) = D(X) \\ &\le D(K) = \sup_{x\in K} U^{\mu'}(x) = U^{\mu'}(z) < U^{\mu'}(x_{n+1}) \end{aligned}$$

for all $y \in X$, $z \in K$.

$\Longrightarrow U^{\nu}(y) \le U^{\mu'}(y)$ for all $y \in X$ and even "$<$" for $y = x_{n+1}$

Integrate w.r.t. ν

$$\int\limits_X \int\limits_X k \,\mathrm{d}\nu \,\mathrm{d}\nu = M(X) < \int\limits_X \int\limits_X k \,\mathrm{d}\mu' \,\mathrm{d}\nu = \int\limits_X \int\limits_X k \,\mathrm{d}\nu \,\mathrm{d}\mu' = M(X),$$

hence a contradiction, unless $\nu(\{x_{n+1}\}) = 0$

If $\nu(\{x_{n+1}\}) = 0$ held, then ν would be an energy minimising measure on K:

$$M(X) = \int\limits_K \int\limits_K k \,\mathrm{d}\nu \,\mathrm{d}\mu' = \int\limits_K \int\limits_K k \,\mathrm{d}\mu' \,\mathrm{d}\nu = M(K) \quad \text{holds.}$$

A contradiction since ν satisfies the maximum principle.

Second case: X arbitrary compact set: follows by approximation.

For compact sets and continuous kernels

- ❑ $r(K) = M(K) = q(K)$ (uniform energy)
- ❑ $D(K) = w(K)$ (Wiener energy)
- ❑ $D(K) \leq M(K)$ always
- ❑ $D(K) = M(K) \iff$ the kernel satisfies the maximum principle

Thank you for listening!