Universal bounds on energy of codes and designs in Hamming spaces

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Budapest, Hungary, August 18-24, 2015

Codes in $H(n, q)$

- Let $H(n,q) = \{(x_1, x_2, \ldots, x_n) : x_i \in \{0, 1, \ldots, q-1\}\}$, with distance

$$d(x,y) = |\{i : x_i \neq y_i\}|,$$

$x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n) \in H(n,q)$, and "inner" product

$$\langle x, y \rangle = 1 - \frac{2d(x,y)}{n}.$$

- nonempty $C \subset H(n,q)$ – code
- Three main parameters of codes: length n, cardinality $M = |C|$ and minimum distance

$$d = d(C) = \min\{d(x,y) : x, y \in C, x \neq y\}.$$

Energy of codes in $H(n, q)$

- For a code $C \subset H(n,q)$ and a given function $h(t) : [-1,1) \to [0,+\infty)$, we define the *h-energy* (or potential energy) of C by

$$W(n, C; h) := \frac{1}{|C|} \sum_{x,y \in C, x \neq y} h(\langle x, y \rangle).$$

- At times we require h to be *absolutely monotone* on [-1,1); i.e., the k-th derivative (continuous or discrete) of h satisfies $h^{(k)}(t) \geq 0$ for all $k \geq 0$ and $t \in [-1,1)$.
- A commonly arising problem is to minimize the potential energy provided the cardinality $|C|$ of C is fixed; that is, to determine

$$\mathcal{W}(n, M; h) := \min\{W(n, C; h) : |C| = M\}$$

the minimum possible h-energy of a code of cardinality M.

Some interesting potentials

- $h(t) = \left(\frac{2}{n(1-t)}\right)^{\alpha}$, for $\alpha \to \infty$ the optimal codes maximize the minimum distance d for fixed cardinality M.
- $h_s(t) = \begin{cases} 0, & \text{if } t \in [-1, s], \\ +\infty, & \text{if } t \in (s, 1]. \end{cases}$, the optimal codes maximize the cardinality for fixed length and minimum distance (maximal inner product); i.e. attain the LP bound.
- $h(t) = \gamma^{2/n(1-t)}$, where γ is Bhattacharyya parameter, the optimal codes minimize the union (upper) bound for the decoding error.
- $h_j(t) = \binom{n-t(n-1)/2}{j}$, $j = 0, 1, 2, \ldots, n$, span the cone of absolute monotone discrete (on T_n – the set of inner products) functions.

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Designs in $H(n, q)$

- **Definition** Let τ and λ be positive integers. A τ-design $C \subset H(n, q)$ of strength τ and index λ is a code such that the $M \times n$ matrix obtained from the codewords of C as rows has the following property: every $M \times \tau$ submatrix contains all ordered τ-tuples of $H(\tau, q)$, each one exactly $\lambda = \frac{M}{q^\tau}$ times as rows.
  $\tau = d' - 1$ (d' is the dual distance of C).

- Example of $(3, 4, 8)$ BOA:

0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	1

Energy of τ-designs

- Let $C \subset H(n,q)$ be a τ-design in $H(n,q)$;
- Denote by

$$\mathcal{L}(n,M,\tau;h) = \min\{W(n,C;h) : |C| = M, C \subset H(n,q), C \text{ is } \tau\text{-design}\}$$

the minimum possible h-energy of τ-designs in $H(n,q)$ of M points,

$$\mathcal{U}(n,M,\tau;h) = \max\{W(n,C;h) : |C| = M, C \subset H(n,q), C \text{ is } \tau\text{-design}\}$$

the maximum possible h-energy of τ-designs in $H(n,q)$ of M points.

Energy of codes of minimum distance d

- Denote by

$$\mathcal{F}(n,d;h) = \min\{W(n,C;h) : |C \subset H(n,q), d(C) = d\}$$

the minimum possible h-energy of codes in $H(n,q)$ of minimum distance d,

$$\mathcal{G}(n,d;h) = \max\{W(n,C;h) : |C \subset H(n,q), d(C) = d\}$$

the maximum possible h-energy of codes in $H(n,q)$ of minimum distance d.

- Allows involving the function $A_q(n,d)$.

Rao's bound

- (Rao, 1947) For fixed strength τ and dimension n denote by

$$B(n,\tau) = \min\{|C| : \exists\ \tau\text{-design } C \subset H(n,q)\}.$$

$$B(n,\tau) \geq R(n,\tau) = \begin{cases} q\sum_{i=0}^{k-1} \binom{n}{i}(q-1)^i, & \text{if } \tau = 2k-1, \\ \sum_{i=0}^{k} \binom{n-1}{i}(q-1)^i, & \text{if } \tau = 2k. \end{cases}$$

- Can be derived by LP with $h(t) = 0$.

Krawtchouk polynomials (1)

- For fixed n and q, the (normalized) Krawtchouk polynomials are defined by
$$Q_i^{(n,q)}(t) = \frac{1}{r_i} K_i^{(n,q)}(n(1-t)/2),$$
where $r_i = (q-1)^i \binom{n}{i}$, $t = 1 - \dfrac{2d}{n} \iff d = \dfrac{n(1-t)}{2}$,
$$K_i^{(n,q)}(d) = \sum_{j=0}^{i} (-1)^j (q-1)^{i-j} \binom{d}{j} \binom{n-d}{i-j}, \quad i = 0, 1, \ldots, n,$$
be the (usual) Krawtchouk polynomials corresponding to $H(n,q)$.

Krawtchouk polynomials (2)

- Adjacent polynomials

$$Q_i^{(1,0,n,q)}(t) = \frac{K_i^{(n-1,q)}(d-1)}{\sum_{j=0}^{i} \binom{n}{j}(q-1)^j}, \qquad (1)$$

$$Q_i^{(1,1,n,q)}(t) = \frac{K_i^{(n-2,q)}(d-1)}{\sum_{j=0}^{i} \binom{n-1}{j}(q-1)^j}, \qquad (2)$$

$$Q_i^{(0,1,n,q)}(t) = \frac{K_i^{(n-1,q)}(d-1)}{\binom{n}{i}(q-1)^i}, \qquad (3)$$

where $d = n(1-t)/2$.

Krawtchouk polynomials (3)

- If $f(t) \in \mathbb{R}[t]$ is a real polynomial of degree $m \leq n$ then $f(t)$ can be uniquely expanded in terms of the Krawtchouk polynomials as

$$f(t) = \sum_{i=0}^{n} f_i Q_i^{(n,q)}(t).$$

- Coefficients f_i can be found in (at least) two different ways – by direct comparison of the coefficients; by using the orthogonality relations for the Krawtchouk polynomials.

- f_0 – the most important coefficient.

Useful quadrature (1)

- V. I. Levenshtein, Designs as maximum codes in polynomial metric spaces, Acta Appl. Math. 25, 1992, 1-82.
- (odd case $\tau = 2k-1$) For every fixed (cardinality) $M > R(n, 2k-1)$ there exist uniquely determined real numbers

$$-1 < \alpha_0 < \alpha_1 < \cdots < \alpha_{k-1} < 1$$

and $\rho_0, \rho_1, \ldots, \rho_{k-1}$, $\rho_i > 0$ for $i = 0, 1, \ldots, k-1$, such that the equality

$$f_0 = \frac{f(1)}{M} + \sum_{i=0}^{k-1} \rho_i f(\alpha_i)$$

holds for every real polynomial $f(t)$ of degree at most $2k-1$.

Useful quadrature (2)

- The numbers α_i, $i = 0, 1, \ldots, k-1$, are the roots of the equation

$$P_k(t)P_{k-1}(s) - P_k(s)P_{k-1}(t) = 0,$$

  where $s = \alpha_{k-1}$, $P_i(t) = Q_i^{1,0,n,q}(t)$.
- In fact, α_i, $i = 0, 1, \ldots, k-1$, are the roots of the Levenshtein's polynomial $f_{2k-1}^{(n,\alpha_{k-1})}(t)$.
- Similarly, nodes β_i and weights are defined in the even case $\tau = 2k$.

Levenshtein bound

- Levenshtein bound on
  $A_q(n,s) := \max\{|C| : C \subset H(n,q), \langle x,y\rangle \le s,\ x \ne y \in C\}$

$$A_q(n,s) \le \begin{cases} L_{2k-1}(n,s) = \left(1 - \dfrac{Q_{k-1}^{(1,0,n,q)}(s)}{Q_k^{(n,q)}(s)}\right) \displaystyle\sum_{j=0}^{k-1} \binom{n}{j}(q-1)^j, \\ \qquad \text{if } s \in \mathcal{I}_{2k-1}, \\ L_{2k}(n,s) = q\left(1 - \dfrac{Q_{k-1}^{(1,1,n,q)}(s)}{Q_k^{(0,1,n,q)}(s)}\right) \displaystyle\sum_{j=0}^{k-1} \binom{n-1}{j}(q-1)^j, \\ \qquad \text{if } s \in \mathcal{I}_{2k}, \end{cases}$$

where $[-1,1] = \mathcal{I}_1 \cup \mathcal{I}_2 \cup \cdots \cup \mathcal{I}_{2n}$, the intervals $\mathcal{I}_\tau$ are determined by largest roots of adjacent polynomials.

Connection between Rao and Levenshtein bounds

- The connection between the Rao bound and the Levenshtein bound is given by the equalities

$$L_{2k-2}(n, t_{k-1}^{1,1}) = L_{2k-1}(n, t_{k-1}^{1,1}) = R(n, 2k-1),$$
$$L_{2k-1}(n, t_k^{1,0}) = L_{2k}(n, t_k^{1,0}) = R(n, 2k)$$

and the ends of the intervals $\mathcal{I}_\tau$.

Location of the cardinality M

- In what follows we always take care where the cardinality M is located with respect to the Rao bound. If

$$M \in (R(n,\tau), R(n,\tau+1)]\,,$$

then we consider the equation

$$M = L_\tau(n,s)$$

as source to the necessary parameters, i.e. we can always associate M with the corresponding (unique) numbers:

$$\alpha_0, \alpha_1, \ldots, \alpha_{k-1}, \rho_0, \rho_1, \ldots, \rho_{k-1} \text{ when } M \in (R(n,2k-1), R(n,2k)]$$

or

$$\beta_0, \beta_1, \ldots, \beta_k, \gamma_0, \gamma_1, \ldots, \gamma_k \text{ when } M \in (R(n,2k), R(n,2k+1)]\,.$$

General bounds – lower bounds for $\mathcal{W}(n, M; h)$ and $\mathcal{L}(n, M, \tau; h)$

- **Theorem 1.** Let n, M, h (and τ) be fixed and $f(t)$ be a real polynomial such that

  (A1) $f(t) \leq h(t)$ for every $t \in T_n = \{-1 + 2i/n \,|i = 0, 1, \ldots, n\}$;

  (A2) the coefficients in the Krawtchouk expansion $f(t) = \sum_{i=0}^{\deg(f)} f_i Q_i^{(n,q)}(t)$ satisfy $f_i \geq 0$ for $i \geq 1$ (for $i \geq \tau + 1$, respectively).

  Then $\mathcal{W}(n, M; h) \geq f_0 M - f(1)$ $(\mathcal{L}(n, M, \tau; h) \geq f_0 M - f(1)$, respectively).

- $A_{n,M;h}$ $(A_{n,M,\tau;h})$ – the set of good polynomials

- **Theorem 2.** Let n, M, τ and h be fixed and $g(t)$ be a real polynomial such that

  (B1) $g(t) \geq h(t)$ for $t \in T_n \cap [-1, t_0]$; where t_0 is such that no τ-design in $H(n, q)$ of M points can have inner products in the interval $(t_0, 1)$.

  (B2) the coefficients in the Krawtchouk expansion $g(t) = \sum_{i=0}^{\deg(g)} g_i Q_i^{(n,q)}(t)$ satisfy $g_i \leq 0$ for $i \geq \tau + 1$.

  Then $\mathcal{U}(n, M, \tau; h) \leq g_0 M - g(1)$.

- $B_{n,M,\tau;h}$ – the set of good polynomials

- **Theorem 3.** Let n, d and h be fixed and $f(t)$ be a real polynomial that satisfies (A2) and
  (A1′) $f(t) \leq h(t)$ for every $t \in T_n \cap [-1, 1-2d/n]\}$;
  Then $\mathcal{F}(n,d;h) \geq f_0 M - f(1)$, where M is a feasible size of a code of minimum distance d. In particular,
  $$\mathcal{F}(n,d;h) \geq f_0 A_q(n,d) - f(1).$$
- **Theorem 4.** Let n, d and h be fixed and $g(t)$ be a real polynomial such that:
  (B1′) $g(t) \geq h(t)$ for every $t \in T_n \cap [-1, 1-2d/n]\}$;
  (B2′) the coefficients in the expansion $g(t) = \sum_{i=0}^{n} g_i Q_i^{(n,q)}(t)$ satisfy $g_i \leq 0$ for every $i \geq 1$.
  Then $\mathcal{G}(n,d;h) \geq g_0 M - g(1)$, where M is a feasible size of a code of minimum distance d. In particular,
  $$\mathcal{G}(n,d;h) \leq g_0 A_q(n,d) - g(1).$$

Interpolation (1)

- We use Hermite's interpolation to $h(t)$ as follows. Define
- (i) the polynomial $f(t)$ of degree $\tau = 2k-1$ by

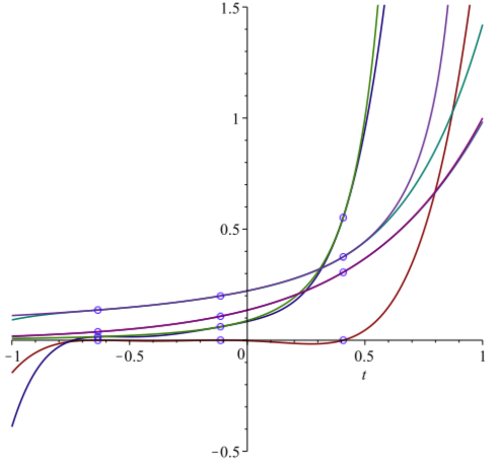
$$f(\alpha_i) = h(\alpha_i),\ f'(\alpha_i) = h'(\alpha_i),\ i = 0, 1, \ldots, k-1.$$

- (ii) the polynomial $f(t)$ of degree $\tau = 2k$ by

$$f(\beta_0) = h(\beta_0),\ f(\beta_i) = h(\beta_i),\ f'(\beta_i) = h'(\beta_i),\ i = 1, \ldots, k.$$

- These conditions define a Hermite's interpolation problem for $f(t)$ to intersect and touch the graph of the potential function $h(t)$.
- This implies that the conditions of LP bounds are satisfied.

Interpolation (2)



Some Hermite interpolants at the universal nodes for $n = 9$, $M = 128$

Lower bounds – universal bound (1)

- **Theorem.** Let n, τ, $M \in (R(n,\tau), R(n,\tau+1)]$ and h be fixed. Then the polynomials from the interpolations (i) and (ii) belong to $A_{n,M,\tau;h}$ and give the bounds

$$\mathcal{L}(n, M, 2k-1; h) \geq M \sum_{i=0}^{k-1} \rho_i h(\alpha_i),$$

$$\mathcal{L}(n, M, 2k; h) \geq M \sum_{i=0}^{k} \gamma_i h(\beta_i),$$

respectively.

These bounds can not be improved by using admissible polynomials $f(t) \leq h(t)$ for $t \in [-1, 1)$ of degree at most $2k-1$ (at most $2k$, respectively).

Lower bounds – universal bound (2)

- **Theorem.** Let n, $M \in (R(n,\tau), R(n,\tau+1)]$ and h be fixed. Then the polynomials from the interpolations (i) and (ii) belong to $A_{n,M;h}$ and give the bounds

$$\mathcal{W}(n,M;h) \geq M \sum_{i=0}^{k-1} \rho_i h(\alpha_i),$$

$$\mathcal{W}(n,M;h) \geq M \sum_{i=0}^{k} \gamma_i h(\beta_i),$$

  respectively.
  These bounds can not be improved by using admissible polynomials $f(t) \leq h(t)$ for $t \in [-1,1)$ of degree at most $2k-1$ (at most $2k$, respectively).

Lower bounds – universal bound (3)

Some remarks

- The bounds do not depend (in certain sense) from the potential function h. The same nodes work for every potential.
- The bounds are attained by all maximal codes which attain the Levenshtein bound (universally optimal in the sense of Cohn-Zhao's paper).
- However, the bounds can be improved in other cases. For example, if one takes care for the discrete nature of the possible inner products (i.e. the notion of admissibility can be widened).
- There are necessary and sufficient conditions for the global optimality of our bounds (with $f(t) \leq h(t)$ in the whole interval $[-1, 1)$).

Interpolation (3)

- For every $i = 0, 1, \ldots, k-1$ (recall that $t_j = 1 - 2j/n$) let

$$t_{j(i)} \le \alpha_i < t_{j(i)+1}.$$

- If necessary, adjust $t_{j(i)}$, $t_{j(i)+1}$, etc.
- Use Lagrange interpolation of $h(t)$ in $t_{j(i)}$ and $t_{j(i)+1}$ instead of Hermite interpolation in α_i (i.e. the double "zero" in α_i will be replaced by two intersections – in $t_{j(i)}$ and $t_{j(i)+1}$). The degree of the interpolating polynomial remains the same.

- Cohn-Zhao's paper – pair covering (the set of interpolation nodes). Therefore, the α_i's just say which pairs must be covered.

- This approach gives better bounds in many cases. However, it is difficult to express the bounds by formulas.

Higher degrees (1)

- Let n, M, $\tau = \tau(n, M)$ be fixed and the equation $L_\tau(n, s) = M$, $s = \alpha_{k-1}$, define all necessary parameters. Let j be a positive integer. We consider the following *test-functions* in n and s:

$$P_j(n, s) := \frac{1}{M} + \sum_{i=0}^{k-1} \rho_i Q_j^{(n,q)}(\alpha_i) \quad \text{for} \quad s \in \mathcal{I}_{2k-1},$$

introduced in 1998 for the so-called polynomial metric spaces (include $H(n, q)$) by B.-Danev (the binary case $q = 2$ considered in detail).

Higher degrees (2)

- **Theorem.** Let h be strictly absolutely monotone function. The ULB can be improved by a polynomial $f \in A_{n,h}$ (or $A_{n,\tau,h}$) satisfying $f(t) \leq h(t)$ in the whole interval $[-1,1)$ of degree at least $\tau+1$ if and only if $P_j(n,s) < 0$ for some $j \geq \tau+1$. Furthermore, if $P_j(n,s) < 0$ for some $j \geq \tau+1$, then ULB can be improved by a polynomial as above of degree exactly j.
- We develop algorithms for deriving better bounds by higher degree polynomials.

Higher degrees bounds (3)

- LP-universally optimal codes – codes which universal optimality can be proved by suitable polynomial in the LP bound.

  Corollary. If $P_j(n,s) \geq 0$ for every $j \in \{\tau, \tau+1, \ldots, n\}$ then ULB can not be improved by linear programming (with $f(t) \leq h(t)$ in $[-1,1)$).

  Corollary. If $C \subset \mathbb{H}(n,q)$ has energy $W(n,C,h) >$ ULB and $P_j(n,s) \geq 0$ for every $j \in \{\tau+1, \ldots, n\}$ then C is can not be proved to be LP-universally optimal with $f(t) \leq h(t)$ in $[-1,1)$.

  Proof. There are finitely many, $n-\tau$, namely, test functions to be checked.

- Cohn-Zhao (2012) note that proof of LP-universal optimality can be given by brute force by solving n linear programs (one per degree).

Higher degrees bounds (4)

- Following the analogy with the upper LP bounds on $A_q(n,s)$ we have to target:
  (1) zero coefficients in the Krawtchouk expansion of the optimal polynomials $f(t)$;
  (2) touching points of the graphs of the optimal polynomials $f(t)$ and the potential function $h(t)$.
- We assume that n and M are such that $P_j(n,s) < 0$ for some $j \in \{\tau+3, \tau+4\}$.
- $q = 2$ – we know that for $P_{\tau+1}(n,s) > 0$ and $P_{\tau+2}(n,s) > 0$ in the open interval $\mathcal{I}_\tau$.

Higher degrees bounds (5)

- **Theorem.** (for $\tau = 2k - 1$) If an optimal polynomial of degree $\tau + 4$ has $f_i > 0$ for $i = 1, 2, \ldots, \tau = 2k - 1$ then the graph of $f(t)$ touches the graph of $h(t)$ at $k + 1 = \left[\frac{\tau+3}{2}\right]$ points.
- For particular potentials – lemmas that show that $f_i = 0$ is necessary for some indexes i.
- Algorithm for finding optimal improving polynomials of degrees $\tau + 3$ and $\tau + 4$.

Examples (1)

- $q = 2$, $n = 19$, $M = 20$, $\tau = 2$ – there are two non-isomorphic binary 2-designs of 20 points in $H(19, 2)$. They have the same unique distance distribution

$$(1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 19, 0, 0, 0, 0, 0, 0, 0, 0, 0).$$

All bounds (the universal, with Lagrange interpolations, the combinatorial) coincide with the actual energy – these designs are optimal.

Examples (2)

- The Best code – optimal (nonlinear) binary code of length 10 with 40 codewords and minimum distance 4; $q = 2$, $n = 10$, $M = 40$, $\tau = 3$ – unique distance distribution $(1, 0, 0, 0, 22, 0, 12, 0, 5, 0, 0)$. Therefore the combinatorial bound gives the actual energy, i.e. the Best code is optimal.

  The other bounds are very close, for example if $h = \dfrac{1}{2(1-t)}$ then the actual energy is 812.5, the universal bound is ≈ 807.222, the pair-covering bound is ≈ 808.571, obtained by

$$
\begin{aligned}
f(t) &= \frac{125}{448}t^3 + \frac{225}{448}t^2 + \frac{115}{224}t + \frac{1}{2} \\
&= \frac{45}{224}Q_0^{10,2}(t) + \frac{405}{896}Q_1^{10,2}(t) + \frac{265}{448}Q_2^{10,2}(t) + \frac{493}{896}Q_3^{10,2}(t)
\end{aligned}
$$

Examples (3)

Moreover, better bound of ≈ 809.167 can be obtained by a polynomial of degree 7

$$\begin{aligned} f(t) &= 0.27680344120t^7 - 0.3875248178t^5 + 0.4402281803t^3 \\ &\quad +0.52083333375t^2 + 0.5038265303t + 0.5 \\ &= 0.552083333375Q_0^{10,2}(t) + 0.598958333500Q_1^{10,2}(t) \\ &\quad +0.468750000375Q_2^{10,2}(t) + 0.217633928625Q_3^{10,2}(t) \\ &\quad +0.016741072125Q_7^{10,2}(t). \end{aligned}$$

Examples (4)

- $q = 2,\ n = 9,\ M = 128,\ \tau = 5$ – there is a unique binary 5-design of 128 points in $H(9,2)$. It has a unique distance distribution $(1,0,9,27,27,27,27,9,0,1)$.
  For $h = \dfrac{1}{2(1-t)}$ the actual energy is 9085.49, the universal bound is ≈ 9054.82, the pair-covering bound is ≈ 9073.03, obtained by

$$
\begin{aligned}
f(t) &= 0.4118931362t^5 + 0.7780203683t^4 + 0.6407226562t^3 \\
&\quad + 0.4870396205t^2 + 0.4862234933t + 0.4987792969 \\
&= 0.1054687500Q_0^{9,2}(t) + 0.3585937500Q_1^{9,2}(t) \\
&\quad + 0.6890625000Q_2^{9,2}(t) + 0.8256696429Q_3^{9,2}(t) \\
&\quad + 0.7443080357Q_4^{9,2}(t) + 0.5795758929Q_5^{9,2}(t)
\end{aligned}
$$

Examples (5)

Here, the best lower bound is equal to the actual energy 9085.49 can be obtained by a polynomial of degree 9

$$
\begin{aligned}
f(t) &= 3.202285254t^9 + 1.824557875t^8 - 5.538491637t^7 \\
&\quad -3.153556821t^6 + 2.745477514t^5 + 1.917797959t^4 \\
&\quad +0.302460339t^3 + 0.373690142t^2 + 0.502107814t \\
&\quad +0.501350130 \\
&= 0.575453404Q_0^{9,2}(t) + 0.708272879Q_1^{9,2}(t) \\
&\quad +0.708649553Q_2^{9,2}(t) + 0.478710937Q_3^{9,2}(t) \\
&\quad +0.164355468Q_4^{9,2}(t) + 0.023856026Q_7^{9,2}(t) \\
&\quad +0.015380859Q_8^{9,2}(t) + 0.002999441Q_9^{9,2}(t)
\end{aligned}
$$

Therefore this code is optimal.

Future work

- Formulas for higher degree bounds
- Bounds for inner products in $[\ell, u] \subset [1-, 1]$
- Upper bounds (for codes and design separately)
- List of optimal codes
- Asymptotic bounds – the asymptotic of the universal bounds must be studied.

THANK YOU FOR YOUR ATTENTION !