Energy bounds for spherical designs and for codes and designs in Hamming spaces

Peter Boyvalenkov
Institute of Mathematics and Informatics, Bulgarian Academy of Sciences

(joint work with Peter Dragnev, Douglas Hardin, Edward Saff, Maya Stoyanova)

Budapest, Hungary, August 18-24, 2015

Contents

- Definitions and notations (spherical codes and designs, energy, problems, some references)
- Preliminaries (Delsarte-Goethals-Seidel bound, Levenshtein bound, useful quadrature)
- LP bounds on energy of spherical codes/designs (folklore)
- Universal lower bound (ULB)
- Two ways of improving ULB for designs – shrinking the interval (bounds on inner products, LP in shorter interval), and higher degrees (test-functions, optimal codes/designs)
- Upper bounds on energy of spherical designs
- Asymptotic bounds (ULB and LP in shorter interval)
- Parallels for codes and designs in Hamming spaces

Spherical codes and designs (1)

- Let $\mathbb{S}^{n-1}$ denote the unit sphere in $\mathbb{R}^n$.
- We work with the the usual distance

$$d(x,y) = \left((x_1 - y_1)^2 + \cdots + (x_n - y_n)^2\right)^{1/2},$$

where $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{S}^{n-1}$, and the usual inner product

$$\langle x, y \rangle = x_1 y_1 + \cdots + x_n y_n.$$

- On $\mathbb{S}^{n-1}$, we have

$$\langle x, y \rangle = 1 - \frac{d^2(x,y)}{2}, \quad d(x,y) = \sqrt{2(1 - \langle x, y \rangle)}$$

and we prefer to work with the inner products.

Spherical codes and designs (2)

- We refer to a finite set $C \subset \mathbb{S}^{n-1}$ as a spherical code.
- A spherical τ-design is a spherical code such that

$$\frac{1}{\mu(\mathbb{S}^{n-1})} \int_{\mathbb{S}^{n-1}} f(x) d\mu(x) = \frac{1}{|C|} \sum_{x \in C} f(x)$$

($\mu(x)$ is the Lebesgue measure) holds for all polynomials $f(x) = f(x_1, x_2, \ldots, x_n)$ of total degree at most τ. The maximal number $\tau = \tau(C)$ such that C is a spherical τ-design is called the *strength* of C.

Spherical codes and designs (3)

- Parameters of spherical designs: the dimension n, the strength τ, the cardinality $|C| = M$, the minimum distance

$$d(C) = \min\{d(x,y) : x,y \in C, x \neq y\},$$

the maximal inner product (or maximal cosine)

$$s(C) = \max\{\langle x,y\rangle : x,y \in C, x \neq y\},$$

the covering radius

$$\rho(C) = \min_{y \in \mathbb{S}^{n-1}} \max_{x \in C} \langle x,y\rangle,$$

the mesh ratio

$$MR(C) = 2\sqrt{\frac{1-\rho(C)}{1-s(C)}}.$$

Energy of spherical codes/designs

- Let $h(t) : [-1,1) \to (0,+\infty)$ be given function. The h-energy (or potential energy) of C is defined by

$$E(n,C;h) := \sum_{x,y\in C, x\neq y} h(\langle x,y\rangle).$$

- A commonly arising problem is to minimize the potential energy provided the cardinality $|C|$ of C is fixed; that is, to determine

$$\mathcal{E}(n,N;h) := \inf\{E(n,C;h) : |C| = N\}$$

  the minimum possible h-energy of a spherical design (or code) of cardinality N.

- For our main results we require h to be (strictly) *absolutely monotone* on [-1,1); i.e., the k-th derivative of h satisfies $h^{(k)}(t) \geq 0$ ($h^{(k)}(t) > 0$) for all $k \geq 0$ and $t \in [-1,1)$.

Energy of spherical τ-designs

- Let $C \subset \mathbb{S}^{n-1}$ be a spherical τ-design and $E(n, C; h)$ be the h-energy of C.
- Denote by

$$L(n, N, \tau; h) = \inf\{E(n, C; h) : |C| = N, C \subset \mathbb{S}^{n-1}, C \text{ is } \tau\text{-design}\}$$

  the minimum possible h-energy of spherical τ-designs on $\mathbb{S}^{n-1}$ of N points,
- Similarly,

$$U(n, N, \tau; h) = \sup\{E(n, C; h) : |C| = N, C \subset \mathbb{S}^{n-1}, C \text{ is } \tau\text{-design}\}$$

  the maximum possible h-energy of spherical τ-designs on $\mathbb{S}^{n-1}$ of N points.

Some references

- P. Delsarte, An Algebraic Approach to the Association Schemes in Coding Theory, *Philips Res. Rep. Suppl.* **10**, (1973).
- P. Delsarte, J.-M. Goethals, J. J. Seidel, Spherical codes and designs, *Geom. Dedicata* **6**, 1977, 363-388.
- G. A. Kabatiansky, V. I. Levenshtein, Bounds for packings on a sphere and in space, *Problems of Information Transmission* **14**, 1–17, (1978).
- V. I. Levenshtein, Universal bounds for codes and designs, *Handbook of Coding Theory*, V. S. Pless and W. C. Huffman, Eds., Elsevier, Amsterdam, 1998, Ch. 6, 499–648.
- V. A. Yudin, Minimal potential energy of a point system of charges, *Discr. Math. Appl.* **3**, 75-81, 1993.
- H. Cohn, A. Kumar, Universally optimal distribution of points on spheres. Journal of AMS, **20**, no. 1, 99-148, 2006.
- P. Boyvalenkov, P. Dragnev, D. Hardin, E. Saff, M. Stoyanova, Universal energy bounds for potential energy of spherical codes (http://arxiv.org/abs/1503.07228)

Preliminaries – Delsarte-Goethals-Seidel bounds

For fixed strength τ and dimension n denote by

$$B(n,\tau) = \min\{|C| : \exists \ \tau\text{-design} \ C \subset \mathbb{S}^{n-1}\}$$

the minimum possible cardinality of spherical τ-designs $C \subset \mathbb{S}^{n-1}$. Then Delsarte-Goethals-Seidel bound is

$$B(n,\tau) \geq D(n,\tau) = \begin{cases} 2\binom{n+k-2}{n-1}, & \text{if } \tau = 2k-1, \\ \binom{n+k-1}{n-1} + \binom{n+k-2}{n-1}, & \text{if } \tau = 2k. \end{cases}$$

Preliminaries – Gegenbauer polynomials

- For fixed dimension n, the Gegenbauer polynomials are defined by

$$P_0^{(n)} = 1, \quad P_1^{(n)} = t$$

and the three-term recurrence relation (for $k \geq 1$)

$$(i + n - 2)P_{i+1}^{(n)}(t) = (2i + n - 2)tP_i^{(n)}(t) - iP_{i-1}^{(n)}(t).$$

- If $f(t) \in \mathbb{R}[t]$ is a real polynomial of degree m then $f(t)$ can be uniquely expanded in terms of the Gegenbauer polynomials as

$$f(t) = \sum_{i=0}^{m} f_i P_i^{(n)}(t).$$

Preliminaries – main identity

- The identity

$$|C|f(1) + \sum_{x,y\in C, x\neq y} f(\langle x,y\rangle) = |C|^2 f_0 + \sum_{i=1}^{m} \frac{f_i}{r_i} \sum_{j=1}^{r_i} \left(\sum_{x\in C} v_{ij}(x) \right)^2.$$

  holds true for: any $C \subset \mathbf{S}^{n-1}$ – a spherical code, any $f(t) = \sum_{i=0}^{m} f_i P_i^{(n)}(t)$, where $\{v_{ij}(x) : j = 1, 2, \ldots, r_i\}$ is an orthonormal basis of the space $\mathrm{Harm}(i)$ of homogeneous harmonic polynomials of degree i and $r_i = \dim \mathrm{Harm}(i)$.

- An equivalent definition of spherical designs says that

$$\sum_{x\in C} v_{ij}(x) = 0$$

  for every $i \leq \tau$ and every $j \leq r_i$.

- This suggests that polynomials of degree at most τ could be useful – the right hand side of main identity is then reduced to $|C|^2 f_0$.

- For every positive integer m we consider the intervals

$$\mathcal{I}_m = \begin{cases} \left[t_{k-1}^{1,1}, t_k^{1,0}\right], & \text{if } m = 2k-1, \\ \left[t_k^{1,0}, t_k^{1,1}\right], & \text{if } m = 2k. \end{cases}$$

- Here $t_0^{1,1} = -1$, $t_i^{a,b}$, $a, b \in \{0,1\}$, $i \geq 1$, is the greatest zero of the Jacobi polynomial $P_i^{(a+\frac{n-3}{2}, b+\frac{n-3}{2})}(t)$.
- The intervals $\mathcal{I}_m$ define partition of $\mathcal{I} = [-1, 1)$ to countably many non-overlapping closed subintervals.

Preliminaries – Levenshtein bounds for spherical codes (2)

- For every $s \in \mathcal{I}_m$, Levenshtein used a polynomial $f_m^{(n,s)}(t)$ of degree m which satisfy all conditions of the linear programming bounds for spherical codes. This yields the bound

$$A(n,s) \leq \begin{cases} L_{2k-1}(n,s) = \binom{k+n-3}{k-1}\left[\frac{2k+n-3}{n-1} - \frac{P_{k-1}^{(n)}(s)-P_k^{(n)}(s)}{(1-s)P_k^{(n)}(s)}\right] \\ \qquad \text{for } s \in \mathcal{I}_{2k-1}, \\ L_{2k}(n,s) = \binom{k+n-2}{k}\left[\frac{2k+n-1}{n-1} - \frac{(1+s)(P_k^{(n)}(s)-P_{k+1}^{(n)}(s))}{(1-s)(P_k^{(n)}(s)+P_{k+1}^{(n)}(s))}\right] \\ \qquad \text{for } s \in \mathcal{I}_{2k}. \end{cases}$$

- For every fixed dimension n each bound $L_m(n,s)$ is smooth and strictly increasing with respect to s. The function

$$L(n,s) = \begin{cases} L_{2k-1}(n,s), & \text{if } s \in \mathcal{I}_{2k-1}, \\ L_{2k}(n,s), & \text{if } s \in \mathcal{I}_{2k}, \end{cases}$$

is continuous in s.

Preliminaries – connections between DGS- and L-bounds (1)

- The connection between the Delsarte-Goethals-Seidel bound and the Levenshtein bounds are given by the equalities

$$L_{2k-2}(n, t_{k-1}^{1,1}) = L_{2k-1}(n, t_{k-1}^{1,1}) = D(n, 2k-1),$$

$$L_{2k-1}(n, t_k^{1,0}) = L_{2k}(n, t_k^{1,0}) = D(n, 2k)$$

and the ends of the intervals $\mathcal{I}_m$.

Preliminaries – connections between DGS- and L-bounds (2)

- For every fixed (cardinality) $N > D(n, 2k-1)$ there exist uniquely determined real numbers $-1 < \alpha_0 < \alpha_1 < \cdots < \alpha_{k-1} < 1$ and $\rho_0, \rho_1, \ldots, \rho_{k-1}$, $\rho_i > 0$ for $i = 0, 1, \ldots, k-1$, such that the equality

$$f_0 = \frac{f(1)}{N} + \sum_{i=0}^{k-1} \rho_i f(\alpha_i)$$

  holds for every real polynomial $f(t)$ of degree at most $2k-1$.
- The numbers α_i, $i = 0, 1, \ldots, k-1$, are the roots of the equation

$$P_k(t)P_{k-1}(s) - P_k(s)P_{k-1}(t) = 0,$$

  where $s = \alpha_{k-1}$, $P_i(t) = P_i^{(n-1)/2,(n-3)/2}(t)$ is a Jacobi polynomial.
- In fact, α_i, $i = 0, 1, \ldots, k-1$, are the roots of the Levenshtein's polynomial $f_{2k-1}^{(n,\alpha_{k-1})}(t)$.

Preliminaries – connections between DGS- and L-bounds (3)

- Similarly, for every fixed (cardinality) $N > D(n, 2k)$ there exist uniquely determined real numbers $-1 = \beta_0 < \beta_1 < \cdots < \beta_k < 1$ and $\gamma_0, \gamma_1, \ldots, \gamma_k$, $\gamma_i > 0$ for $i = 0, 1, \ldots, k$, such that the equality

$$f_0 = \frac{f(1)}{N} + \sum_{i=0}^{k} \gamma_i f(\beta_i) \qquad (1)$$

  is true for every real polynomial $f(t)$ of degree at most $2k$.
- The numbers β_i, $i = 0, 1, \ldots, k$, are the roots of the Levenshtein's polynomial $f_{2e}^{(n,\beta_k)}(t)$.

- V. I. Levenshtein, Designs as maximum codes in polynomial metric spaces, Acta Appl. Math. 25, 1992, 1-82.

Preliminaries – connections between DGS- and L-bounds (4)

- In what follows we always take care where the cardinality N is located with respect to the Delsarte-Goethals-Seidel bound. It follows from the properties of the bounds $D(n,\tau)$ and $L_m(n,s)$ that

$$N \in [D(n,\tau), D(n,\tau+1)] \iff s \in \mathcal{I}_m,$$

where s and N are connected by the equality

$$N = L_\tau(n,s).$$

- Therefore we can always associate N with the corresponding numbers:

$$\alpha_0, \alpha_1, \ldots, \alpha_{k-1}, \rho_0, \rho_1, \ldots, \rho_{k-1} \text{ when } N \in [D(n,2k-1), D(n,2k))$$

or with

$$\beta_0, \beta_1, \ldots, \beta_k, \gamma_0, \gamma_1, \ldots, \gamma_k \text{ when } N \in [D(n,2k), D(n,2k+1)).$$

LP bounds – lower bounds for $L(n, N, \tau; h)$

- **Theorem 1.**
  Let N, n, τ and h be fixed and $f(t)$ be a real polynomial such that

  (A1) $f(t) \leq h(t)$ for $-1 \leq t \leq 1$.

  (A2) the coefficients in the Gegenbauer expansion
  $f(t) = \sum_{i=0}^{\deg(f)} f_i P_i^{(n)}(t)$ satisfy $f_i \geq 0$ for $i \geq \tau + 1$.

  Then $L(n, N, \tau; h) \geq N(f_0 N - f(1))$.

- $A_{n,N,\tau;h}$ – the set of good polynomials

LP bounds – upper bounds for $U(n, N, \tau; h)$

- **Theorem 2.**
  Let N, n, τ and h be fixed. Suppose that there exists $t_0 \in [-1, 1]$ such that no τ-design on $\mathbb{S}^{n-1}$ of N points can have inner products in the interval $(t_0, 1)$. Let $g(t)$ be a real polynomial such that

  (B1) $g(t) \geq h(t)$ for every $t \in [-1, t_0]$,

  (B2) the coefficients in the Gegenbauer expansion $g(t) = \sum_{i=0}^{\deg(g)} g_i P_i^{(n)}(t)$ satisfy $g_i \leq 0$ for $i \geq \tau + 1$.

  Then $U(n, N, \tau; h) \leq N(g_0 N - g(1))$.

- $B_{n,N,\tau;h}$ – the set of good polynomials

Why spherical designs? (1)

- Spherical designs allow nontrivial upper bounds on their energy.
- In general, a spherical code can have close points (inner products close to 1) and for h tending to infinity as t tends to 1 from below we obtain energies tending to infinity.

Lower bounds (ULB) (1)

- We need Hermite's interpolation to $h(t)$ as follows. Define
  (i) the polynomial $f(t)$ of degree $\tau = 2k-1$ by

$$f(\alpha_i) = h(\alpha_i),\ f'(\alpha_i) = h'(\alpha_i),\ i = 0, 1, \ldots, k-1.$$

  (ii) the polynomial $f(t)$ of degree $\tau = 2k$ by

$$f(\beta_0) = h(\beta_0),\ f(\beta_i) = h(\beta_i),\ f'(\beta_i) = h'(\beta_i),\ i = 1, \ldots, k.$$

- These conditions define a Hermite's interpolation problem for $f(t)$ to intersect and touch the graph of the potential function $h(t)$.

Lower bounds (ULB) (2)

- **Theorem 3.** Let n, τ, $N \in (D(n,\tau), D(n,\tau+1)]$ and h be fixed. Then the polynomials from (i) and (ii) give the bounds

$$L(n,N,2k-1;h) \geq N^2 \sum_{i=0}^{k-1} \rho_i h(\alpha_i),$$

$$L(n,N,2k;h) \geq N^2 \sum_{i=0}^{k} \gamma_i h(\beta_i),$$

respectively.
These bounds can not be improved by using polynomials from $A_{n,N,2k-1;h}$ of degree at most $2k-1$ and $A_{n,N,2k;h}$ of degree at most $2k$, respectively.

Lower bounds (ULB) (3)

- Proof of Theorem 3 (case $\tau = 2k-1$).

  – The condition (A2) is trivially satisfied since $\deg(f) = 2k-1 = \tau$.

  – The condition (A1) follows from the Hermite interpolation and the Role theorem.

  – The calculation of the bound: we have $f_0 = \dfrac{f(1)}{N} + \displaystyle\sum_{i=0}^{k-1} \rho_i f(\alpha_i)$

  whence $N(f_0 N - f(1)) = N^2 \displaystyle\sum_{i=0}^{k-1} \rho_i f(\alpha_i) = N^2 \sum_{i=0}^{k-1} \rho_i h(\alpha_i)$. Therefore

$$L(n, N, 2k-1; h) \geq N(f_0 N - f(1)) = N^2 \sum_{i=0}^{k-1} \rho_i h(\alpha_i).$$

  – The optimality of the bound: s can not be improved by using polynomials from $A_{n,N,2k-1;h}$ of degree at most $2k-1$ and $A_{n,N,2k;h}$ of degree at most $2k$, respectively.

Lower bounds (ULB) (4)

– The optimality of the bound: if $g(t) \in A_{n,N,2k-1;h}$ is another polynomial of degree at most $2k-1$, then

$$\begin{aligned} N(g_0 N - g(1)) &= N^2 \sum_{i=0}^{k-1} \rho_i g(\alpha_i) \leq N^2 \sum_{i=0}^{k-1} \rho_i h(\alpha_i) = N^2 \sum_{i=0}^{k-1} \rho_i f(\alpha_i) \\ &= N(f_0 N - f(1)). \end{aligned}$$

i.e. the bound by g is not better than the bound by f.

Why spherical designs? (2)

- Spherical designs allow very easy proof of our bound.
- For spherical codes we need to prove in addition the positive Gegenbauer expansion.

Two ways for improving ULB – Shorter intervals and Higher degrees

- The ULB bounds are optimal in some sense – they can not be improved by polynomials from $A_{n,M,\tau;h}$ of degree τ or lower.
- First way for obtaining better bounds – making better LP by subintervals of $[-1,1)$ based on preliminary (nontrivial) information on inner products (of τ-designs of N points on $\mathbb{S}^{n-1}$). This is exactly the case when τ is even.
- Second way for obtaining better bounds – using LP with higher degree polynomials. There are necessary and sufficient conditions for the global optimality of ULB, and we can do better when the ULB in not globally optimal.

Why spherical designs? (3)

- Spherical designs allow nontrivial subintervals of $[-1, 1]$ to be used. Indeed, in all cases $(n, \tau = 2k, N \in (D(n, 2k), D(n, 2k + 1)))$ it can be proved that all inner products belong to some subinterval $[\ell, u]$ of $[-1, 1]$, $-1 < \ell < s < 1$.
- For spherical codes any shrinking of the interval $[-1, 1]$ means that we do not consider all possible codes (of that dimension and cardinality).

Improving ULB – Shorter interval (1)

- Denote

$$u(n,N,\tau) = \sup\{s(C) : C \subset \mathbb{S}^{n-1} \text{ is a } \tau\text{-design}, |C| = N\},$$

and

$$\ell(n,N,\tau) = \inf\{\ell(C) : C \subset \mathbb{S}^{n-1} \text{ is a } \tau\text{-design}, |C| = N\},$$

where $\ell(C) = \min\{\langle x,y\rangle : x,y \in C, x \neq y\}$.

- For every n, τ and cardinality $N \in [D(n,\tau), D(n,\tau+1)]$ non-trivial upper bounds on $u(n,N,\tau)$ can be obtained. Similarly, for even $\tau = 2k$ and cardinality $N \in [D(n,2k), D(n,2k+1))$ non-trivial lower bounds on $\ell(n,N,2k)$ are possible. We describe here explicitly the cases $\tau = 2$ and $\tau = 4$.

Improving ULB – Shorter interval (2)

- Further equivalent definition: A spherical τ-design $C \subset \mathbb{S}^{n-1}$ is a spherical code such that

$$\sum_{y \in C} f(\langle x, y \rangle) = f_0 |C|.$$

holds for any point $x \in \mathbb{S}^{n-1}$ and any real polynomial $f(t) = \sum_{i=0}^{r} f_i P_i^{(n)}(t)$ of degree at most τ.

- **Lemma.** a) For every $n \geq 3$ and every $N \in [D(n,2), D(n,3)] = [n+1, 2n]$ we have

$$u(n,N,2) \leq \frac{N-2}{n} - 1.$$

b) For every $n \geq 3$ and every $N \in [D(n,4), D(n,5)] = [n(n+3)/2, n(n+1)]$ we have

$$u(n,N,4) \leq \frac{2(3+\sqrt{(n-1)[(n+2)N-3(n+3)]})}{n(n+2)} - 1.$$

Improving ULB – Shorter interval (3)

- **Lemma.** a) For every $n \geq 3$ and every $N \in [D(n,2), D(n,3)] = [n+1, 2n]$ we have

$$\ell(n,N,2) \geq 1 - \frac{N}{n}.$$

  b) For every $n \geq 3$ and for every $N \in [D(n,4), D(n,5)] = [n(n+3)/2, n(n+1)]$ we have

$$\ell(n,N,4) \geq 1 - \frac{2}{n}\left(1 + \sqrt{\frac{(n-1)(N-2)}{n+2}}\right).$$

- Further bounds on $u(n,N,\tau)$ and $\ell(n,N,\tau)$ can be obtained by a technique from B.-Boumova-Danev, Necessary conditions for existence of some designs in polynomial metric spaces, *Europ. J. Combin.*, **20** 213-225, 1999. For $\tau \geq 4$ such bounds are better in higher dimensions than these from the Lemmas.

Improving ULB – Shorter interval (4)

- **Theorem.** Let n, $N \in [D(n,2k), D(n,2k+1)]$, $\tau = 2k$ and h be fixed. Let $f(t)$ be a real polynomial which satisfies (A2) and $(\mathrm{A1}')$ $f(t) \leq h(t)$ for $\ell(n,N,2k) \leq t \leq u(n,N,2k)$. Then $L(n,N,2k;h) \geq N(f_0 N - f(1))$.
- **Theorem.** (degree two) For every n, $N \in [n+1, 2n]$ and h:

$$L(n,N,2;h) \geq \frac{N[h(0)N(N-n-1)+nh(1-N/n)]}{N-n}.$$

Sketch of proof. Use $f(t)$: $f(\ell) = h(\ell)$, $f(a) = h(a)$ and $f'(a) = h'(a)$ for some $a \in (\ell, 1)$. The optimization of a to maximize $f_0 N - f(1)$ gives best value at $a_0 = \frac{n(1-\ell)-N}{n(1-\ell)+\ell Nn}$ which turns to $a_0 = 0$ for $\ell = 1 - N/n$. Plug in $f_0 N - f(1)$ to get the desired bound. □

Improving ULB – Shorter interval (5)

- **Corollary.** If n and $N = \xi n$, $\xi \in (1,2)$ is constant, tend simultaneously to infinity then

$$L(n, N, 2; h) \gtrsim h(0)N^2 + \frac{N[h(1-\xi) - \xi h(0)]}{\xi - 1}.$$

Proof. Plug $n = N/\xi$ and $\ell = 1 - \xi$. □

- Lower bounds for the energy of 4-designs in shorter intervals can be obtained by interpolation with polynomials of degree four:

$$f(\ell) = h(\ell),\ \ f(a) = h(a),\ \ f'(a) = h'(a),\ \ f(b) = h(b),\ \ f'(b) = h'(b),$$

where the touching points a and b must be chosen to maximize $f_0 N - f(1)$ – as in the previous talk.

Improving ULB – Higher degrees (1)

- Let n and N be fixed, $N \in [D(n, 2k-1), D(n, 2k))$, $L_\tau(n,s) = N$ and j be positive integer.
- We introduce the following functions in n and $s \in \mathcal{I}_{2k-1}$

$$Q_j(n,s) = \frac{1}{N} + \sum_{i=0}^{k-1} \rho_i P_j^{(n)}(\alpha_i) \qquad (2)$$

(note that $P_j^{(n)}(1) = 1$).

- It follows that $Q_j(n,s) = 0$ for every $1 \le j \le 2k-1$ and every $s \in \mathcal{I}_{2k-1}$ (since this is the coefficient $f_0 = 0$ in the Gegenbauer expansion of $P_j^{(n)}(t)$). So the functions $Q_j(n,s)$ are not interesting in these cases and we assume below that $j \ge 2k$.
- The next theorem shows that the functions $Q_j(n,s)$ give necessary and sufficient conditions for existence of improving polynomials of higher degrees.

Improving ULB – Higher degrees (2)

- **Theorem.** Assume that h is completely monotone. Then the bound

$$L(n,N,2k-1;h) \geq N^2 \sum_{i=0}^{k-1} \rho_i h(\alpha_i)$$

can be improved by a polynomial from $A_{n,N,2k-1;h}$ of degree at least $2k$ if and only if $Q_j(n,s) < 0$ for some $j \geq 2k$.
Moreover, if $Q_j(n,s) < 0$ for some $j \geq 2k$, then that bound can be improved by a polynomial from $A_{n,N,2k-1;h}$ of degree exactly j.

Improving ULB – Higher degrees (3)

- The test functions $Q_j(n,s)$ coincide algebraically with the functions with the same name which were introduced and investigated in B.-Danev-Bumova, Upper bounds on the minimum distance of spherical codes, IEEE Trans. Inform. Theory **42**, 1996, 1576-1581.
- **Theorem.** The bounds $L(n,N,2k-1;h) \geq N^2 \sum_{i=0}^{k-1} \rho_i h(\alpha_i)$ can not be improved by using polynomials of degrees $2k$ and $2k+1$.
- **Corollary.** Any improving polynomial must have degree at least $2k+2$.

  Algorithm for finding improving polynomials of degrees $2k+2$ and $2k+3$ – as in the previous talk.

Improving ULB – Higher degrees (4)

- **Theorem.** The function $Q_{2k+3}(n,s)$ has the following properties for $n \geq 3$, $k \geq 2$ and $s \in \mathcal{I}_{2k-1}$:
  If $k \geq 9$ and $3 \leq n \leq \frac{k^2-4k+5+\sqrt{k^4-8k^3-6k^2+24k+25}}{4}$, then $Q_{2k+3}(n,s) < 0$ for every $s \in \left(t_{e-1}^{1,1}, t_e^{1,0}\right]$.
  Thus in fixed dimension all ULBs $L(n,N,2k-1;h) \geq N^2 \sum_{i=0}^{k-1} \rho_i h(\alpha_i)$ with sufficiently large k can be improved.
- **Corollary.** There could exist only finitely many sharp configurations (the same as codes attaining the odd Levenshtein bound $L_{2k-1}(n,s)$ for some $n \geq 3$ and $s \in \mathcal{I}_{2k-1}$ and $k \geq 2$) in any fixed dimension.

Upper bounds (1)

- **Theorem.** Let n, $N \in [D(n,2k), D(n,2k+1)]$, $\tau = 2k$ and h be fixed and $g(t)$ be a real polynomial which satisfies (B2) and (B1′) $g(t) \geq h(t)$ for $\ell(n,N,2k) \leq t \leq u(n,N,2k)$. Then $U(n,N,\tau;h) \leq N(g_0 N - g(1))$.
- Upper bound for 2-designs:

$$U(n,N,2;h) \leq \frac{N[(N-1)(uh(\ell)-\ell h(u))+h(\ell)-h(u)]}{u-\ell}$$

for every n, $N \in [n+1, 2n]$ and h.
Proof. Use the linear polynomial which graph passes through the points $(\ell, h(\ell))$ and $(u, h(u))$. □

Upper bounds (2)

- If n and $N=\xi n$, $\xi \in (1,2)$ is constant, tend simultaneously to infinity, then

$$\frac{U(n,N,2;h)}{N^2} \lesssim \frac{h(1-\xi)+h(\xi-1)}{2} + \frac{(2-\xi)h(1-\xi)-\xi h(\xi-1)}{2N(\xi-1)}.$$

- We now have an asymptotic strip for the energy E of spherical 2-designs of $N=\xi n \in [n+1, 2n-1]$ points:

$$h(0)+\frac{h(1-\xi)-\xi h(0)}{(1-\xi)N} \lesssim \frac{E}{N^2} \lesssim \frac{h(1-\xi)+h(\xi-1)}{2} + \frac{(2-\xi)h(1-\xi)-\xi h(\xi-1)}{2(\xi-1)N}.$$

Upper bounds (3)

- Upper bound for 3- and 4-designs:

$$U(n,N,\tau;h) \leq N(N-1)h(a_0) +$$
$$+ \frac{(h(\ell)-h(a_0))\left[uN(1+na_0^2)+2Na_0+n(1-u)(1-a_0)^2\right]}{n(u-\ell)(\ell-a_0)^2}$$
$$+ \frac{(h(u)-h(a_0))\left[\ell N(1+na_0^2)+2Na_0+n(1-\ell)(1-a_0)^2\right]}{n(u-\ell)(u-a_0)^2},$$

for $\tau=3$, every n and $N \in [2n, \frac{n(n+3)}{2}]$, and for $\tau=4$, every n and $N \in [\frac{n(n+3)}{2}, n^2+n]$, where $a_0 = \frac{N(\ell+u)+n(1-\ell)(1-u)}{n(1-\ell)(1-u)-N(1+\ell un)}$.
Proof. Use third degree polynomial which graph passes through the points $(\ell, h(\ell))$ and $(u, h(u))$ and touches the graph of $h(t)$ (from above) at the point $(a_0, h(a_0))$. □

Upper bounds (4)

- **Theorem.** If n and N tend to infinity in relation $N = n^2\xi$, where $\xi \in [1/2, 1)$ is a constant, then

$$U(n, N, 4; h) \lesssim h(0)N^2 - h(0)N + c_1\sqrt{N} + c_2,$$

where c_1 and c_2 are certain constants.
Proof. The asymptotic forms of our parameters is:

$$\begin{aligned} u(n, N, 4) &\sim 2\sqrt{\xi} - 1 \ \text{(from Lemma)}, \\ \ell(n, N, 4) &\sim 1 - 2\sqrt{\xi} \ \text{(from Lemma)}, \\ a_0 &\sim 0 \ \text{(from above)}, \end{aligned}$$

now plug these. □

Example: compare lower and upper bounds for 2-designs (1)

- The upper bounds coincide exactly when $N = n + 1$ or $N = n + 2$ for every n and h.
- The case $N = n + 1$ leads to the regular simplex on $\mathbb{S}^{n-1}$.
- The case $N = n + 2$ is more interesting – Mimura (1990) has proved that spherical 2-designs with $n + 2$ points on $\mathbb{S}^{n-1}$ do exists if and only if n is even. The nonexistence result follows easily from the coincidence.
- It also follows that the 2-designs of $n + 2$ points for even n are unique (firstly proved by Sali in 1993) and optimal – they have simultaneously minimum and maximum possible energy.

Example: compare lower and upper bounds for 2-designs (2)

- Furthermore, the case $N = n + 2$ ($n \geq 4$ is even) is suitable to show that the design property is important.
- There are codes with $N = n + 2$ points which are not 2-designs and which have smaller or greater energy.
- Take two simplices of $\dfrac{n-2k}{2}$ and $\dfrac{n+2k+4}{2}$ points, $k = 0, 1, 2, \ldots$, and place them orthogonal to each other. Then the different distances will prevail for suitable functions $h(t)$. We can arrange to obtain less or larger energies.

Asymptotic bounds (1)

- Let the dimension n and the cardinality N tend simultaneously to infinity in the relation

$$\lim \frac{N}{n^{k-1}} = \frac{1}{(k-1)!} + \gamma,$$

where $\gamma \geq 0$ is a constant, i.e. $N \sim n^{k-1}(\frac{1}{(k-1)!} + \gamma)$.

- We know the asymptotic behaviour of the parameters:

$$\alpha_i \sim 0, \quad \text{for } i = 1, 2, \ldots, k-1,$$

$$\alpha_0 \sim -\frac{1}{1+\gamma(k-1)!},$$

$$\rho_0 N \sim (1+\gamma(k-1)!)^{2k-1}.$$

Applications – asymptotic bounds (2)

- Now the bounds are easy to be calculated –

$$L(n,N,2k-1;h) \geq N^2 \sum_{i=0}^{k-1} \rho_i h(\alpha_i) \sim h(0)N^2.$$

- The bound of $h(0)N^2 \sim ch(0)n^4$ is attained by the spherical realization of the Kerdock codes (in dimensions $2^{2\ell}$).
- Similarly, in the even case we obtain

$$L(n,N,2k;h) \gtrsim h(0)N^2.$$

Work in progress and future work

- General (universal) bounds in shorter intervals
- General (universal) bounds by higher degrees
- General (universal) upper bounds
- Bounds for codes and designs in Hamming spaces, with special interest to the binary case
- Bounds for codes and designs in Johnson spaces
- Bounds for codes and designs in infinite projective spaces
- Bounds for energy of Euclidean designs

THANK YOU FOR YOUR ATTENTION !