The generalized translation operator and Nikol'skii type inequality for algebraic polynomials on an interval

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Statement of the problem

We use the standard notation $L_q^{(\alpha,\beta)} = L_q^{(\alpha,\beta)}(-1,1)$, $1 \le q < \infty$, for the space of complex-valued functions f measurable on $(-1,1)$ and such that the function $|f|^q$ is integrable over $(-1,1)$ with Jacobi weight $\phi^{(\alpha,\beta)}(x) = (1-x)^\alpha(1+x)^\beta, \ \alpha, \beta > -1$.
This is a Banach space with respect to the norm

$$\|f\|_{L_q^{(\alpha,\beta)}(-1,1)} = \left(\int_{-1}^{1} |f(x)|^q (1-x)^\alpha (1+x)^\beta \, dx \right)^{1/q}.$$

Statement of the problem

In the limiting case $q = \infty$, we assume that $L_q^{(\alpha,\beta)}$ is the classical space $L_\infty = L_\infty(-1,1)$ of measurable (complex-valued) functions essentially bounded on $(-1,1)$ with the norm

$$\|f\|_{L_\infty} = \operatorname{ess\,sup}\{|f(x)| : \ x \in (-1,1)\}. \qquad (*)$$

This space contains the space $C = C[-1,1]$ of (complex-valued) functions continuous on $[-1,1]$ where $(*)$ becomes the uniform norm

$$\|f\|_{C[-1,1]} = \max\{|f(x)| : x \in [-1,1]\}.$$

Statement of the problem

Let $\mathscr{P}_n = \mathscr{P}_n(\mathbb{C})$, $n \geq 0$, be the set of algebraic polynomials (in one variable) of degree (at most) n with complex coefficients.

Denote by $M_n = M_{n,q}^{(\alpha,\beta)}$ the best (i.e., the least possible) constant in the inequality

$$\|p_n\|_{C[-1,1]} \leq M_n \|p_n\|_{L_q^{(\alpha,\beta)}(-1,1)}, \qquad p_n \in \mathscr{P}_n. \tag{1}$$

Statement of the problem

Our aim is to study polynomials extremal in inequality (1), i.e., polynomials $\rho_n \in \mathscr{P}_n, \ \rho_n \not\equiv 0$, at which the inequality turns into an equality.

In particular, we are interested in the uniqueness property of an extremal polynomial.

It is clear that if a polynomial ρ_n is extremal, then the polynomial $c\rho_n$ with any constant $c \neq 0$ is also extremal.

If ρ_n is an extremal polynomial in inequality (1) and any other extremal polynomial has the form $c\rho_n, \ c \neq 0$, then ρ_n is said to be the *unique* extremal polynomial in inequality (1).

A background

Inequality (1) is a special case of a different metrics inequality or Nikol'skii type inequality. Such inequalities and, more generally, inequalities between the uniform norm and integral norms with weights (especially with Jacobi weights) of derivatives of algebraic polynomials and the polynomials themselves were studied during a century and a half. A number of results is huge.

References

=икольский Т. i. iриближение функций многих переменнvх и теоремv вложения. =аука, iосква, 1977.

iорнейчук =. i., +абенко Т. L., Тигун L. L. iкстремальнvе свойства полиномов и сплайнов. =аукова думка, iиев, 1992.

Milovanović G. V., Mitrinovic D. S., Rassias Th. M. Topics in polynomials: extremal problems, inequalities, zeros. World Scientific, Singapore, 1994.

Borwein P., Erdelyi T. Polynomials and polynomial inequalities, Grad. Texts in Math., 161, Springer-Verlag, New York, NY, 1995.

Rahman Q. I., Schmeisser G. Analytic theory of polynomials. Oxford Univ. Press, Oxford, 2002.

A background

Let us describe results related to inequality (1) that are known to the present.

A.Lupas (1974) obtained sharp inequality (1) for $q = 2$ and Jacobi weight with parameters $\alpha, \beta \geq -1/2$. He found the best constant and an extremal polynomial.

The Jacobi weight with $\alpha = \beta = -1/2$ is called the Chebyshev weight. Inequality (1) for the Chebyshev weight can be written as the classical Nikol'skii inequality between the uniform norm and the L_q-norm (with unit weight) on the set of trigonometric polynomials of degree (at most) n.

Probably, Jackson was the first to study such inequality (1933, $q = 2$).

A background

The case $q = 1$ is the most completely studied. This version of the different metrics inequality was studied by Stechkin (1965), Taikov (1965, 1993), V.Babenko, Kofanov, and Pichugov (2002), and Gorbachev (2005).

Glazyrina and Simonov (2015) constructed a polynomial extremal in inequality (1) with the Chebyshev weight for $q = 1$ (in the form of a linear combination of Chebyshev polynomials of the first kind), proved its uniqueness, and showed that its uniform norm is attained at an end point of the interval $[-1, 1]$.

Restrictions on parameters

In 2014–2015, the authors studied inequality (1) for the ultraspherical weight ($\alpha = \beta > -1/2$). The present talk is devoted to an extension of the result.

Studying inequality (1), it is sufficient to consider the case $\alpha \geq \beta$. Besides, our methods can be applied only for $\alpha, \beta \geq -1/2$. Therefore, in what follows, we basically suppose that $\alpha \geq \beta \geq -1/2$.

End point version of the Nikol'skii inequality

Along with inequality

$$\|p_n\|_{C[-1,1]} \leq M_n \|p_n\|_{L_q^{(\alpha,\beta)}(-1,1)}, \qquad p_n \in \mathscr{P}_n, \tag{1}$$

consider an auxiliary inequality

$$|p_n(1)| \leq D_n \|p\|_{L_q^{(\alpha,\beta)}(-1,1)}, \qquad p_n \in \mathscr{P}_n, \tag{2}$$

with the best constant $D_n = D_{n,q}^{(\alpha,\beta)}$. This inequality is also of independent interest. It is clear that $D_n \leq M_n$. We will show below that, in fact, $D_n = M_n$ at least for $\alpha \geq \beta \geq -1/2$.

Polynomials that deviate least from zero

Given the Jacobi weight

$$\phi^{(\alpha+1,\beta)}(x) = \phi^{(\alpha,\beta)}(x)(1-x) = (1-x)^{\alpha+1}(1+x)^{\beta},$$

a parameter q, $1 \le q < \infty$, and an integer $n \ge 1$, denote by $\varrho_n = \varrho_{n,q}^{(\alpha+1,\beta)}$ the polynomial of degree n with unit leading coefficient that deviates least from zero in the space $L_q^{(\alpha+1,\beta)}$. The polynomial ϱ_n is a solution of the problem

$$\min\{\|p_n\|_{L_q^{(\alpha+1,\beta)}} : p_n \in \mathscr{P}_n^1\} = \|\varrho_n\|_{L_q^{(\alpha+1,\beta)}}, \tag{3}$$

where $\mathscr{P}_n^1$ is the set of polynomials $p_n(x) = x^n + \sum_{k=0}^{n-1} a_k x^k$ of degree n with leading coefficient 1.

Polynomials that deviate least from zero

Polynomials that deviate least from zero first appeared in Chebyshev's studies. He found (1854) polynomials that deviate least from zero in the space $C[-1,1]$. These polynomials are called the Chebyshev polynomials of the first kind.

A.N.Korkin and E.I.Zolotarev (1873) solved such problem in $L(-1,1)$; the extremal polynomials are the Chebyshev polynomials of the second kind.

Later on, E.I.Zolotarev, Ya.L.Geronimus, F.Peherstorfer, V.E.Geit, A.L.Lukashov, I.E.Simonov, and many others studied algebraic and trigonometric polynomials with several leading coefficients fixed and under some other restrictions that deviate least from zero with respect to the uniform and integral norms.

Polynomials that deviate least from zero

The solution of the problem

$$\min\{\|p_n\|_{L_q^{(\alpha+1,\beta)}} : p_n \in \mathscr{P}_n^1\} = \|\varrho_n\|_{L_q^{(\alpha+1,\beta)}} \tag{3}$$

for $q = 2$ is well known. In this case, the Jacobi polynomial $R_n^{(\alpha+1,\beta)}$ divided by its leading coefficient solves the problem.

For $q = 1$ and non-negative integer α and β, problem (3) reduces to studying a system of n polynomial equations in n variables which can be solved at least for small n immediately or by finding a Grobner basis.

Main result

Theorem 1

For $\alpha > \beta \geq -1/2$, $1 \leq q < \infty$, and $n \geq 1$, the following statements are valid.
1. *The best constants in inequalities* (1) *and* (2) *coincide*:

$$M_{n,q}^{(\alpha,\beta)} = D_{n,q}^{(\alpha,\beta)}.$$

2. *The polynomial ϱ_n that deviates least from zero with respect to the norm of the space $L_q^{(\alpha+1,\beta)}$ is the unique extremal polynomial in both inequalities* (1) *and* (2).
3. *The polynomial ϱ_n and, hence, any polynomial extremal in inequality* (1) *attain their uniform norms at the unique point $x = 1$.*

Remarks on the proof

The fact that a polynomial extremal in inequality (1) attains its uniform norm only at the right end point 1 of the interval plays an important role in proving Theorem 1. To prove this fact we apply the generalized translation operator generated by the Jacobi weight.

Background

For the ultraspherical case $\beta = \alpha$, a statement similar to Theorem 1 was proved in

[AD-2015] **Arestov V., Deikalova M.** Nikol'skii Inequality Between the Uniform Norm and L_q-Norm with Ultraspherical Weight of Algebraic Polynomials on a Segment // Computational Methods and Function Theory. 2015.
DOI: 10.1007/s40315-015-0134-y

Background

For

$$\alpha = \frac{m-3}{2}, \quad \text{where } m \text{ is an integer}, \quad m \geq 3,$$

such statement was proved earlier in [AD-2013] simultaneously with studying the Nikol'skii inequality between the uniform norm and the L_q-norm of algebraic polynomials on the unit sphere of the Euclidean space $\mathbb{R}^m$, $m \geq 3$.

[AD-2013] **Arestov, V.V., Deikalova,M.V.** Nikol'skii inequality for algebraic polynomials on a multidimensional Euclidean sphere. Trudy Inst. Mat. Mekh. Ural'sk. Otdel. Ross. Akad. Nauk 19(2) 34–47 (2013) = Proc. Steklov Inst. Math. 284 (Suppl. 1), S9–S23 (2014).

In [AD-2015] for $\beta = \alpha$, we argued using the generalized translation generated by the ultraspherical weight.

Arguments in the case $\beta \neq \alpha$ (the Jacobi weight) are similar but with unique features.

Theorem 1 reduces the problem of studying inequality

$$\|p_n\|_{C[-1,1]} \leq M_n \|p_n\|_{L_q^{(\alpha,\beta)}(-1,1)}, \qquad p_n \in \mathscr{P}_n. \tag{1}$$

to studying the problem

$$\min\{\|p_n\|_{L_q^{(\alpha+1,\beta)}} \colon p_n \in \mathscr{P}_n^1\} = \|\varrho_n\|_{L_q^{(\alpha+1,\beta)}}, \tag{3}$$

which, in our opinion, is considerably simpler.

An auxiliary result

Lemma

For $n \geq 1$, $\alpha, \beta > -1$, and $1 \leq q < \infty$, the polynomial ϱ_n is the **unique** *extremal polynomial in inequality*

$$|p_n(1)| \leq D_n \, \|p\|_{L_q^{(\alpha,\beta)}(-1,1)}, \qquad p_n \in \mathscr{P}_n. \tag{2}$$

In fact [AD-2013], the lemma is valid for an arbitrary weight ϕ nonzero almost everywhere.

Generalized translation operator

We prove Theorem 1 by means of the generalized translation operator generated by the Jacobi weight $\phi^{(\alpha,\beta)}$. Initially, the generalized translation operator is defined in the space $L_2^{(\alpha,\beta)}$. This is a Hilbert space with the inner product

$$(f,g) = (f,g)_{L_2^{(\alpha,\beta)}} = \int_{-1}^{1} f(x)\overline{g(x)}(1-x)^\alpha(1+x)^\beta dx. \qquad (4)$$

The operator is defined on the base of expansion of functions into series with respect to Jacobi polynomials. Let us list some properties of the polynomials, which we use in what follows.

Let $R_\nu = R_\nu^{(\alpha,\beta)}, \ \nu \geq 0$, be a system of algebraic Jacobi polynomials of degree ν orthogonal on the interval $[-1,1]$ with the Jacobi weight; more exactly, orthogonal with respect to inner product (4) and normalized by the condition $R_\nu(1) = 1, \ \nu \geq 0$.

Jacobi polynomials

In the case

$$\alpha \geq \beta > -1, \quad \alpha \geq -\frac{1}{2}, \tag{5}$$

the following relation holds:

$$\max\{|R_\nu(x)|: x \in [-1,1]\} = R_\nu(1) = 1, \qquad \nu \geq 0. \tag{6}$$

If the conditions

$$\alpha > \beta > -1 \quad \text{and} \quad \alpha \geq -\frac{1}{2}$$

hold, which are more rigid than (5), then, for $\nu \geq 1$, the property

$$|R_\nu(x)| < R_\nu(1) = 1, \qquad x \in [-1,1).$$

is valid, which is stronger than (6).

Jacobi polynomials

The system of Jacobi polynomials $\{R_\nu\}_{\nu\geq 0}$ forms an orthogonal basis in the space $L_2^{(\alpha,\beta)}$. Thus, an arbitrary function $f \in L_2^{(\alpha,\beta)}$ is expanded into the Fourier series

$$f(x) = \sum_{\nu=0}^{\infty} f_\nu R_\nu(x), \qquad f_\nu = \frac{(f, R_\nu^{(\alpha,\beta)})}{(R_\nu^{(\alpha,\beta)}, R_\nu^{(\alpha,\beta)})}. \tag{7}$$

For a pair of functions $f, g \in L_2^{(\alpha,\beta)}$, the generalized version of Parseval's identity holds:

$$(f, g) = \sum_{\nu=0}^{\infty} \delta_\nu f_\nu \overline{g}_\nu, \qquad \delta_\nu = (R_\nu, R_\nu) = \|R_\nu\|^2_{L_2^{(\alpha,\beta)}}.$$

In particular, the norm of a function $f \in L_2^{(\alpha,\beta)}$ can be expressed in terms of its Fourier coefficients $\{f_\nu\}$ by Parseval's identity:

$$\|f\|^2_{L_2^{(\alpha,\beta)}} = \sum_{\nu=0}^{\infty} \delta_\nu \, |f_\nu|^2.$$

Generalized translation operator in the space $L_2^{(\alpha,\beta)}$

The generalized translation operator with step $t \in [-1, 1]$ is a linear operator Θ_t defined on functions $f \in L_2^{(\alpha,\beta)}$ with Fourier series (7) by the relation

$$\Theta_t f(x) = \sum_{\nu=0}^{\infty} f_\nu R_\nu(t) R_\nu(x). \tag{8}$$

Probably, Legendre was the first to study the generalized translation operator. It was in the early 18th century.

The generalized translation operator was used in studying several problems of function theory on an interval: convergence of Fourier–Jacobi series, direct and inverse theorems of approximation of functions by algebraic polynomials in the spaces $L_q^{(\alpha,\beta)}(-1,1)$, and others; see, for example, the monograph

Askey, R. Orthogonal Polynomials and Special Functions. SIAM, Philadelphia, 1975.

Generalized translation operator in the space $L_2^{(\alpha,\beta)}$

(1) For $\alpha \geq \beta > -1,\ \alpha \geq -1/2$, and any $t \in [-1,1]$, the generalized translation operator Θ_t is a bounded linear operator in the space $L_2^{(\alpha,\beta)}$, and the norm of this operator is 1:

$$\|\Theta_t\|_{L_2^{(\alpha,\beta)} \to L_2^{(\alpha,\beta)}} = 1.$$

(2) For $\alpha > \beta > -1,\ \alpha \geq -1/2$, and $t \in [-1,1)$, the norm of the operator Θ_t in the space $L_2^{(\alpha,\beta)}$ is attained only at functions that are constant almost everywhere on $(-1,1)$.

Generalized translation operator in the space $L_2^{(\alpha,\beta)}$

The operator Θ_t can be extended by continuity to a bounded linear operator in the space $L_q^{(\alpha,\beta)}$, $\alpha \geq \beta \geq -1/2$, $1 \leq q < \infty$.

Norm of the generalized translation operator

Theorem 2

For $\alpha > \beta \geq -1/2$, $1 \leq q < \infty$, and any $t \in [-1, 1]$, the following statements are valid.
1. *The generalized translation operator Θ_t is a bounded linear operator in the space $L_q^{(\alpha,\beta)}$ and*

$$\|\Theta_t\|_{L_q^{(\alpha,\beta)} \to L_q^{(\alpha,\beta)}} = 1. \tag{9}$$

2. *For $\alpha > \beta \geq -1/2$, $1 < q < \infty$, and $t \in [-1, 1)$, the norm of the operator Θ_t is attained at a polynomial f if and only if f is a constant.*
3. *For $\alpha > \beta \geq -1/2$, $q = 1$, and $t \in [-1, 1)$, the norm of the operator Θ_t is attained at a polynomial $f \not\equiv 0$ if and only if f is of constant sign on $[-1, 1]$.*

Norm of the generalized translation operator

The value of the norm of the generalized translation operator, i.e., the assertion (9), is known. It was proved in

[G-1971] **Gasper G.** Positivity and the Convolution Structure for Jacobi Series. The Annals of Mathematics, Second Series, Vol. 93, No. 1 (Jan., 1971), pp. 112–118.

Bavinck H. A special class of Jacobi series and some applications. J. Math. Anal. Appl. 37, 767–797 (1972).

Norm of the generalized translation operator

R. Askey and S. Wainger, *A convolution structure for Jacobi series, Amer. J. Math. 91 (1969), 463-485*

They proved that the formula

$$(\Theta_t f)(x) = \int_{-1}^{1} f(z)\, F^{(\alpha,\beta)}(x,t,z)(1-z)^{\alpha}(1+z)^{\beta} dz, \qquad (10)$$

holds for the generalized translation in the space $L_1^{(\alpha,\beta)}$ with $\alpha \geq \beta \geq -1/2$, $\alpha > -1/2$, and $-1 \leq x, t < 1$; besides, the function $F^{(\alpha,\beta)}$ has a number of good properties; in particular, the integral $\int_{-1}^{1} |F^{(\alpha,\beta)}(x,t,z)|(1-z)^{\alpha}(1+z)^{\beta} dz$ is bounded by a constant independent of x and t.

Norm of the generalized translation operator

G. Gasper [G-1971] showed that the function $F^{(\alpha,\beta)}$ is nonnegative and

$$\int_{-1}^{1} F^{(\alpha,\beta)}(x,t,z)(1-z)^{\alpha}(1+z)^{\beta}dz = 1.$$

This easily implies that the operator Θ_t has norm 1 in all spaces $L_q^{(\alpha,\beta)}$, $1 \le q < \infty$.

For us, it is important to find extremal functions, more precisely, polynomials at which the norm is attained. This is a nontrivial question. The point is that the support of the function $F^{(\alpha,\beta)}$ in (10) with respect to z does not coincide with $(-1,1)$ and depends on x and t.

Norm of the generalized translation operator

To prove Theorem 1, we used an integral representation of the generalized translation from

Koornwinder T. Jacobi polynomials, II. An analytic proof of the product formula. SIAM J. Math. Anal. 5 (1), 125–137 (1974).

Integral representation of the generalized translation operator

More precisely, Koornwinder proved that the generalized translation admits the following integral representation for $\alpha > \beta > -1/2$:

$$(\Theta_t f)(x) =$$

$$= \frac{1}{\kappa(\alpha,\beta)} \int_0^1 \int_{-1}^1 f\left(U\right) \left(1-\rho^2\right)^{\alpha-\beta-1} \rho^{2\beta+1} \left(1-\xi^2\right)^{\beta-1/2} d\xi d\rho, \tag{11}$$

$$U = U(x,t,\rho,\xi) = tx + \rho\xi\sqrt{1-t^2}\sqrt{1-x^2} + \frac{1}{2}(\rho^2-1)(1-t)(1-x),$$

$$\kappa(\alpha,\beta) = \frac{\sqrt{\pi}\,\Gamma\left(\alpha-\beta\right)\Gamma\left(\beta+1/2\right)}{2\Gamma(\alpha+1)}.$$

Formula (11) is valid at least on the set $\mathscr{P} = \mathscr{P}(\mathbb{C})$ of all algebraic polynomials with complex coefficients.

Conclusion remark

We consider the application of the generalized translation operator in an extremal problem for algebraic polynomials to be one of the key results of this research.

Thank you for your attention!