Levenshtein-type bounds for codes with inner products in prescribed interval

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Outline

- Levenshtein framework revisited
- Positive definite signed measures of given degree and orthogonality
- Gauss-type Radau quadrature
- ULB Theorem
- Levenshtein-type bound for “restricted” maximal codes

Recall: Minimal h-energy - preliminaries

- Spherical Code: A finite set $C \subset \mathbb{S}^{n-1}$ with cardinality $|C|$;
- Let the *interaction potential* $h : [-1, 1] \to \mathbb{R} \cup \{+\infty\}$ be an *absolutely monotone*[1] function;
- The *h-energy* of a spherical code C:

$$E(n, C; h) := \sum_{x,y \in C, y \neq x} h(\langle x, y \rangle), \quad |x - y|^2 = 2 - 2\langle x, y \rangle = 2(1 - t),$$

where $t = \langle x, y \rangle$ denotes Euclidean inner product of x and y.

Problem

Determine

$$\mathcal{E}(n, N; h) := \min\{E(n, C; h) : |C| = N, C \subset \mathbb{S}^{n-1}\}$$

and find (prove) *optimal h-energy codes*.

[1] A function f is *absolutely monotone on I* if $f^{(k)}(t) \geq 0$ for $t \in I$ and $k = 0, 1, 2, \ldots$.

Recall: Delasarte-Yudin method (modified)

Problem

Let C_ℓ be a generic code with inner products in $[\ell, 1]$. Denote

$$\mathcal{E}(n, N, \ell; f) := \min_{C_\ell}\{E(n, C_\ell; h)\}.$$

GOAL: **Find lower bounds on** $\mathcal{E}(n, N, \ell; h)$**.**

Thm (Modified Delsarte-Yudin LP Bound)

Let $A_{n,h} = \{f : f(t) \leq h(t), t \in [\ell, 1], f_k \geq 0, k = 1, 2, \dots\}$. Then

$$\mathcal{E}(n, N, \ell; h) \geq N^2(f_0 - f(1)/N), \qquad f \in A_{n,h}. \tag{1}$$

An N-point spherical code C_ℓ satisfies $E(n, C_\ell; h) = N^2(f_0 - f(1)/N)$ if and only if both of the following hold:

(a) $f(t) = h(t)$ for all $t \in \{\langle x, y\rangle : x \neq y,\ x, y \in C\}$.

(b) for all $k \geq 1$, either $f_k = 0$ or $\sum_{x,y\in C} P_k^{(n)}(\langle x, y\rangle) = 0$.

Recall: Delasarte-Yudin method (modified)

Thm (Modified Delsarte-Yudin LP Bound)

Let $A_{n,h} = \{f : f(t) \leq h(t), t \in [\ell, 1], f_k \geq 0, k = 1, 2, \dots\}$. Then

$$\mathcal{E}(n, N, \ell; h) \geq N^2(f_0 - f(1)/N), \qquad f \in A_{n,h}. \tag{2}$$

Here $f(t) = \sum_{k=0}^{\infty} f_k P_k^{(n)}(t)$ is the Gegenbauer expansion of f.

Linear program

Maximizing $f_0 - f(1)/N$ leads to the truncated the program

$$(LP) \qquad \text{Maximize } F_m(f_0, f_1, \dots, f_m) := N\left(f_0(N-1) - \sum_{k=1}^{m} f_k\right),$$

subject to $f \in \mathcal{P}_m \cap A_{n,h}$.

Levenshtein framework revisited

Linear program 1

$$(LP\ 1) \qquad \text{Maximize } f_0 - \frac{f(1)}{N} \quad \text{subject to} \quad f \in \mathcal{P}_m \cap A_{n,h}.$$

If a Gauss-type Quadrature Formula (QF with positive weights $\{\rho_i\}$) with $x_{k+1} = 1$ (called right end-point Radau quadrature) exists:

$$f_0 = \gamma_n \int_{-1}^{1} f(t)(1-t^2)^{(n-3)/2} dt = \frac{f(1)}{N} + \sum_{i=1}^{k} \rho_i f(\alpha_i), \quad f \in \mathcal{P}_m,$$

Linear program 2

$$(LP\ 2) \qquad \text{Maximize} \sum_{i=1}^{k} \rho_i f(\alpha_i), \quad f \in \mathcal{P}_m \cap A_{n,h} \quad \left(\leq \sum_{i=1}^{k} \rho_i h(\alpha_i) \right).$$

Hence, we seek such Radau Quadrature.

Positive definite signed measures of degree m

Definition

A signed Borel measure ν on $\mathbb{R}$ for which all polinomials are integrable is called *positive definite up to degree m* if for all polinomials $p \in \mathcal{P}_m$ we have

$$\int p(t)^2 d\nu(t) > 0, \quad \text{for all} \quad p \in \mathcal{P}_m, \ \ p(t) \not\equiv 0.$$

For such measures Gram-Schmidt orthogonalization provides unique orthogonal polynomials $p_0, p_1, \ldots, p_{m+1}$, where $p_i(1) = 1$.

They satisfy three-term requrrence, have simple zeros in the support of ν, and zeros of p_i and p_{i+1} interlace.

Three positive definite signed measures

Let $t_{1,k} < t_{2,k} < \cdots < t_{k,k}$ be the zeros of the adjacent Gegenbauer polynomials

$$P_k^{1,0}(t) := \frac{P_k^{(\frac{n-1}{2},\frac{n-3}{2})}(t)}{P_k^{(\frac{n-1}{2},\frac{n-3}{2})}(1)} = \eta_k^{1,0} t^k + \cdots, \quad \eta_k^{1,0} > 0, \tag{3}$$

orthogonal w.r.t. $d\chi(t) := (1-t)(1-t^2)^{(n-3)/2}\,dt$.

For any $-1 < \ell < t_{1,k} < t_{k,k} < s < 1$ define

$d\nu_\ell(t) := (t-\ell)d\chi(t),\ \nu_s(t) := (s-t)d\chi(t),\ d\nu_{\ell,s}(t) := (t-\ell)(s-t)d\chi(t).$

Lemma

The measures $d\nu_\ell(t)$ and $d\nu_s(t)$ are positive definite up to degree $k-1$ and the measure $d\nu_{\ell,s}(t)$ is positive definite up to degree $k-2$.

Proof: (on the board).

Three orthogonal polynomial classes $\{P_j^{1,\ell}, P_j^{1,s}, P_j^{1,\ell,s}\}$

Applying Gram-Schmidt orthogonalization one derives the existence and uniqueness (for the so-chosen normalization) of the following classes of orthogonal polynomials.

Corollary

Let $\ell < t_{1,k} < t_{k,k} < s$. The following classes of orthogonal polynomials are well-defined:

$$\{P_j^{1,\ell}(t)\}_{j=0}^{k}, \quad \text{orthogonal w.r.t.} \quad d\nu_\ell(t), \quad P_j^{1,\ell}(1) = 1; \tag{4}$$

$$\{P_j^{1,s}(t)\}_{j=0}^{k}, \quad \text{orthogonal w.r.t.} \quad d\nu_s(t), \quad P_j^{1,s}(1) = 1; \tag{5}$$

$$\{P_j^{1,\ell,s}(t)\}_{j=0}^{k-1}, \quad \text{orthogonal w.r.t.} \quad d\nu_{\ell,s}(t), \quad P_j^{1,\ell,s}(1) = 1. \tag{6}$$

The polynomials in each class satisfy a three-term recurrence relation and their zeros interlace.

Christofel-Darboux formula and $\{P_j^{1,\ell}\}$, $\{P_j^{1,s}\}$, $\{P_j^{1,\ell,s}\}$

Recall the Christoffel-Darboux formula (associated with $\{P_j^{1,0}(t)\}$)

$$T_i(x,y) := \sum_{j=0}^{i} r_j^{1,0} P_j^{1,0}(x) P_j^{1,0}(y) = r_i^{1,0} b_i^{1,0} \frac{P_{i+1}^{1,0}(x)P_i^{1,0}(y) - P_{i+1}^{1,0}(y)P_i^{1,0}(x)}{x-y}.$$

Then

$$P_i^{1,\ell}(t) = \frac{T_i(t,\ell)}{T_i(1,\ell)}, \quad P_i^{1,s}(t) = \frac{T_i(t,s)}{T_i(1,s)}, \quad i = 0, 1, \ldots, k.$$

Christoffel-Darboux kernel associated with $\{P_i^{1,\ell}\}_{i=0}^{k}$

$$R_i(x,y;\ell) := \sum_{j=0}^{i} r_j^{1,\ell} P_j^{1,\ell}(x) P_j^{1,\ell}(y) = r_i^{1,\ell} b_i^{1,\ell} \frac{P_{i+1}^{1,\ell}(x)P_i^{1,\ell}(y) - P_{i+1}^{1,\ell}(y)P_i^{1,\ell}(x)}{x-y}$$

yields

$$P_i^{1,\ell,s}(t) := \frac{R_i(t,s;\ell)}{R_i(1,s;\ell)}, \quad i = 0, 1, \ldots, k-1.$$

Positive definiteness of some polynomials

Lemma

Under the condition that $-1 \leq \ell < t_{1,k}$ *the polynomials* $(t-\ell)P_i^{1,\ell}(t)$, $i = 0, 1, \ldots, k$, *have positive Gegenbauer coefficients.*

Denote the zeros of $P_k^{1,\ell}(t)$ with $y_{1,k} < y_{2,k} < \cdots < y_{k,k}$.

Lemma

Let $t_{k,k}^{1,0} \leq s \leq y_{k,k}$. *Then the polynomials* $(t-\ell)(t-s)P_i^{1,\ell,s}(t)$ *and* $(t-\ell)P_i^{1,\ell,s}(t)$, $i = 0, 1, \ldots, k-1$, *have positive Gegenbauer coefficients.*

The Levenshtein-type polynomial in this case is:

$$f_{2k}^{(n,s,\ell)}(t) := (t-\ell)(t-s)\left(P_{k-1}^{1,\ell,s}(t)\right)^2. \tag{7}$$

Gauss-type Radau Quadrature

Theorem

Let $\{\beta_1 < \beta_2 < \cdots < \beta_{k-1}\}$ be the zeros of $P_{k-1}^{1,\ell,s}(t)$. Then the Radau quadrature formula

$$f_0 = \int_{-1}^{1} f(t)(1-t^2)^{\frac{n-3}{2}}\, dt = \rho_0^\ell f(\ell) + \sum_{i=1}^{k-1} \rho_i^\ell f(\beta_i) + \rho_k^\ell f(s) + \rho_{k+1}^\ell f(1) \quad (8)$$

is exact for all polynomials of degree at most $2k$ and has positive weights $\rho_i > 0$, $i = 0, 1, \ldots, k+1$.

Proof: (on the board).

Definition - Levenshtein function for subinterval

For any fixed $1 < \ell \leq t_{1,k}$ the Levenshein-type function is defined to be

$$L_k^\ell(n,s) := 1/\rho_{k+1}^\ell.$$

Universal Lower Bound (ULB) on subinterval $[\ell, 1]$.

ULB Subinterval Theorem - (BDHSS - 2015)

Let h be a fixed absolutely monotone potential, $-1 < \ell < 1$, n and N be fixed. Let k be the largest integer such that $-1 < \ell \leq t_{1,k}$. Let $s = s(\ell)$ be the unique real number such that $L_k^\ell(n, s) = N$. Then the Levenshtein-type nodes $\{\beta_i\}$ defined in the previous theorem, provide the bounds

$$\mathcal{E}(n, N, \ell, h) \geq N^2 \left(\rho_0^\ell h(\ell) + \sum_{i=1}^{k-1} \rho_i^\ell h(\beta_i) + \rho_k^\ell h(s) \right).$$

The Hermite interpolant of order 1 at ℓ and 2 at β_i, $i = 1, \ldots, k-1$ and s provides the optimal polinomial solving the finite LP in the class $\mathcal{P}_{2k} \cap A_{n,h}$.

Levenshtein-type bound for “restricted” maximal codes

Bounds on restricted maximal codes - (BDHSS - 2015)

Suppose that $C = C_{\ell,s}$ is a spherical code with distinct inner products in the interval $[\ell, s]$. Then the following Levenshtein-type bound on its cardinality holds

$$|C_{\ell,s}| \leq L_k^{\ell}(n, s).$$

THANK YOU!