External Field Problems on the Sphere and Minimal Energy Points Separation

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External Field Problem in $\mathbb{C}$ - overview

Classical energy problems

- Electrostatics - capacity, equilibrium measures;
- Geometry - transfinite diameter;
- Polynomials - Chebyshev constant discrete orthogonal polynomials
- Classical theorem in potential theory

External field problems

- Characterization theorem of weighted equilibrium
- Examples
- Applications to orthogonal polynomials on the real line

Constrained energy problems

- Characterization theorem of constrained equilibrium
- Examples
- Applications to discrete orthogonal polynomials

Classical energy problem and equilibrium measure

Electrostatics - capacity of a conductor $\mathrm{cap}(E)$

E - compact set in $\mathbb{C}$, $\mu \in \mathcal{M}(E)$ - probability measure on E;

Equilibrium occurs when potential (**logarithmic**) energy $I(\mu)$ is minimized.

$$V_E := \inf\{I(\mu) := -\int\int \log|x-y|\, d\mu(x)d\mu(y)\}, \ \ \mathrm{cap}(E) := \exp(-V_E)$$

Remark: For **Riesz energy** we use **Riesz kernel** $|x-y|^{-s}$ instead.

Equilibrium measure μ_E

If $\mathrm{cap}(E) > 0$, there exists unique $\mu_E \ : \ I(\mu_E) = V_E$.
Potential satisfies $U^{\mu_E}(x) = -\int \log|x-y|\, d\mu(y) = C$ on E.

Examples

- $E = \mathbb{T}$, $d\mu_E = d\theta/(2\pi)$
- $E = [-1, 1]$, $d\mu_E = dx/\pi\sqrt{1-x^2}$

Classical theorem in potential theory

Geometry - transfinite diameter of a set $\delta(E)$

E - compact set in $\mathbb{C}$, $Z_n = \{z_1, z_2, \dots, z_n\} \subset$ of E;

Maximize Vandermond (product of all mutual distances)

$$\delta_n(E) := \max_{Z_n \subset E} \left(\prod_{1 \le i < j \le n} |z_i - z_j| \right)^{2/(n(n-1))}, \quad \delta(E) := \lim \delta_n(E)$$

Approximation Theory - Chebyshev constant $\tau(E)$

E - compact set in $\mathbb{C}$, $T_n(x)$ - monic polynomial of minimal uniform norm;

$$t_n(E) := \min\{\|x^n - p_{n-1}(x)\| \ : p_{n-1} \in \mathbb{P}_{n-1}\}, \quad \tau(E) = \lim t_n^{1/n}(E)$$

Classical theorem (Fekete, Szegö)

$$\mathrm{cap}(E) = \delta(E) = \tau(E)$$

External field problem - Characterization theorem

Electrostatics - add external field

E - **closed** set in $\mathbb{C}$, Q - lower semi-continuous on E (growth cond.);

$$V_Q := \inf\{I_Q(\mu) := I(\mu) + 2\int Q(x)\, d\mu(x)$$

Theorem - Weighted equilibrium μ_Q

There exists unique $\mu_Q \;:\; I_Q(\mu_Q) = V_Q$.

Potential satisfies: $U^{\mu_Q}(x) + Q(x) \geq C$ q.e. on E

$U^{\mu_Q}(x) + Q(x) \leq C$ on $\mathrm{supp}(\mu_Q)$.

Applications

- Orthogonal polynomials on real line
- Approximation of functions by weighted polynomials
- Integrable systems
- Random matrices

Proof of the characterization theorem

Let E - compact, Q - continuous. Then $I_Q : \mathcal{M}(E) \to \mathbb{R}$ is lower semi-continuous functional, i.e. if $\mu_n \to \mu$ weak* then

$$\liminf I_Q(\mu_n) \geq I_Q(\mu).$$

Let $\{\mu_n\}$ s.t. $I_Q(\mu_n) \to V_Q$. Select a weak* convergent subsequence $\mu_{n_k} \to \mu$, $\mu \in \mathcal{M}(E)$. Then $I_Q(\mu) = V_Q$.

The positive definiteness of the energy functional implies uniqueness.

To show the first characterization inequlity, suppose

$$\mathrm{cap}\{x : U^{\mu_Q}(x) + Q(x) < V_Q - \int Q(x)\, d\mu_Q(z) =: F_Q\} > 0.$$

Then there is n s.t. $\mathrm{cap}(K_n) = \mathrm{cap}(\{x : U^{\mu_Q}(x) + Q(x) \leq F_Q - \frac{1}{n}\}) > 0$. Then for small enough $\alpha > 0$, $I_Q(\alpha\mu_{K_n} + (1-\alpha)\mu_Q) < I_Q(\mu_Q)$.

Finally, if there is $x_0 \in \mathrm{supp}(\mu_Q)$, s.t. $U^{\mu_Q}(x_0) + Q(x_0) > F_Q$ then $I_Q(\mu_Q) > V_Q$, a contradiction.

Constrained energy problem

Electrostatics - add external field and upper constraint

Add constraint measure σ: $\sigma(E) > 1$

$$V_Q^\sigma := \inf\{I_Q(\mu) := I(\mu) + 2\int Q(x)\,d\mu(x) \ : \mu \leq \sigma$$

Applications: Discrete orthogonal polynomials, random walks, numerical linear algebra methods, etc.

Theorem (Saff-D. '97) - Constrained equilibrium λ_Q^σ

There exists unique $\lambda_Q^\sigma \ : \ I_Q(\lambda_Q^\sigma) = V_Q^\sigma$.

Potential satisfies: $U^{\lambda_Q^\sigma}(x) + Q(x) \geq C$ on $\mathrm{supp}(\sigma - \lambda_Q^\sigma)$

$U^{\lambda_Q^\sigma}(x) + Q(x) \leq C$ on $\mathrm{supp}(\mu)$.

Theorem (Saff-D. '97) - Constrained vs. weighted equilibrium

If $Q \equiv 0$, then $\sigma - \lambda^\sigma = (\|\sigma\| - 1)\mu_Q$ for $Q(x) = -U^\sigma(x)/(\|\sigma\| - 1)$

Discrete Energy on $\mathbb{S}^d$

Recall from yesterday

Why search for minimal energy optimal) configurations on the sphere?

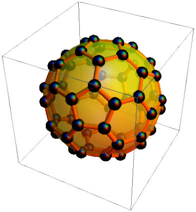
Numerous applications in:

- Physics
- Biology
- Chemistry
- Computer Science

Optimal Configurations in Physics

Thomson Problem (1904) -
("plum pudding" model of an atom)

Find the (most) stable (ground state) energy configuration of N classical electrons (Coulomb law) constrained to move on the sphere $\mathbb{S}^2$.



Generalized Thomson Problem ($1/r^s$ potentials and $\log(1/r)$)

A configuration $\omega_N := \{\mathbf{x}_1, \ldots, \mathbf{x}_N\} \subset \mathbb{S}^2$ that minimizes **Riesz** ***s*-energy**

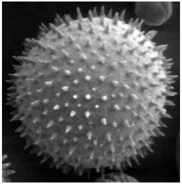
$$E_s(\omega_N) := \sum_{j \neq k} \frac{1}{|\mathbf{x}_j - \mathbf{x}_k|^s}, \quad s > 0, \qquad E_0(\omega_N) := \sum_{j \neq k} \log \frac{1}{|\mathbf{x}_j - \mathbf{x}_k|}$$

is called an **optimal *s*-energy configuration**.

Optimal Configurations in Biology

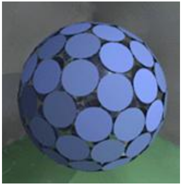
Tammes Problem (1930)

A Dutch botanist that studied modeling of the distribution of the orifices in pollen grain asked the following.



Tammes Problem (Best-Packing)

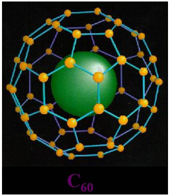
Place N points on the unit sphere so as to maximize the minimum distance between any pair of points, **or**, where to situate hostile dictators?



Optimal Configurations in Chemistry

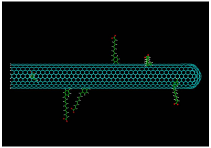
Fullerenes (1985) - (Buckyballs)

Vaporizing graphite, Curl, Kroto, Smalley, Heath, and O'Brian discovered C_{60} (Chemistry 1996 Nobel prize)

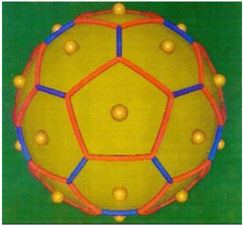


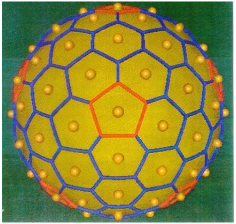
Nanotechnology - Nanowire (R. Smalley)

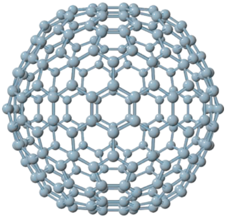
A giant fullerene molecule few nanometers in diameter, but hundreds of microns (and ultimately meters) in length, with electrical conductivity similar to copper's, thermal conductivity as high as diamond and tensile strength about 100 times higher than steel.



32 and 122 Electrons and C_{60} and C_{240} Buckyballs





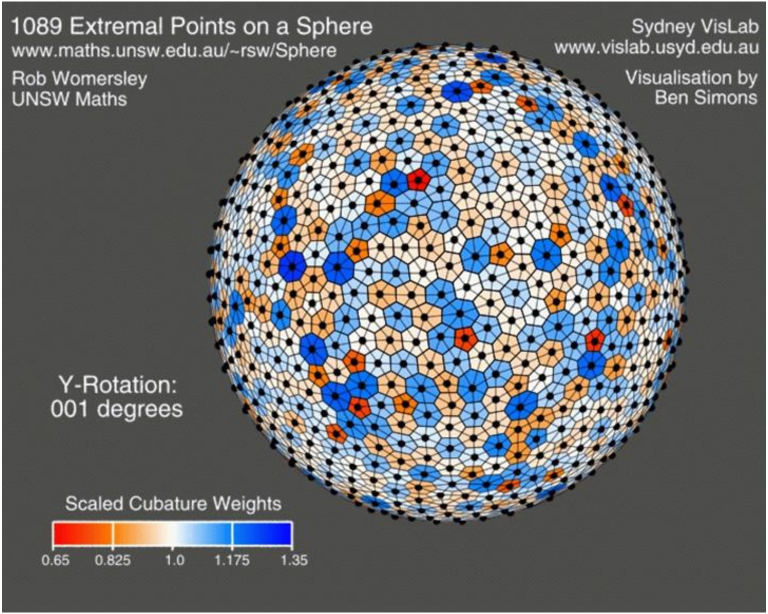


Other "Fullerenes"

Under the lion paw

Montreal biosphere

Computational "Fulerene" - Rob Womersley



Known Optimal Configurations

Recall: Riesz Oprimal Configurations

A configuration $\omega_N := \{\mathbf{x}_1, \ldots, \mathbf{x}_N\} \subset \mathbb{S}^2$ that minimizes **Riesz** ***s*-energy**

$$E_s(\omega_N) := \sum_{j \neq k} \frac{1}{|\mathbf{x}_j - \mathbf{x}_k|^s}, \quad s > 0, \qquad E_0(\omega_N) := \sum_{j \neq k} \log \frac{1}{|\mathbf{x}_j - \mathbf{x}_k|}$$

is called an **optimal *s*-energy configuration**.

- $s = 0$, Smale's problem, logarithmic points (known for $N = 1 - 6,\ 12$);
- $s = 1$, Thomson Problem (known for $N = 1 - 6,\ 12$)
- $s = -1$, Fejes-Toth Problem (known for $N = 1 - 6,\ 12$)
- $s \to \infty$, Tammes Problem (known for $N = 1 - 12,\ 13,\ 14,\ 24$)

Separation Problem for $\mathbb{S}^d$

Separation Distance

$$\delta(\omega_N) := \min_{j \neq k} |\mathbf{x}_j - \mathbf{x}_k|, \quad \omega_N = \{\mathbf{x}_1, \ldots, \mathbf{x}_N\}$$

Expect: $\delta(\omega_N^{(s)}) \asymp N^{-1/d}$ as $N \to \infty$, where $\omega_N^{(s)}$ optimal for $\mathbb{S}^d$

Definition

A **sequence** of N-point configurations $\{\omega_N\}_{N=2}^{\infty} \subset \mathbb{S}^d$ is **well-separated** if there exists some $c > 0$ **not** depending on N s.t. $\delta(\omega_N) \geq c\, N^{-1/d}$ for all N.

Separation Problem for $\mathbb{S}^d$

Separation Results for Optimal Configurations on $\mathbb{S}^d$

$d = 2, s = 0$	$\delta(\omega_N^{(0)}) \geq \mathcal{O}(N^{-1/2})$	R-S-Z (1995)
$0 < s < d - 2$	$\delta(\omega_N^{(s)}) \geq ?$	
$s = d - 1$	$\delta(\omega_N^{(d-1)}) \geq \mathcal{O}(N^{-1/d})$	Dahlberg (1978)
$d - 1 \leq s < d$	$\delta(\omega_N^{(s)}) \geq \mathcal{O}(N^{-1/d})$	K-S-S (2007)
$d - 2 \leq s < d$	$\delta(\omega_N^{(s)}) \geq \beta_{s,d} N^{-1/d}$	D-S (2007)
$s = d$	$\delta(\omega_N^{(d)}) \geq \mathcal{O}((N \log N)^{-1/d})$	K-S (1998)
$s > d$	$\delta(\omega_N^{(s)}) \geq \mathcal{O}(N^{-1/d})$	K-S (1998)
$s = \infty$	$\delta(\omega_N^{(\infty)}) \geq \mathcal{O}(N^{-1/d})$	Conway-Sloane

Asymptotic Results (H-vdW (1951), Bo-H-S (2007))

Logarithmic Points on $\mathbb{S}^2$ ($d = 2, s = 0$)

Separation Results for Logarithmic Configurations on $\mathbb{S}^2$

$\delta(\omega_N^{(0)}) \geq (3/5)/\sqrt{N}$	R-S-Z (1995)
$\delta(\omega_N^{(0)}) \geq (7/4)/\sqrt{N}$	Dubickas (1997)
$\delta(\omega_N^{(0)}) \geq 2/\sqrt{N-1}$	Dragnev(2002)

Logarithmic Points on $\mathbb{S}^2$ ($d = 2, s = 0$)

Separation Results for Logarithmic Configurations on $\mathbb{S}^2$

$\delta(\omega_N^{(0)}) \geq (3/5)/\sqrt{N}$	R-S-Z (1995)
$\delta(\omega_N^{(0)}) \geq (7/4)/\sqrt{N}$	Dubickas (1997)
$\delta(\omega_N^{(0)}) \geq 2/\sqrt{N-1}$	Dragnev(2002)

Proof.

- R-S-Z, Dubickas: Stereographical projection with South Pole in ω_N.
- Dragnev: Stereographical projection with North Pole in ω_N. This creates external field on projections of remaining $N-1$ points $\{z_k\}$. All weighted Fekete points are contained in support of continuous MEP, i.e. $|z_k| \leq \sqrt{N-2}$, which implies estimate. $\square$

Separation Problem for $\mathbb{S}^d$ for $d-2 \leq s < d$

Approach for $\mathbb{S}^d$

- Fix a point of $\omega_N^{(s)}$ and consider external field Q_N it generates on the remaining $n = N-1$ points.
- Study continuous energy problem for this external field Q_N.
- Discrete energy points for Q_N are **contained** in CEP equilibrium support.

Theorem (D-Saff 2007)

$$\delta(\omega_N^{(s,d)}) \geq \frac{K_{s,d}}{N^{1/d}}, \quad K_{s,d} := \left(\frac{2\mathcal{B}(d/2,1/2)}{\mathcal{B}(d/2,(d-s)/2)}\right)^{1/d},$$

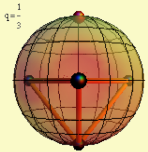
where $\mathcal{B}(x,y)$ denotes the Beta function. In particular,

$$K_{d-1,d} = 2^{1/d}, \quad K_{s,2} = 2\sqrt{1-s/2}.$$

Remark: We need Principle of Domination, de la Valleè-Pousin type theorem, and Riesz balayage, hence the restriction on s.

Discrete MEP on $\mathbb{S}^d$ for $d-2 \le s < d$

Q-optimal points



$$Q(\mathbf{x}) = \frac{q}{|\mathbf{x} - R\mathbf{p}|^s}$$

Let Q be an **external field**. Find Q-optimal configuration of n points on $\mathbb{S}^d$, that solve

$$\min \left\{ \sum_{j \neq k}^{n} \left[\frac{1}{|\mathbf{x}_j - \mathbf{x}_k|^s} + Q(\mathbf{x}_j) + Q(\mathbf{x}_k) \right] : \mathbf{x}_k \in \mathbb{S}^d \right\}$$

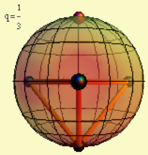
2007 Separation: $q = 1/(N-2)$, $R = 1$, $n = N - 1$.

What do ***Q-Fekete points*** *look like?*

Example ($\mathbb{S}^2$, $s = 1$, $q = 1/3$ and $q = 1$, $n = 4$)

Discrete MEP on $\mathbb{S}^d$ for $d-2 \leq s < d$

Q-optimal points



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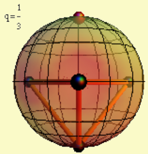
Key idea:

Discrete MEP on $\mathbb{S}^d$ for $d-2 \leq s < d$

Q-optimal points



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2007 Separation: $q = 1/(N-2)$, $R = 1$, $n = N - 1$.

Key idea:

Theorem

Q-optimal points are contained in $\mathrm{supp}(\mu_Q)$.

cf. continuous problem

Example ($\mathbb{S}^2$, $s = 0$, $Q = -\log|\mathbf{x} - R\mathbf{n}|$, $20 > R > 1.1$, $n = 1000$)

External field Continuous MEP on $\mathbb{S}^d$ for $d-2 \le s < d$

$K \subset \mathbb{S}^d$ compact; $\mathcal{M}(K)$ class of positive unit Borel measures μ supported on K

$$U_s^\mu(\mathbf{x}) := \int |\mathbf{x}-\mathbf{y}|^{-s}\, \mathrm{d}\,\mu(\mathbf{y}) \qquad \mathcal{I}_s[\mu] := \int\int |\mathbf{x}-\mathbf{y}|^{-s}\, \mathrm{d}\,\mu(\mathbf{x})\, \mathrm{d}\,\mu(\mathbf{y})$$

Riesz s-potential of μ **Riesz s-energy of μ**

$$W_s(K) := \inf\{\mathcal{I}_s[\mu] : \mu \in \mathcal{M}(K)\}$$

Riesz s-energy of K

Extremal measure

Given an **external field** Q on K, there exists unique **extremal measure** μ_Q that minimizes the weighted energy

$$\mathcal{I}_s[\mu] + 2\int Q\, \mathrm{d}\,\mu, \qquad \mu \in \mathcal{M}(K),$$

characterized by $U_s^\mu(\mathbf{x}) + Q(\mathbf{x}) \ge C$ on $\mathbb{S}^d$ with "$=$" on supp(μ_Q).

cf. discrete problem

Physicist's Problem (Signed Equilibrium)

Given compact $K \subset \mathbb{S}^d$, Q external field on K, find a **signed measure** η_Q s.t.

$$U_s^{\eta_Q}(\mathbf{x}) + Q(\mathbf{x}) = \text{const.} \qquad \text{everywhere on } K$$
$$\eta_Q(K) = 1$$

Definition

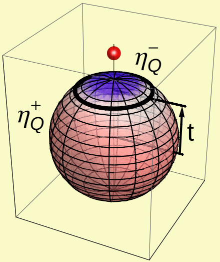
$\eta_Q = \eta_{Q,K}$ is called **signed equilibrium on K associated with Q**.

Proposition

If η_Q exists, then it is unique. Go to proof details

Theorem

Let $\eta_{Q,K} = \eta_{Q,K}^+ - \eta_{Q,K}^-$. Then $\operatorname{supp}(\mu_{Q,K}) \subseteq \operatorname{supp}(\eta_{Q,K}^+)$



Example (Brauchart-Saff-D., 2009)

$$K = \mathbb{S}^d, \quad Q_{\mathbf{a}}(\mathbf{x}) = q/\left|\mathbf{x} - \mathbf{a}\right|^s, \quad R = |\mathbf{a}| \geq 1$$

$$\eta_{Q_{\mathbf{a}}} = \eta^+_{Q_{\mathbf{a}}} - \eta^-_{Q_{\mathbf{a}}}$$

Let Σ_t be spherical cap centered at South Pole of height $-1 \leq t \leq 1$

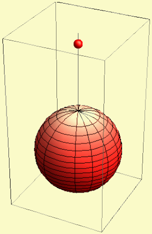
$$\operatorname{supp}(\eta^+_{Q_{\mathbf{a}}}) = \Sigma_{t(Q_{\mathbf{a}})}, \ \ \operatorname{supp}(\eta^-_{Q_{\mathbf{a}}}) = \overline{\mathbb{S}^d \setminus \Sigma_{t(Q_{\mathbf{a}})}}.$$

Remark

If $\eta_{Q_{\mathbf{a}}} \geq 0$, then $\mu_{Q_{\mathbf{a}}} = \eta_{Q_{\mathbf{a}}}$. If not, then $\operatorname{supp}(\mu_{Q_{\mathbf{a}}}) \subseteq \operatorname{supp}(\eta^+_{Q_{\mathbf{a}}})$.

Finding μ_Q when $\text{supp}(\mu_Q) = \mathbb{S}^d$

Gonchar's Problem for $\mathbb{S}^d$



Let $q = 1$, $s = d - 1$ (Newton potential).
Find $R_0 > 0$ s.t. for $Q_{\mathbf{a}}(\mathbf{x}) = |\mathbf{x} - \mathbf{a}|^{1-d}$, $\mathbf{a} = R\mathbf{p}$

$$\text{supp}(\mu_{Q_{\mathbf{a}}}) \begin{cases} = \mathbb{S}^d & \text{if } R \geq R_0, \\ \subsetneq \mathbb{S}^d & \text{if } R < R_0. \end{cases}$$

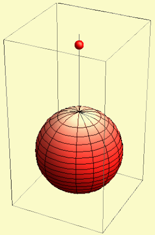
Proposition

For $s = d - 1$,

$$\mathrm{d}\,\eta_{Q_{\mathbf{a}}}(\mathbf{x}) = \left[1 + \frac{1}{R^{d-1}} - \frac{R^2 - 1}{|\mathbf{x} - \mathbf{a}|^{d+1}}\right] \mathrm{d}\,\sigma_d(\mathbf{x})$$

Finding μ_Q when $\mathrm{supp}(\mu_Q) = \mathbb{S}^d$

Gonchar's Problem for $\mathbb{S}^d$



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If $d = 2$, then $R_0 - 1 = \dfrac{1 + \sqrt{5}}{2}$. When $d = 4$, $R_0 - 1 =$ Plastic number from architecture (see Padovan sequence $P_{n+3} = P_{n+1} + P_n$).

Finding μ_Q when $\text{supp}(\mu_Q) \subsetneq \mathbb{S}^d$; B-D-S (2009)

Definition ($\mathcal{F}_s$-Mhaskar-Saff functional for general Q)

$$\mathcal{F}_s(K) := W_s(K) + \int Q \, \mathrm{d}\mu_K, \qquad K \subset \mathbb{S}^d \text{ compact.}$$

Theorem

If $d-2 \leq s < d$ with $s > 0$, then $\mathcal{F}_s$ is minimized for $S_Q := \text{supp}(\mu_Q)$.

Proposition (Connection to signed equilibrium)

If $d-2 < s < d$ with $s > 0$, $Q : K \to \mathbb{R}$ continuous and $W_s(K) < \infty$, then $U_s^{\eta_{Q,K}} + Q \equiv \mathcal{F}_s(K)$ on K.

Proof.

By definition $\qquad U_s^{\eta_{Q,K}}(\mathbf{x}) + Q(\mathbf{x}) = C \qquad$ on K

□

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Proof.

$$\int U_s^{\eta_{Q,K}}(\mathbf{x}) \, \mathrm{d}\, \mu_K(\mathbf{x}) + \int Q(\mathbf{x}) \, \mathrm{d}\, \mu_K(\mathbf{x}) = \int C \, \mathrm{d}\, \mu_K(\mathbf{x})$$

□

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Proposition (Connection to signed equilibrium)

If $d-2 < s < d$ with $s > 0$, $Q : K \to \mathbb{R}$ continuous and $W_s(K) < \infty$, then $U_s^{\eta_{Q,K}} + Q \equiv \mathcal{F}_s(K)$ on K.

Proof.

$$\int U_s^{\mu_K}(\mathbf{x}) \, \mathrm{d}\, \eta_{Q,K}(\mathbf{x}) + \int Q(\mathbf{x}) \, \mathrm{d}\, \mu_K(\mathbf{x}) = C \int \mathrm{d}\, \mu_K(\mathbf{x})$$

□

Finding μ_Q when $\mathrm{supp}(\mu_Q) \subsetneq \mathbb{S}^d$; B-D-S (2009)

Definition ($\mathcal{F}_s$-Mhaskar-Saff functional for general Q)

$$\mathcal{F}_s(K) := W_s(K) + \int Q \, \mathrm{d}\, \mu_K, \qquad K \subset \mathbb{S}^d \text{ compact.}$$

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$$\mathcal{F}_s(K) = W_s(K) + \int Q \, \mathrm{d}\, \mu_K = C$$

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Theorem

*Let $d-2 \leq s < d$, $s>0$. If Q is axially symmetric, i.e. $Q(\mathbf{z}) = f(\xi)$, where $\xi =$ height of $\mathbf{z}$, with f **convex and increasing**, then*

$$\operatorname{supp}(\mu_Q) = \Sigma_{t_0} \qquad \textit{for some } t_0.$$

Note: $Q_{\mathbf{a}}(\mathbf{z}) = q/\left|\mathbf{z}-\mathbf{a}\right|^s = f(\xi)$ on $\mathbb{S}^d$ for $s>0$

Consequently

$\operatorname{supp}(\mu_{Q_{\mathbf{a}}}) = \Sigma_{t_0}$ for some t_0.

Theorem (for $Q_{\mathbf{a}}$)

If $d-2 \leq s < d$, $s>0$, and $\mathbf{a} = R\mathbf{p}$, then $\mathcal{F}_s$ is minimized over Σ_t's when $t = t_0$ is the unique solution of

$$W_s(\mathbb{S}^d)\frac{1+q\left\|\epsilon_t\right\|}{\left\|\nu_t\right\|} = \frac{q\left(R+1\right)^{d-s}}{\left(R^2-2Rt+1\right)^{d/2}}, \qquad \begin{aligned}\epsilon_t &= \mathrm{Bal}_s(\delta_{\mathbf{a}}, \Sigma_t),\\ \nu_t &= \mathrm{Bal}_s(\sigma_d, \Sigma_t)\end{aligned}$$

or $t_0 = 1$ when such a solution does not exist.

The Signed Equilibrium on Σ_t

Theorem

Let $d-2<s<d$. $Q_{\mathbf{a}}(\mathbf{x}) = q/|\mathbf{x}-\mathbf{a}|^s$. Signed equilibrium on Σ_t is

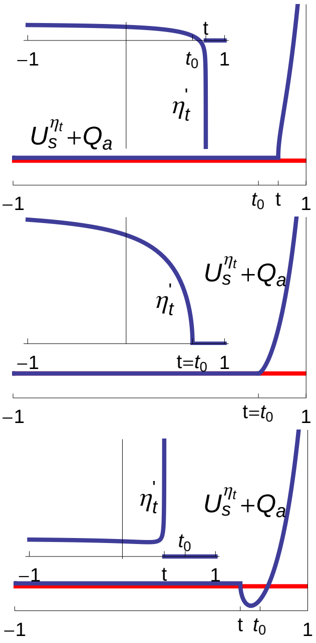
$$\eta_t := \eta_{Q_{\mathbf{a}},\Sigma_t} = \frac{1+q\|\epsilon_t\|}{\|\nu_t\|}\nu_t - q\,\epsilon_t, \qquad \begin{array}{l}\epsilon_t = \mathrm{Bal}_s(\delta_{\mathbf{a}},\Sigma_t),\\ \nu_t = \mathrm{Bal}_s(\sigma_d,\Sigma_t).\end{array}$$

Moreover,

$$\mathrm{d}\,\eta_t(\mathbf{x}) = \eta_t'(u)\,\mathrm{d}\,\sigma_d(\mathbf{x}), \quad \mathbf{x} = (\sqrt{1-u^2}\,\overline{\mathbf{x}}, u) \in \Sigma_t, \quad \overline{\mathbf{x}} \in \mathbb{S}^{d-1}.$$

The weighted s-potential is

$$\begin{array}{ll} U_s^{\eta_t}(\mathbf{z}) + Q_{\mathbf{a}}(\mathbf{z}) = \mathcal{F}_s(\Sigma_t) & \text{on} \quad \Sigma_t, \\ U_s^{\eta_t}(\mathbf{z}) + Q_{\mathbf{a}}(\mathbf{z}) = \mathcal{F}_s(\Sigma_t) + [\cdots] & \text{on} \quad \mathbb{S}^d \setminus \Sigma_t. \end{array}$$



Compare with $s = 0$, $d = 2$ case

$$t > t_0,$$
$$U_s^{\eta_t}(\mathbf{z}) + Q_a(\mathbf{z}) \geq \mathcal{F}_s(\Sigma_t) \quad \text{on } \mathbb{S}^d \setminus \Sigma_t,$$
$$U_s^{\eta_t}(\mathbf{z}) + Q_a(\mathbf{z}) = \mathcal{F}_s(\Sigma_t) \quad \text{on } \Sigma_t,$$
$$\eta_t' \not\geq 0 \quad \text{on } \Sigma_t.$$

$$t = t_0,$$
$$U_s^{\eta_t}(\mathbf{z}) + Q_a(\mathbf{z}) \geq \mathcal{F}_s(\Sigma_t) \quad \text{on } \mathbb{S}^d \setminus \Sigma_t,$$
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$$t < t_0,$$
$$U_s^{\eta_t}(\mathbf{z}) + Q_a(\mathbf{z}) \not\geq \mathcal{F}_s(\Sigma_t) \quad \text{on } \mathbb{S}^d \setminus \Sigma_t,$$
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$$\eta_t' \geq 0 \quad \text{on } \Sigma_t.$$

Axis-supported external fields; B-D-S (2009)

Balayage of a measure is superposition of balayages of Dirac-delta's

Definition

Q **positive-axis supported**, if

$$Q(\mathbf{x}) = \int |\mathbf{x} - R\mathbf{p}|^{-s} \, \mathrm{d}\, \lambda(R), \qquad \mathbf{x} \in \mathbb{S}^d,$$

for some finite pos. meas. λ supp. on a compact subset of $(0, \infty)$.

Theorem (Signed equilibrium on Σ_t for positive-axis supported Q)

Let Q be as above with $\operatorname{supp}(\lambda) \subset [1, \infty)$ *and $d - 2 < s < d$. Then*

$$\tilde{\eta}_t = \frac{1 + \|\tilde{\epsilon}_t\|}{\|\nu_t\|} \nu_t - \tilde{\epsilon}_t,$$

where

$$\nu_t = \mathrm{Bal}_s(\sigma_d, \Sigma_t)$$

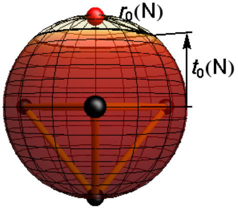
$$\tilde{\epsilon}_t := \mathrm{Bal}_s(\lambda, \Sigma_t) = \int \mathrm{Bal}_s(\delta_{R\mathbf{p}}, \Sigma_t) \, \mathrm{d}\, \lambda(R)$$

Separation of Q-Fekete points on $\mathbb{S}^d$; B-D-S (2014)

Set $Q(\mathbf{x}) := q/|\mathbf{x}-\mathbf{b}|^s$, $|\mathbf{b}|>1$, let $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N\}$ be a Q-Fekete point set. If $\mathbf{x}_N$ is the fixed, then $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{N-1}\}$ is a $\widetilde{Q}$-Fekete set with $\widetilde{Q}(\mathbf{x}) = Q(\mathbf{x}) + |\mathbf{x}-\mathbf{x}_N|^{-s}/(N-2)$.

Theorem

- *If $d-2<s<d$, then **all** $\widetilde{Q}$-Fekete points are in $\operatorname{supp}(\mu_{\widetilde{Q}})$.*
- *In addition, $\operatorname{supp}(\mu_{\widetilde{Q}}) \subseteq \operatorname{supp}(\eta^{+}_{\widetilde{Q},K})$ for any compact $\operatorname{supp}(\mu_{\widetilde{Q}}) \subseteq K \subseteq \mathbb{S}^d$.*



Theorem

If $d-2<s<d$, then

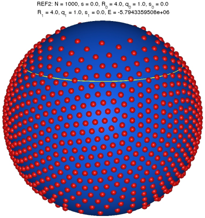
$$\delta(\omega^{(s)}_{Q,N}) \geq \left(\frac{2\mathcal{B}(d/2, 1/2)}{(1+q)\mathcal{B}(d/2,(d-s)/2)} \right)^{1/d} N^{-1/d}.$$

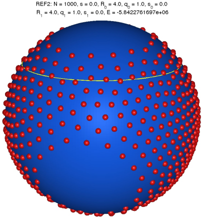
Logarithmic external fields with discrete support - B-D-S-W, in progress

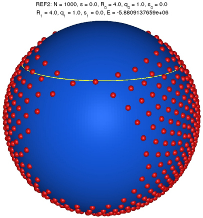
Let $Q(\mathbf{x}) := \sum q_i \log 1/(|\mathbf{x} - \mathbf{b}_i|)$, $\mathbf{b}_i \in \mathbb{S}^2$. (If $d > 2$, then $s = d - 2$)

Theorem

For small enough q_i, the support $\mathrm{supp}(\mu_Q)$ is found explicitly by removing suitable nonintersecting spherical caps around $\mathbf{b}_i$ and the extremal measure is the normalized surface area measure for $\mathrm{supp}(\mu_Q)$.







THANK YOU!

Proof.

Suppose η_1 and η_2 are two signed s-equilibria on K. Then

$$U_s^{\eta_1}(\mathbf{x}) + Q(\mathbf{x}) = F_1, \quad U_s^{\eta_2}(\mathbf{x}) + Q(\mathbf{x}) = F_2 \qquad \text{for all } \mathbf{x} \in K.$$

Subtracting the two equations and integrating with respect to $\eta_1 - \eta_2$ we obtain

$$\mathcal{I}_s(\eta_1 - \eta_2) = \int [U_s^{\eta_1}(\mathbf{x}) - U_s^{\eta_2}(\mathbf{x})] \, \mathrm{d}(\eta_1 - \eta_2)(\mathbf{x}) = 0.$$

We used that $\int (F_2 - F_1) \, \mathrm{d}(\eta_1 - \eta_2)(\mathbf{x}) = 0$, since $(\eta_1 - \eta_2)(K) = 0$.

But $\mathcal{I}_s(\eta) \geq 0$ for any signed measure η with equality iff $\eta \equiv 0$.

Therefore $\eta_1 = \eta_2$. □