Dualities in Algebraic Logic

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This talk is dedicated to Hajnal Andréka

Overview

- Introduction
- Modal Dualities
- Subdirectly irreducible algebras and rooted structures
- Vietoris via modal logic
- Final remarks

Algebraic Logic

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  interpolation: amalgamation
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  completeness: representation
- abstract algebraic logic:
  study Logic using methods from (universal) algebra

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- C and D are dual(ly equivalent) if C and D° are equivalent
  i.e. there are contravariant functors linking C and D

A Fundamental Duality

verbal visual

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verbal	visual
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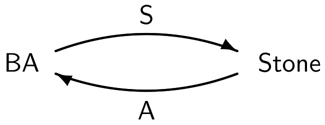
Stone duality:



Variants of Stone duality

- Heyting algebra vs Esakia spaces
- compact regular frames vs compact Hausdorff spaces
- distributive lattices vs Priestley spaces
- modal algebras vs topological Kripke structures
- cylindric algebras vs ...
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Contravariance In all these examples both categories are concrete!

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Main characters

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Aim:

- introduce TKS
- develop duality between MA and TKS

Modal Algebras

- $\mathbb{A} = (A, \vee, -, \perp, \Diamond)$ is a modal algebra if
  - $(A, \vee, -, \perp)$ is a Boolean algebra
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- $h : \mathbb{A}' \to \mathbb{A}$ is an MA-morphism if it preserves all operations:
  - $h(a' \vee' b') = h(a') \vee h(b')$, ..., $h(\Diamond' a') = \Diamond h(a')$.

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- A modal logic L can be algebraized by a variety V_L of modal algebras
- Modal algebras are (the simplest) Boolean Algebras with Operators

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- $\mathbb{S}$ is rooted if its collection $W_\mathbb{S}$ of roots is non-empty

Stone spaces

- A (topological) space is a pair (S, τ) where τ is a topology on S
- A Stone space is a space (S, τ) where τ is
  - compact,
  - Hausdorff
  - zero-dimensional (i.e. it has a basis of clopen sets)
- Stone is the category of Stone spaces and continuous functions

Stone duality

From Stone spaces to Boolean algebras: $(\cdot)^*$

Objects Given (S, τ) take $(S, \tau)^* := (Clp(\tau), \cup, \sim_S, \varnothing)$

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Stone duality 2

Theorem

The functors $(\cdot)^*$ and $(\cdot)_*$ witness the dual equivalence of BA and Stone.

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Ultrafilter structures

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- Open Problem characterize the ultrafilter structures modulo isomorphism

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- TKS is the category with
  - objects: topological Kripke structures
  - arrows: continuous bounded morphism

Topological modal duality

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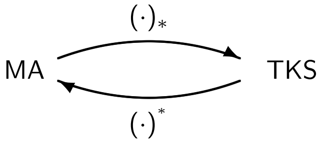

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- (completeness) Which varieties are generated by their PMAs?

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Folklore Subdirect irreducibility is related to rootedness

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- Let $T_{\mathbb{A}_*}$ denote the collection of topo-roots of $\mathbb{A}_*$

Observations

Proposition For any modal algebra $\mathbb{A}$:
(1) $Q^\star$ is transitive
(2) $Q^\omega \subseteq Q^\star$
(3) $Q^\star(u)$ is hereditary for any ultrafilter u
(4) $Q^\star(u)$ is closed for any ultrafilter u
(5) $Q^\star(u) = \overline{Q^\omega(u)}$ for any ultrafilter u
(6) $\langle Q^\star \rangle$ maps opens to opens
(7) If Q is transitive then $Q = Q^\omega = Q^\star$

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Suggestion Develop the modal theory of $Q^\star$

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- $\mathrm{V}(\mathbb{X}) := \langle K(\mathbb{X}), \upsilon_\tau \rangle$ is the Vietoris space of $\mathbb{X}$.

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Different presentation:

- For $a \in \tau$, define

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Fact The Vietoris construction preserves various properties, including:

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- zero-dimensionality

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From now on we restrict to the category KHaus of

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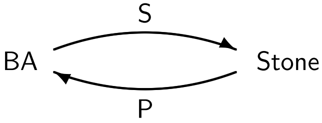

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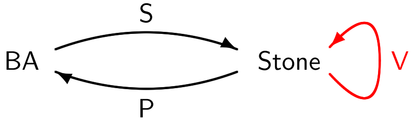

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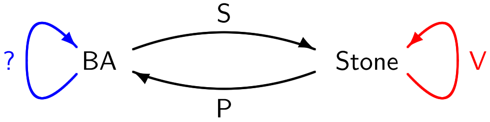

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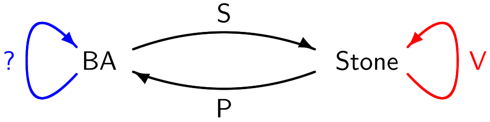

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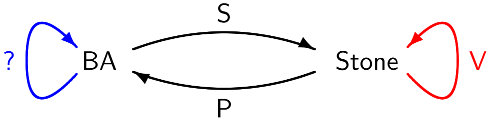

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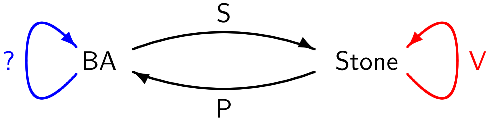

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Theorem
Topological Kripke frames are Vietoris coalgebras over Stone

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- Sufficiently general to model notions like:
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Examples:

- Kripke structures are P-coalgebras over Set
- deterministics finite automata are coalgebras over Set

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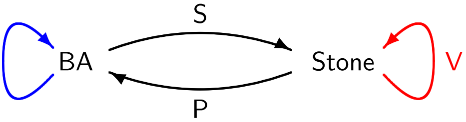

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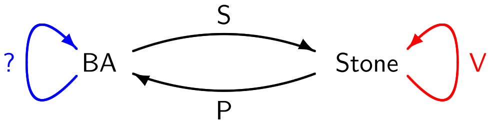
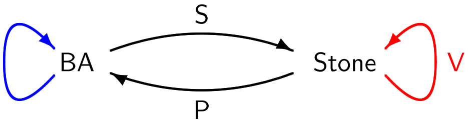


Modal Logic Dualizes the Vietoris Functor



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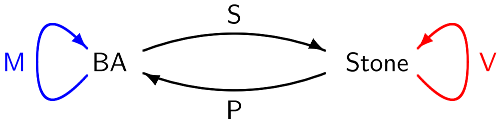

- Johnstone: describe M via generators and relations
- Given a BA $\mathbb{B}$, $\mathrm{M}\mathbb{B}$ is the Boolean algebra
  - generated by the set $\{\underline{\Diamond b} : b \in B\}$
  - modulo the relations $\underline{\Diamond(a \vee b)} = \underline{\Diamond a} \vee \underline{\Diamond b}$ and $\underline{\Diamond \top} = \top$

Theorem (Kupke, Kurz & Venema) $\mathrm{MA} \cong \mathrm{ALg_{BA}(M)}$.

The topological modal duality is an algebra|coalgebra duality

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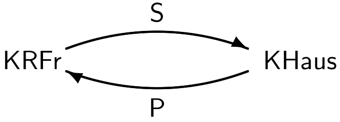

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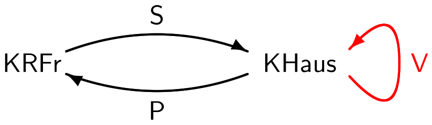

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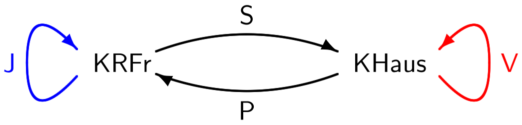

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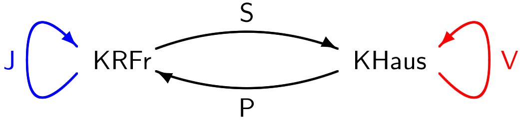

Geometric modal logic dualizes/axiomatizes the Vietoris functor
(Johnstone)

Vietoris pointfree (Johnstone Functor)

Given a frame $\mathbb{L}$, define $L_\Box := \{\Box a \mid a \in L\}$ and $L_\Diamond := \{\Diamond a \mid a \in L\}$.

$$
\begin{array}{ll}
V\mathbb{L} := \mathrm{Fr}\langle L_\Box \uplus L_\Diamond \mid & \Box(\bigwedge A) = \bigwedge_{a\in A} \Box a \qquad (A \in \mathrm{P}_\omega L) \\
& \Diamond(\bigvee A) = \bigvee_{a\in A} \Diamond a \qquad (A \in \mathrm{P}_\omega L) \\
& \\
& \Box a \wedge \Diamond b \leq \Diamond(a \wedge b) \\
& \Box(a \vee b) \leq \Box a \vee \Diamond b \\
& \\
& \Box(\bigsqcup A) = \bigsqcup_{a\in A} \Box a \qquad (A \in \mathrm{P}L \text{ directed}) \\
& \Diamond(\bigsqcup A) = \bigsqcup_{a\in A} \Diamond a \qquad (A \in \mathrm{P}L \text{ directed}) \\
& \\
& \rangle
\end{array}
$$

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- ... and formulate M & J accordingly, in terms of ∇

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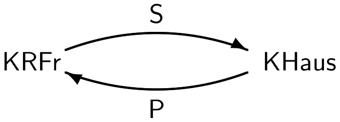

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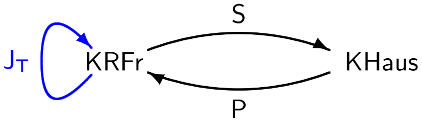

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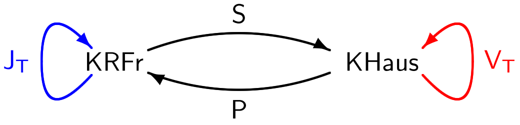

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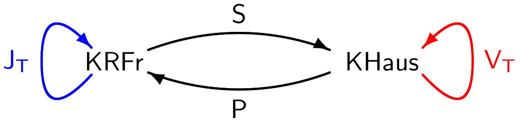

The Vietoris functor is the power set instantiation of the functor V_T
Describe the functor V_T for an arbitrary set functor T!

Overview

- Introduction
- Modal Dualities
- Subdirectly irreducible algebras and rooted structures
- Vietoris via modal logic
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References

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