Relativistic theories of dissipative fluids

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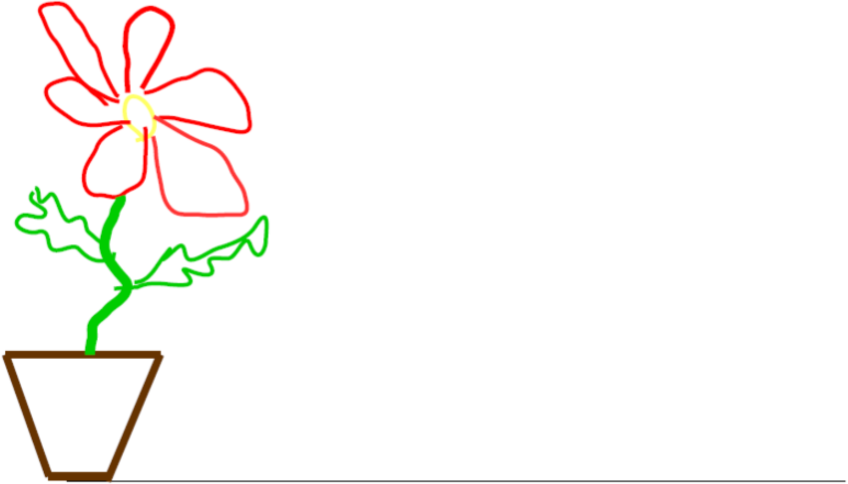
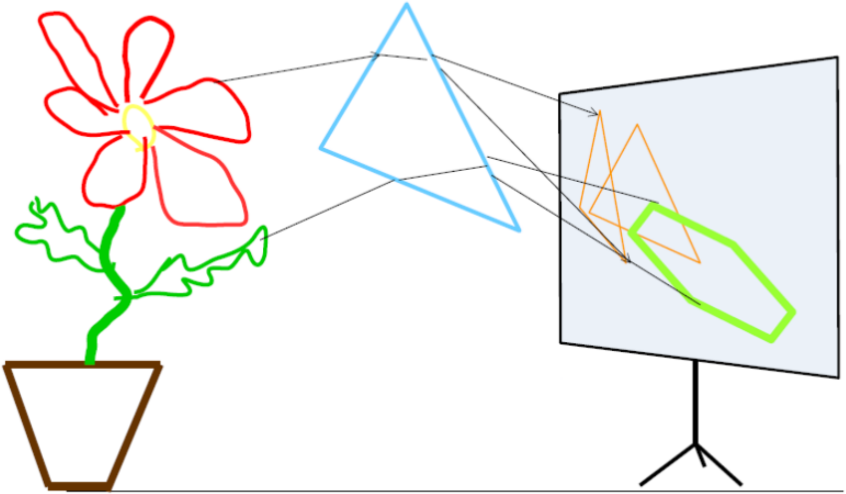
Logic, Relativity and Beyond 2017

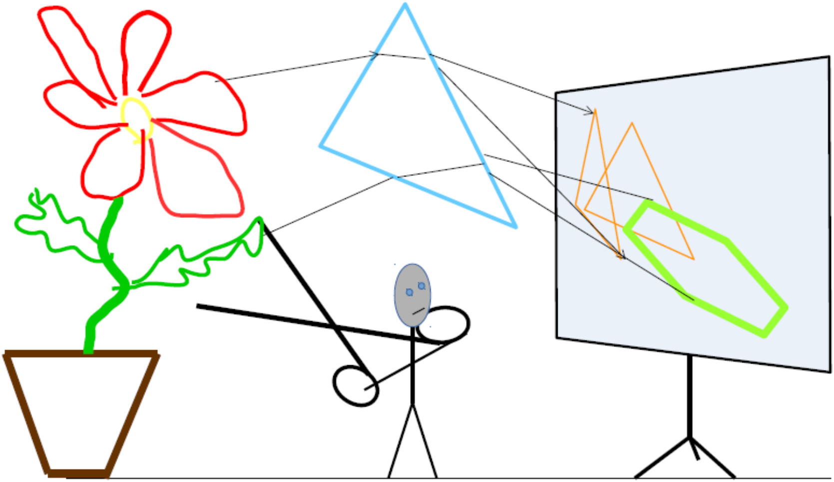
Outline

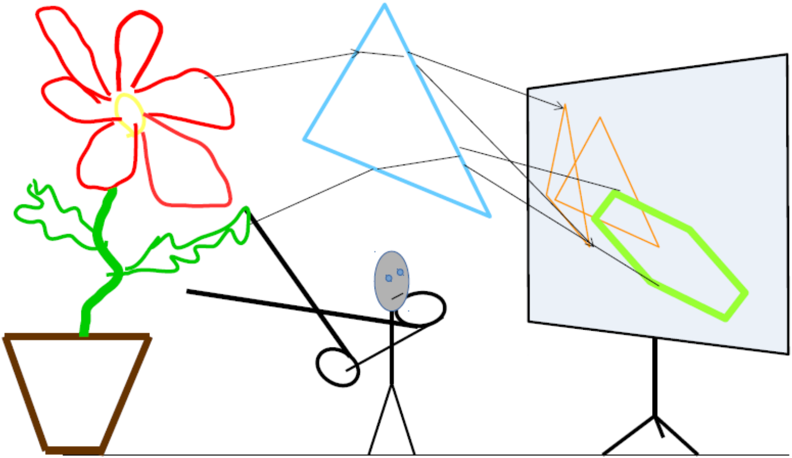
1. Philosophy: projector epistemology
2. Galilean relativistic fluids
3. Special relativistic dissipative fluids

Reality









Mathematics is the light.

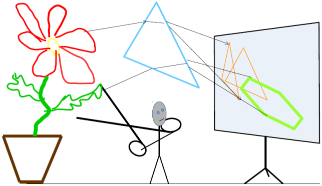
True, unbiased and sharp vision

Beamer technology and photographic art

- Sharpness: use mathematical building blocks
- Polishing the projection: Cleaning of the prism – removing the artificial and arbitrary elements (e.g. units, reference frames)
- Distorsion free: Projector rebuilding – adapted building blocks. E.g. differential equations, from circles of Copernicus to ellipses of Kepler.
- Fragment unification: minimal number of assumptions and axioms.

Human objectivity: double control

- Negative feedback: viable model
- Physical: observations and experiments
- Mathematical: paradoxes and inconsistencies
- Constructive: prediction machine



True, unbiased and sharp vision

Model concepts of Matolcsi:

1. Sharp: every element of the model is a mathematical object.
2. True: only those elements and properties are accepted that have counterparts in the reality.

Mathematical equivalence and physical difference. E.g. physical units.

Light version for physicists: focus on important elements

1. Spacetimes without reference frames and relative notions: Galilean relativistic and special relativistic.
2. Thermodynamics is responsible for material stability. Entropy is a Ljapunov function(al).

Fluids are more fundamental than you think

- Spacetime is a fluid (Geroch), or not (Etesi)
- Quantum Field Theories are fluid theories (Jackiw)

Objectivity and relativity

Transformation rules

- Galilei invariance (physics)
- Rigid body motion (engineering)

Transformation rule of Noll (1958):

$$x'^a = \begin{pmatrix} t' \\ x'^i \end{pmatrix} = \begin{pmatrix} t \\ h^i(t) + Q^{ij}(t)x^j \end{pmatrix},$$

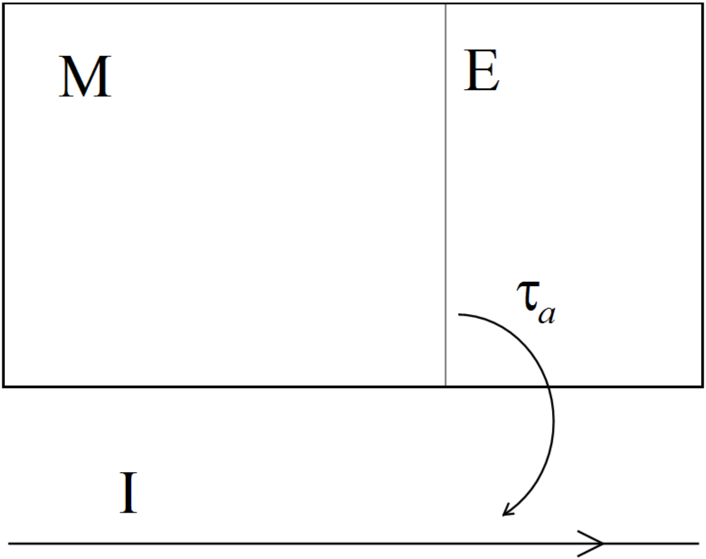
where $Q^{-1} = Q^T$ is an orthogonal tensor, a is abstract index.
Jakobian:

$$J'^{ab} = \frac{\partial x'^a}{\partial x^b} = \begin{pmatrix} 1 & 0^j \\ \dot{h}^i + \dot{Q}^{ij}x^j & Q^{ij} \end{pmatrix}$$

Transformation rule:

$$C'^a = J'^{ab}C^b$$

The four dimensions of Galilean relativistic space-time

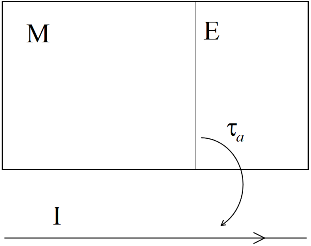


Mathematical structure of Galilean relativistic space-time

1. The *space-time* $\mathbb{M}$ is an oriented four dimensional vector space of the $x^a \in \mathbb{M}$ *world points or events*. There are no Euclidean or pseudoeuclidean structures on $\mathbb{M}$: the length of a space-time vector does not exist.
2. The *time* $\mathbb{I}$ is a one dimensional oriented vector space of $t \in \mathbb{I}$ *instants*.
3. $\tau_a : \mathbb{M} \to \mathbb{I}$ is the *timing* or *time evaluation*, a linear surjection.
4. $\delta_{ij} : \mathbb{E} \times \mathbb{E} \to \mathbb{R} \otimes \mathbb{R}$ Euclidean structure is a symmetric bilinear mapping, where $\mathbb{E} := Ker(\tau) \subset \mathbb{M}$ is the three dimensional vector space of *space vectors*.

- Simplification: space-time and time are affine spaces
- Simplification: measure lines.
- Abstract indexes: $a, b, c, ...$ for $\mathbb{M}$, $i, j, k, ...$ for S
- Reference frames are global and smooth velocity fields.
- Transformation rules can be derived between any reference frames.

Vectors an covectors are different



$$\left(\begin{matrix} t' \\ x'^i \end{matrix}\right) = \left(\begin{matrix} t \\ x^i + v^i t \end{matrix}\right)$$

Vector transformations (extensives):

$$\left(\begin{matrix} A' \\ A'^i \end{matrix}\right) = \left(\begin{matrix} A \\ A^i + v^i A \end{matrix}\right)$$

Covector transformations (derivatives):

$$\left(B' \quad B'_i\right) = \left(B - B_k v^k \quad B_i\right)$$

$$\boxed{A'^a B'_a = A^a B_a = AB + A^i B_i}$$

Balances: absolute, local and substantial

$$\boxed{\partial_a A^a = 0} \qquad \longrightarrow \qquad \begin{aligned} u^a: \quad D_u A + \partial_i A^i &= d_t A + \partial_i A^i = 0, \\ u'^a: \quad D_{u'} A + \partial_i A'^i &= \partial_t A + \partial_i A'^i = 0. \end{aligned}$$

$(a,b,c \in \{0,1,2,3\})$

Transformed: $(d_t - v^i \partial_i)A + \partial_i(A^i + Av^i) = d_t A + A\partial_i v^i + \partial_i A^i = 0$

From relative to absolute fluids

Usual substantial balances

$$
\begin{aligned}
\dot{\rho} + \rho\partial_i v^i &= 0, \\
\rho\dot{v}^i + \partial_k P^{ik} &= 0^i, \\
\dot{e} + e\partial_i v^i + \partial_i q^i + P^{ij}\partial_i v_j &= 0.
\end{aligned}
$$

Energy-momentum-density does not work in Galilean relativity.

Entropy production rate

$$
\frac{1}{T}\left(P^{ij} - p\delta^{ij}\right)\partial_i v_j + q^i\partial_i\frac{1}{T} \geq 0
$$

Products of relative and absolute quantities.

What kind of quantity is the energy?

- Square of the relative velocity $\rightarrow$ 2nd order tensor
- Kinetic theory: trace of a contravariant second order tensor.
- Energy density and flux: additional order

Basic field:

$$\boxed{Z^{abc} = z^{bc}u^a + z^{ibc} : \qquad \text{mass-energy-momentum density-flux tensor}}$$

$a, b, c \in \{0,1,2,3\}, \quad i, j, k \in \{1,2,3\}$

$$z^{bc} \rightarrow \begin{pmatrix} \rho & p^j \\ p^k & e^{jk} \end{pmatrix}, \qquad z^{ibc} \rightarrow \begin{pmatrix} j^i & P^{ij} \\ P^{ik} & q^{ijk} \end{pmatrix}, \qquad e = \frac{e^j{}_j}{2}$$

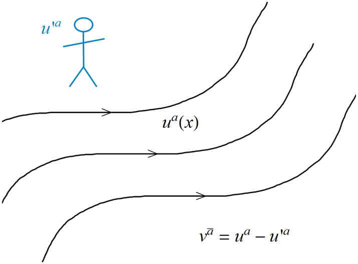
Galilean transformation

$$\boxed{Z'^{abc} = G^a_d G^b_e G^c_f Z^{def}}$$

$$Z^{abc} = \left(\begin{pmatrix} \rho & p^i \\ p^j & e^{ji} \end{pmatrix} \quad \begin{pmatrix} j^k & P^{ki} \\ P^{kj} & q^{kij} \end{pmatrix} \right), \quad G^a_d = \begin{pmatrix} 1 & 0^i \\ v^j & \delta^{ji} \end{pmatrix}, \quad e = \frac{e^i{}_i}{2}$$

Transformation rules follow:

$$\begin{aligned} \rho' &= \rho, \\ p'^i &= p^i + \rho v^i, \\ e' &= e + p^i v_i + \rho \frac{v^2}{2}, \\ j'^i &= j^i + \rho v^i, \end{aligned}$$



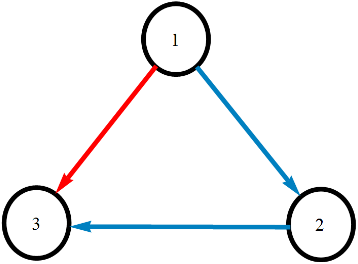
$$\begin{aligned} P'^{ij} &= P^{ij} + \rho v^i v^j + j^i v^j + p^j v^i, \\ q'^i &= q^i + e v^i + P^{ij} v_j + p^j v_j v^i + (j^i + \rho v^i) \frac{v^2}{2}. \end{aligned}$$

Galiean transformation of energy

Transitivity:

$$\left.\begin{aligned} e_2 &= e_1 + p_1 v_{12} + \rho\frac{v_{12}^2}{2} \\ e_3 &= e_2 + p_2 v_{23} + \rho\frac{v_{23}^2}{2} \end{aligned}\right\} \to e_3 = e_1 + p_1 v_{13} + \rho\frac{v_{13}^2}{2}$$

$$p_2 = p_1 + \rho v_{12}, \qquad v_{13} = v_{12} + v_{23}$$



Balance transformations

Absolute

$$\partial_a Z^{abc} = \dot{z}^{bc} + z^{bc}\partial_a u^a + \partial_a z^{ibc} = 0$$

Rest frame

$$\begin{aligned}
\dot{\rho} + \partial_i j^i &= 0, \\
\dot{p}^i + \partial_k P^{ik} &= 0^i, \\
\dot{e} + \partial_i q^i &= 0.
\end{aligned}$$

Inertial reference frame

$$\begin{aligned}
\dot{\rho} + \rho\partial_i v^i + \partial_i j^i &= 0, \\
\dot{p}^i + p^i\partial_k v^k + \partial_k P^{ik} + \rho\dot{v}^i + j^k\partial_k v^i &= 0^i, \\
\dot{e} + e\partial_i v^i + \partial_i q^i + p^i\dot{v}_i + P^{ij}\partial_i v_j &= 0.
\end{aligned}$$

Further consequences

- Fluid mechanics, thermodynamics, including entropy production, are absolute: independent of reference and flow-frames.
- Four-tensors are useful. Transformation rules can be calculated easily. For inertial frames those are the same as in RET.
- Thermodynamics of motion: four-cotensor of intensive quantities. Absolute entropy production with absolute thermodynamic fluxes and forces.
- Second law: (linear) asymptotic stability of homogeneous equilibrium.
- Key concept: flow-frame.

Special relativistic dissipative fluids

Problem set

1. First order and second order fluids: unexpected violent dissipative instability
2. Parabolic or hyperbolic?
3. What is flowing? Particles, the energy or the thermometer?
4. Temperature of moving bodies. The enigma of covariant thermodynamics.
5. Kinetic theory is not a big help.

Relativistic fluid theory

$$T^{ab} = eu^a u^b + q^a u^b + q^b u^a + P^{ab},$$ energy-momentum density

$$N^a = nu^a + j^a.$$ particle number density

$$q^a u_a = j^a u_a = 0, \quad P^{ba} u_a = P^{ab} u_a = 0^b$$

$$T^{ab} = \begin{pmatrix} e & q^i \\ q^j & P^{ij} \end{pmatrix}, \qquad N^a = \begin{pmatrix} n \\ j^i \end{pmatrix}$$

$a, b \in \{0,1,2,3\}; \quad i, j \in \{1,2,3\}; \quad diag(1,-1,-1,-1)$

$$\dot{e} = u^a \partial_a e$$

- u^a - velocity field
- e - energy density
- q^a - momentum density or energy flux??
- P^{ab} - pressure
- n - particle num. density
- j^a - particle current

General, expressed by comoving splitting

$$u_a \partial_b T^{ab} = \dot{e} + e\, \partial_a u^a + \partial_a q^a + u_a \dot{q}^a + u_a \partial_b P^{ab} = 0^a$$ energy balance

$$\partial_b N^b = \dot{n} + n\, \partial_a u^a + \partial_a j^a = 0$$ particle number balance

Dissipative or ideal?

$$P^{ab} = -p\Delta^{ab} + \Pi^{ab} = (-p + \Pi)\Delta^{ab} + \pi^{ab}$$ pressure splitting

What is ideal?

Landau-Lifshitz:

$$N^a=\hat{n}\,\hat{u}^a+\hat{j}^a$$
$$T^{ab}=\hat{e}\,\hat{u}^b\,\hat{u}^a+\hat{P}^{ab}=\hat{e}\,\hat{u}^b\,\hat{u}^a-\hat{p}\,\hat{\Delta}^{ab}+\hat{\Pi}^{ab}$$

$\Leftrightarrow$

Eckart:

$$N^a=nu^a$$
$$T^{ab}=eu^b u^a+q^b u^a+q^a u^b-p\Delta^{ab}+\Pi^{ab}$$

Transformation:

$$\hat{u}^a=\frac{u^a+w^a}{\zeta}$$

What is ideal?

$$N_0^a=nu^a$$
$$T_0^{ab}=eu^b u^a-p\Delta^{ab}$$

$\Rightarrow$

$$N_0^a=\hat{n}\,\hat{u}^a+j^a$$
$$T_0^{ab}=\hat{e}\,\hat{u}^b\,\hat{u}^a+q^b\,\hat{u}^a+q^a\,\hat{u}^b-p\,\hat{\Delta}^{ab}+\Pi^{ab}$$

$$\hat{n}=\frac{n}{\zeta},\quad j^a=n\frac{\hat{w}^a}{\zeta},\quad \hat{e}=\frac{h}{\zeta^2}-p,\quad q^a=h\hat{w}^a,\quad \Pi^{ab}=\frac{\hat{w}^a\hat{w}^b}{h}$$

Ideal fluid is a class of N^a, T^{ab}

Dissipation leads to homogenization? Equilibrium is a submanifold?

Entropy production

Constitutive theory – closure by linear relation, Entropy balance is constrained by the other balances.

$$\Sigma = -j^a \partial_a \alpha - \beta \Pi^{ab} \partial_b u_a + q^a \left(\partial_a \beta + \beta \dot{u}_a \right) \geq 0$$

Closure by linear relations:

$$j^a = \eta \Delta^{ab} \partial_b \alpha$$

$$Pi^{ab} = \eta_v \partial_c u^c + \eta \Delta^{ac} \Delta^{bd} (\partial_c u_d + \partial_d u_c)/2$$

$$q^a = \lambda \Delta^{ab} \left(\partial_a \beta + \beta \dot{u}_a \right)$$

Background: Ideal fluid: $j^a = 0$, $q^a = 0$, $\Pi^{ab} = 0^{ab}$
Entropy flux and Gibbs relation: $J^a = \beta q^a$, $ds = \beta de - \alpha dn$

Fields and equations

Fields:

N^a	4
T^{ba}	10
u^a	3
	Σ17

j^a	3
q^a	3
Π^{ab}	6
	Σ 12

$$q^a u_a = j^a u_a = 0, \quad \Pi^{ba} u_a = \Pi^{ab} u_a = 0^b$$

Equations:

$\partial_a N^a = 0,$	1
$\partial_b T^{ab} = 0^a,$	4

N^a – particle number vector
T^{ab} – energy-momentum density
u^a – velocity field

j^a – particle flux/current
q^a – energy flux and momentum density
Π^{ab} – viscous pressure

n, e, u^a – basic fields

Flow-frames, non-equilibrium thermodynamics, second law

Paradox solved? Second order

$$\partial_a S^a = \dot{s} + s\,\partial_a u^a + \partial_a J^a \geqslant 0$$

Eckart (1940), theory and flow-frame:

$$S^a(T^{ab}, N^a) = s(e,\ n)u^a + \frac{q^a}{T}$$

(Müller)-Israel-Stewart (1969-72) theory in Eckart flow-frame:

$$S^a(T^{ab}, N^a) = \left(s(e,\ n) - \frac{\beta_0}{2\mathrm{T}}\Pi^2 - \frac{\beta_1}{2\mathrm{T}}q_b q^b - \frac{\beta_2}{2\mathrm{T}}\pi^{bc}\pi_{bc}\right)u^a + \\ + \frac{1}{T}\left(q^a + \alpha_0 \Pi q^a + \alpha_1 \pi^{ab} q_b\right)$$

isotropic, Grad compatible

Paradox solved? Divergence type

(Müller)-Israel-Stewart theory: The linearized version is conditionally hyperbolic in Eckart frame.

Classical structure, normal convariant entropy inequality:

$$\partial_a N^a = 0; \qquad \partial_a T^{ab} = 0^b;$$
$$\partial_a S^a + \alpha \partial_a N^a + \beta_b \partial_a T^{ab} \geq 0.$$

Divergence, type theories: hyperbolic by construction (Geroch).

$$\partial_a N^a = 0; \qquad \partial_a T^{ab} = 0^b; \qquad \partial_a A^{abc} = I^{bc}$$
$$\partial_a S^a + \chi \partial_a N^a + \chi_b \partial_a T^{ab} + \chi_{bc} \partial_a A^{abc} = 0,$$
$$\partial_a S^a = -\chi_{bc} I^{bc} \geq 0.$$

Conclusions

- There is a reference frame independent Galilean relativistic fluid theory. Minimal assumptions.
- Flow-frames.
- Flow-frame and reference frame independent entropy production.
- Dissipative theories are stable with proper thermodynamics, that distinguishes between momentum density and energy flux.

We have a sharp and true but still incomplete vision.

Thank you for the attention!

More details are here:

VP, Biró, TS., EPJ–ST, 155:201–212, 2008, (arXiv:0704.2039v2).
VP, MMS, 3(6):1161–1169, 2009, (arXiv:07121437).
Biró, TS., VP. EPL, 89:30001, 2010. (arXiv:0905.1650)
VP, EPJ WoC, 13:07004, 2011, (arXiv:1102.0323).
VP, Biró, TS., PLB, 709(1-2):106–110, 2012, (arXiv:1109.0985).
VP, CMT, 29/2, 133/151, 2017 (arXiv:1510.03900)
VP-Pavelka-Grmela, JNET, 42/2, 133-142, 2017 (arXiv:1508.00121)

Detailed conclusions

Relativistic incomplete conclusions

1. Temperature is not necessarily parallel to the flow.
2. Generic instability is due to momentum density–heat flux identification. Israel-Stewart theory is not necessary.
3. Kinetic theory?
4. Flow-frame independent entropy production.

Galilei and special relativity

1. Mathematically equivalent, physically different: not sharp enough.
2. Flow- and reference frame free thermodynamics and dissipation. Galilean relativity.
3. There is an energy-momentum-mass-....
4. Momentum-flow is the best.

We have a sharp and true but still incomplete vision.

Balances of simple fluids

Local

$$\begin{aligned} \partial_t \rho + \partial_k(\rho v^k) &= 0, \\ \partial_t(\rho v^i) + \partial_k(P^{ik} + \rho v^i v^k) &= 0^i, \\ \partial_t e_{tot} + \partial_k(q^k_{tot} + e_{tot} v^k) &= 0. \end{aligned}$$

Substantial

$$\begin{aligned} \dot{\rho} + \rho \partial_k v^k &= 0, \\ (\rho v^i)^{\cdot} + \rho v^i \partial_k v^k + \partial_k P^{ik} &= 0^i, \\ \dot{e}_{tot} + e_{tot} \partial_k v^k + \partial_k q^k_{tot} &= 0. \end{aligned}$$

Notation:

- $\partial_t = \frac{\partial}{\partial_t}$, $\partial_i = \nabla$, $v^i = \mathbf{v}$, indices are not coordinates. $i, j, k \in \{1,2,3\}$
- e_{tot} is the total energy density.

Transformations

v^i relative velocity,
$\partial_t + v^i \partial_i = \frac{d}{dt}$, comoving derivative,
$\hat{q}^i = q^i + e_{tot} v^i$, conductive and convective

Fluid thermodynamics

total - kinetic = internal , $e = e_{tot} - \rho v^2/2$

$$\frac{d}{dt}\left(\rho\frac{v^2}{2}\right) + \rho\frac{v^2}{2}\partial_i v^i + \partial_i(P^{ik}v_k) - P^{ik}\partial_i v_k = 0.$$

$$\dot{e} + e\partial_k v^k + \partial_k(\underbrace{q^k_{tot} - P^{ik}v_i}_{q^k}) + P^{ik}\partial_i v_k = 0.$$

Thermodynamics:

$s(e,\rho), \quad de = Tds + \mu d\rho; \quad e + p = Ts + \mu\rho, \quad s^i = \frac{q^i}{T}$

$$\dot{s} + s\partial_i v^i + \partial_i s^i = \frac{1}{T}\dot{e} - \frac{\mu}{T}\dot{\rho} + s\partial_i v^i + \partial_i\frac{q^i}{T} =$$

$$-\frac{1}{T}\left(e\partial_i v^i + \partial_i q^i + P^{ij}\partial_i v_j\right) + \frac{\mu}{T}\left(\rho\partial_i v^i\right) + s\partial_i v^i + \frac{\mu}{T}\partial_i q^i + q^i\partial_i\frac{1}{T} =$$

$$\boxed{q^i\partial_i\frac{1}{T} - \frac{1}{T}(P^{ij} - p\delta^{ij})\partial_i v_j \geq 0.}$$

Basic fields: ρ, e, v^i; Constitutive functions: q^i, P^{ij}

Absolute and relative fields

$Z^{abc} = z^{bc} u^a + z^{ibc}$: mass-energy-momentum density-flux tensor

u-form:

$$Z^{abc} = \left(\rho u^b u^c + p^{\bar{b}} u^c + u^b p^{\bar{c}} + e^{\bar{b}\bar{c}}\right) u^a + \left(j^i u^b u^c + P^{i\bar{b}} u^c + P^{i\bar{c}} u^b + q^{i\bar{b}\bar{c}}\right)$$

ρ	$= \tau_b \tau_c z^{bc} = \tau_a \tau_b \tau_c Z^{abc},$	density
$p^{\bar{b}}$	$= \pi^{\bar{b}}{}_d \tau_c z^{dc} = \tau_a \pi^{\bar{b}}{}_d \tau_c Z^{adc},$	momentum density
$e^{\bar{b}\bar{c}}$	$= \pi^{\bar{b}}{}_d \pi^{\bar{c}}{}_e z^{de} = \tau_a \pi^{\bar{b}}{}_d \pi^{\bar{c}}{}_e Z^{ade},$	energy density tensor
j^i	$= \pi^i{}_d \tau_b \tau_c Z^{dbc},$	(self)diffusion flux
$P^{i\bar{b}}$	$= \pi^i{}_d \pi^{\bar{b}}{}_e \tau_c Z^{dec},$	pressure
$q^{i\bar{b}\bar{c}}$	$= \pi^i{}_d \pi^{\bar{b}}{}_e \pi^{\bar{c}}{}_f Z^{def}.$	heat flux tensor

$e = \frac{1}{2} e^i{}_i$ energy density $\qquad$ $q^i = \frac{1}{2} q^{ij}{}_j$ heat flux

Thermodynamics. Gibbs relation I.

$\boxed{ds = Y_{bc} dz^{bc}}$ Y_{bc} chemical potential-thermovelocity-temperature cotensor

Physical definitions

$$Y_{bc} \overset{u}{\prec} \begin{pmatrix} y & y_j \\ y_k & y_{kj} \end{pmatrix} = \frac{\beta}{2} \begin{pmatrix} -2\mu & -w_j \\ -w_k & \delta_{jk} \end{pmatrix},$$

Transformation rules

$$\begin{aligned} \beta' &= \beta, \\ w_i' &= w_i + v_i, \qquad \text{like a vector!} \\ \mu' &= \mu - w_i v^i - \frac{v^2}{2}. \end{aligned}$$

Calculation with classical transformation matrix.

Thermodynamics. Gibbs relation II.

Absolute Gibbs relation: $\boxed{ds = Y_{bc} dz^{bc}}$

Absolute extensivity condition: $\boxed{S^a = Y_{bc} Z^{abc} + p^a}$

Absolute and relative

Pressure decomposition: $p^a = \beta p(u^a + w^i)$

$$
\begin{aligned}
S^a = Y_{bc} Z^{abc} + p^a \quad & \rightarrow \quad Ts = e + p - \mu\rho - w_i p^i, \\
& \rightarrow \quad Ts^i = q^i - \mu j^i - P^{ij} w_j + p w^i,
\end{aligned}
$$

$$
ds = Y_{bc} dz^{bc} \quad \rightarrow \quad de = Tds + \mu d\rho + w_i dp^i + (\rho w_i - p_i) dv^i.
$$

Relative Gibbs relation is Galilean invariant if the inertial reference frame changes.

Thermostat(odynam)ics.

Gibbs relation: $de = Tds + \mu d\rho + w_i dp^i + (\rho w_i - p_i)dv^i$

Maxwell relations

$s(e, \rho, p^i, v^i)$

$$\frac{\partial s}{\partial p^i} = \frac{w_i}{T}, \qquad \frac{\partial s}{\partial v^i} = \frac{\rho w_i - p_i}{T}$$

$$\frac{\partial^2 s}{\partial v^i p^j} = \frac{\partial^2 s}{\partial p^i v^j} = \boxed{\frac{\partial w_i}{\partial v^j} = \delta_{ij} - \rho \frac{\partial w_i}{\partial p^j}}$$

Solution:

$$w_i = \frac{p_i}{\rho} + A_{ij}\left(v^j + \frac{p^j}{\rho}\right) + \overline{w}_i$$

Galilean invariant(!) part:

$$\boxed{p_i = \rho w_i}$$

Termodynamics III. Entropy balance.

$$\partial_a S^a = \partial_a(su^a + s^i) = \sigma \geq 0, \text{ condition: } \partial_a Z^{abc} = 0$$

Entropy production

$$\begin{aligned}
\partial_a S^a &= \dot{s} + s\partial_a u^a + \partial_a s^i \\
&= \dots \\
&= -(j^i - \rho w^i)\partial_a \left(\beta\mu + \beta\frac{w^2}{2}\right) + \\
&\quad \left(q^i - w^i(e - p^j w_j) + (j^i - \rho w^{\bar{a}})\frac{w^2}{2} - P^{ij} w_j\right) \partial_a \beta - \\
&\quad \beta \left(P^i_j + w^i(\rho w_j - p_j) - j^i w_j - p\delta^i_j\right) \partial_a(u^b + w^j) \geq 0
\end{aligned}$$

Entropy production II.

$$\begin{aligned} \dot{\rho} + \rho \partial_i v^i + \partial_i j^i &= 0, \\ \dot{p}^i + p^i \partial_k v^k + \partial_k P^{ik} + \rho \dot{v}^i + j^k \partial_k v^i &= 0^i, \\ \dot{e} + e \partial_i v^i + \partial_i q^i + p^i \dot{v}_i + P^{ij} \partial_i v_j &= 0. \end{aligned}$$

$$\begin{aligned} \Sigma = & -(j^i - \rho w^i) \partial_i \left(\beta\mu + \beta \frac{w^2}{2} \right) + \\ & \left(q^i - w^i (e - p^j w_j) + (j^i - \rho w^i) \frac{w^2}{2} - P^{ij} w_j \right) \partial_i \beta - \\ & \beta \left(P^i_j + w^i (\rho w_j - p_j) - j^i w_j - p \delta^i_j \right) \partial_i (v^j + w^j) \geq 0 \end{aligned}$$

Variables: ρ, p^i, e

Constitutive functions: j^i, P^{ij}, q^i,

Equation of state: μ, T, w^i

v^i? flow-frame

Classical theory

Eos: $w^i = \frac{p^i}{\rho}$
Flow-frame: $A^i = 0$ if $u^a = \frac{A^a}{\tau_a A^a}$

Thermo-frame

$$w^i = 0 \quad \rightarrow \quad p^i = 0$$

$$\begin{aligned}
\dot{\rho} + \rho \partial_i v^i + \partial_i j^i &= 0, \\
\rho \dot{v}^i + \partial_k P^{ik} + j^k \partial_k v^i &= 0^i, \\
\dot{e} + e \partial_i v^i + \partial_i q^i + P^{ij} \partial_i v_j &= 0.
\end{aligned}$$

$$-j^i \partial_i \frac{\mu}{T} + q^i \partial_i \frac{1}{T} - \frac{1}{T}(P^{ij} - p\delta^{ij}) \partial_i v_j \geq 0$$

Flow-frame: hidden Galilean invariance

Constitutive theory

$$-j^i\partial_i\frac{\mu}{T} + q^i\partial_i\frac{1}{T} - \frac{1}{T}(P^{ij} - p\delta^{ij})\partial_i v_j \geq 0$$

	Diffusion	Thermal	Mechanical
Force	$-\partial_i\frac{\mu}{T}$	$\partial_i\frac{1}{T}$	$\partial_i v_j$
Flux	j^i	q^i	$-\frac{1}{T}\left(P^{ij} - p\delta^{ij}\right)$

$$\begin{aligned}
\dot\rho + \rho\partial_i v^i + \partial_i j^i &= 0, \\
\rho\dot v^i + \partial_k P^{ik} + j^k\partial_k v^i &= 0^i, \\
\dot e + e\partial_i v^i + \partial_i q^i + P^{ij}\partial_i v_j &= 0.
\end{aligned}$$

(Self)-diffusion : not Brenner like

The need of four dimensions

$V^i := \dot{h}^i$

Objectivity of spatial vectors

$$\begin{pmatrix} 1 & 0 \\ V^i + \dot{Q}^{ij} x^j & Q^{ij} \end{pmatrix} \begin{pmatrix} 0 \\ C^j \end{pmatrix} = \begin{pmatrix} 0 \\ Q^{ij} C^j \end{pmatrix} \quad \rightarrow \quad C'^i = Q^{ij} C^j.$$

Galilean transformations ($Q^{ij} = \delta^{ij}$) and four-vectors?

$$\begin{pmatrix} \hat{\rho} \\ \hat{j}^i \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ V^i & \delta^{ik} \end{pmatrix} \begin{pmatrix} \rho \\ j^k \end{pmatrix} = \begin{pmatrix} \rho \\ j^i + \rho V^i \end{pmatrix} \quad \rightarrow \quad \begin{matrix} \rho' = \rho \\ j'^i = j^i + \rho V^i \end{matrix}$$

Velocity $v^i := \dot{x}^i(t)$. By definition: $v'^i = \frac{d}{dt} x'^i = V^i + \dot{Q}^{ij} x^j + Q^{ij} v^j$
This is not a transformation of three-vectors.

Velocity as four-vector: $\dot{x}^a = (1, v^i)$

$$\begin{pmatrix} 1' \\ v'^i \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ V^i + \dot{Q}^{ij} x^j & Q^{ij} \end{pmatrix} \begin{pmatrix} 1 \\ v^j \end{pmatrix} = \begin{pmatrix} 1 \\ V^i + \dot{Q}^{ij} x^j + Q^{ij} v^j \end{pmatrix}$$