On Kripke completeness of some predicate modal logics

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General observations

Unlike the propositional case, in first-order modal (and intuitionistic) logic there is a gap between syntax and semantics. It turns out that simply axiomatizable modal logics may have complex semantic descriptions. The standard Kripke semantics does not work properly in the predicate case - "most of" modal predicate logics are Kripke-incomplete.

As the semantics of predicate logics is not clearly understandable, natural questions about properties of logics may be quite difficult.

Reference

D.Gabbay, V. Shehtman, D. Skvortsov. Quantification in Nonclassical Logic, Volume 1. Elsevier, 2009.

Formulas

Modal predicate formulas are built from the following ingredients:

- the countable set of individual variables $Var=\{v_1,v_2,\ldots\}$
- countable sets of n-ary predicate letters (for every $n\geq 0$)
- $\rightarrow$, $\bot$, $\Box$
- $\exists$, $\forall$

The connectives $\neg$, $\wedge$, $\vee$, $\Diamond$ are derived.

No constants or function symbols

NOTATION for the set of formulas MF

Variable and formula substitutions

$[y_1,\ldots,y_n/x_1,\ldots,x_n]$ simultaneously replaces all free occurrences of $x_1,\ldots,x_n$ with $y_1,\ldots,y_n$ (renaming bound variables if necessary)

To obtain $[C(x_1,\ldots,x_n,y_1,\ldots,y_m)/P(x_1,\ldots,x_n)]A$ from A

(1) rename all bound variables of A that coincide with the "new" parameters $y_1,\ldots,y_m$ of C,

(2) replace every occurrence of every atom $P(z_1,\ldots,z_n)$ with $[z_1,\ldots,z_n/x_1,\ldots,x_n]C$

Strictly speaking, all substitutions are defined up to congruence: formulas are congruent if they can be obtained by "legal" renaming of bound variables

$[Q(x,y,z)/P(x)]\,(\exists y P(y) \wedge P(z)) = \exists x Q(x,y,z) \wedge Q(z,y,z)$ or

$\exists u Q(u,y,z) \wedge Q(z,y,z)$

Modal logics

An modal predicate logic (mpl) is a set L of modal formulas

such that L contains

- the classical propositional tautologies
- the axiom of **K:** $\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$
- the standard predicate axioms

L is closed under the rules

- Modus Ponens: $A, A \rightarrow B \,/\, B$
- Necessitation: $A \,/\, \Box A$
- Generalization: $A \,/\, \forall x A$
- Substitution: A/SA (for any formula substitution S)

Propositional logics can be regarded as fragments of predicate logics (with only 0-ary predicate letters, without quantifiers).

Some notation

L+Γ := the smallest logic containing (L and Γ)

K := the minimal modal propositional logic

QL := the minimal predicate logic containing the propositional logic L

$L \vdash A$:= $A \in L$

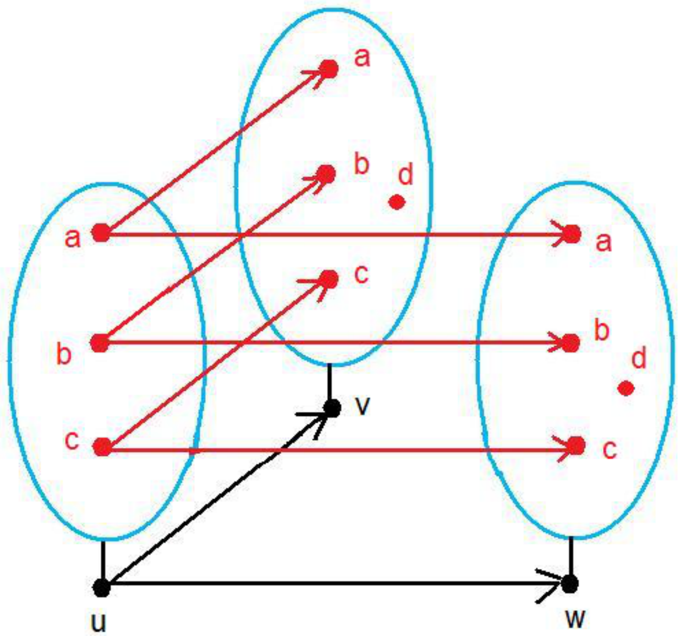
Kripke frame semantics for predicate logics

- A propositional Kripke frame $F=(W, R)$ $(W \neq \varnothing, R \subseteq W^2)$
- A predicate Kripke frame: $\Phi=(F,D)$, where

$D=(D_u)_{u \in W}$ is an expanding family of non-empty sets:

$$\text{if } u \, R \, v, \text{ then } D_u \subseteq D_v$$

D_u is the domain at the world u (consists of existing individuals).



A Kripke model over Φ is a collection of classical models:

$M=(\Phi,\theta)$, where $\theta=(\theta_u)_{u\in W}$ is a valuation

$\theta_u(P)$ is an n-ary relation on D_u for each n-ary predicate letter P

For every modal formula $A(x_1,\ldots, x_n)$ and $d_1,\ldots, d_n \in D_u$

consider a D_u-sentence $A(d_1,\ldots, d_n)$.

<u>Def</u> Forcing (truth) relation $M,u \vDash B$

between the worlds u and D_u-sentences B is defined by induction:

- $M,u \vDash P(d_1,\ldots, d_n)$ iff $(d_1,\ldots, d_n) \in \theta_u(P)$
- $M,u \vDash a=b$ iff a equals b
- $M,u \vDash \Box B$ iff for any v, uRv implies $M,v \vDash B$
- $M,u \vDash \forall x\, B$ iff for any $d \in D_u$ $M,u \vDash [d/x]B$

etc. (the other cases are clear)

Def (truth in a Kripke model; validity in a frame)

$M \vDash A(x_1,\dots,x_n)$ iff for any $u \in W$ $M,u \vDash \forall x_1 \dots \forall x_n A(x_1,\dots,x_n)$

$\Phi \vDash A$ iff for any M over Φ, $M \vDash A$

Soundness theorem

ML$(\Phi) := \{A \in MF \mid \Phi \vDash A\}$ is an mpl

Logics of this form are called *Kripke-complete.*

Remark For propositional formulas we do not need domains. So we can define validity of propositional formulas in propositional frames. Kripke-complete propostional logics are defined as sets of frame-valid propositional formulas.

Def The logic (of a certain type) of a class of frames $\mathscr{C}$ is the intersection of the logics of frames from $\mathscr{C}$.
A logic of a class of Kripke frames is called Kripke ($\mathscr{K}$)-complete.

Examples of Kripke-completeness

Surprisingly, for logics of the form **QL** not so many examples are known:

- for standard logics L (classical results by Kripke, Gabbay, Cresswell et al.):

  K, T, D, KB, K4, S4, S5

K4 : transitive frames

S4: transitive reflexive frames

- for other cases, with more sophisticated proofs

$\mathbf{S4.2} = \mathbf{S4} + \Diamond\Box p \rightarrow \Box\Diamond p$ confluent frames

(Ghilardi&Corsi, 1989)

$\mathbf{K4.3} = \mathbf{K4} + \Box(\Box p \wedge p \rightarrow q) \vee \Box(\Box q \wedge q \rightarrow p)$

non-branching transitive

$\mathbf{S4.3} = \mathbf{K4.3} + \Box p \rightarrow p$

non-branching transitive reflexive (Corsi, 1989)

Barcan formula

$$Ba := \Diamond\exists x A \to \exists x \Diamond A$$

This formula is valid in a Kripke frame iff the domains *remain constant:*

if uRv then $D_u = D_v$

For he same basic cases, **QL**+Ba are also Kripke-complete

(but Ba is derivable in **QKB, QS5**)

However, **QS4.2** + Ba is *K*-incomplete (Shehtman&Skvortsov 1990)

Def A propositional modal logic is called universal if the class of its frames is universal, i.e., the class of models of a universal classical first-order theory.

A propositional logic of a single finite frame is called tabular.

Theorem (Tanaka - Ono, 2001; book09) *If a modal propositional logic* Λ *is universal or tabular and K -complete, then* $L = Q\Lambda + Ba$ *is also K -complete.*

For other examples of Kripke-complete logics see book09, chapters 6,7.

Logics of specific frames

<u>Notation</u> For a class $\mathscr{F}$ of propositional Kripke frames

$$\mathscr{K}\mathscr{F} := \{(F,D) \mid F \in \mathscr{F}\},$$

the class of *predicate frames over $\mathscr{F}$.*

Little is known about logics of $\mathscr{K}\mathscr{F}$ for specific $\mathscr{F}$.

$\mathbf{ML}(\mathscr{K}(\mathbf{Q}, \leq))=\mathbf{QS4.3}$

Follows from the results by G. Corsi (cf. [book09, cor. 6.7.13])

$\mathbf{ML}(\mathscr{K}(\mathbf{R}, \leq)) = ?$

$\mathbf{ML}(\mathscr{K}(\mathbf{Z}, \leq)) = ?$ (probably, not RE)

$\mathbf{ML}(\mathscr{K}(\mathbf{Q}, <))=\mathbf{QD4.3}\mathrm{Ad}$ (G.Corsi, 1993)

Propositional modal logics for relativisitic time

$\mu(a_1,\ldots,a_n)=a_n^2-(a_1^2+\ldots+a_{n-1}^2)$

Causal accessibility: **a** can send a signal to **b**.

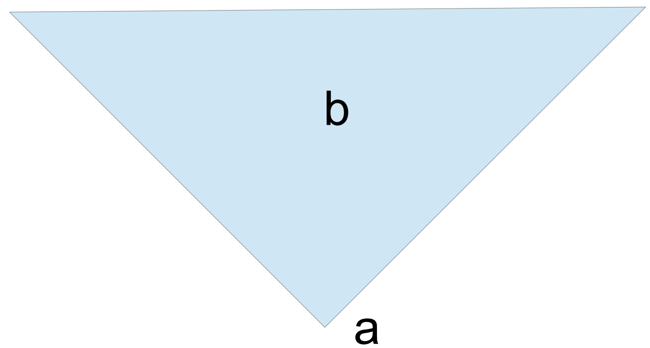
$(a_1,\ldots,a_n) \preccurlyeq (b_1,\ldots,b_n)$ iff $\mu(\mathbf{b}-\mathbf{a}) \geq 0$ & $a_n \leq b_n$

Chronological accessibility: **a** can send a signal to **b** slower than light

$(a_1,\ldots,a_n) \prec (b_1,\ldots,b_n)$ iff $\mu(\mathbf{b}-\mathbf{a}) > 0$ & $a_n < b_n$

$(\mathbf{R}^n)\_ = \{(a_1,\ldots,a_n) \mid a_n<0\}$

For n=2



Theorem (Goldblatt 1980, Shehtman 1976-83)

$\mathbf{ML}((\mathbf{R}^n)_{\_}, \preceq) = \mathbf{S4}$ (the logic of reflexive transitive frames)

$\mathbf{ML}(\mathbf{R}^n, \preceq) = \mathbf{S4.2}$ $(= \mathbf{S4} +$ confluence)

Theorem (Shapirovsky & Shehtman 2003)

$\mathbf{ML}((\mathbf{R}^n)_{\_}, \prec) = \mathbf{D4Ad_2}$ (the logic of serial transitive 2-dense frames)

Seriality condition: $\forall x\, \exists y\, xRy$

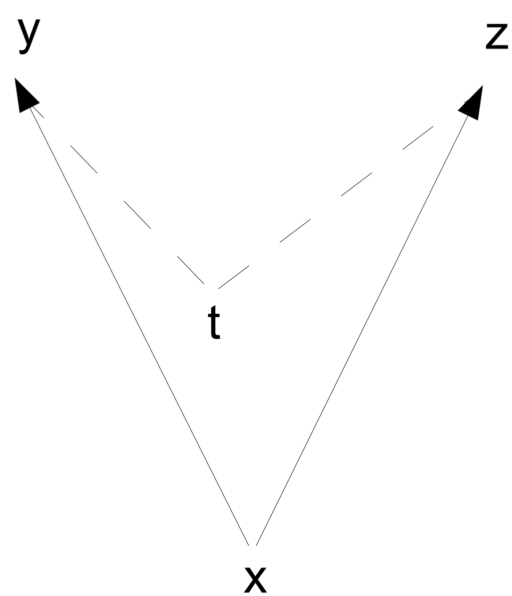
Seriality axiom: $\Diamond \top$

2-density condition:

$$\forall x,y,z\ (xRy\ \&\ xRz \Rightarrow\ \exists t\ (\ xRt\ \&\ tRy\ \&\ tRz))$$

2-density axiom:

$$\Diamond p \land \Diamond q \rightarrow \Diamond(\Diamond p \land \Diamond q)$$



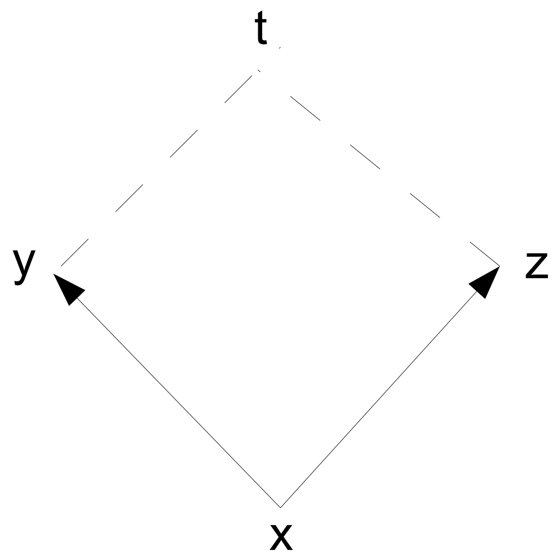
ML(R^n, $\prec$)=D4.2Ad (= **D4Ad** + confluence axiom)

confluence condition:

$$\forall x,y\ (xRy\ \&\ xRz \Rightarrow\ \exists t\,(\ xRt\ \&\ yRt))$$

confluence axiom:

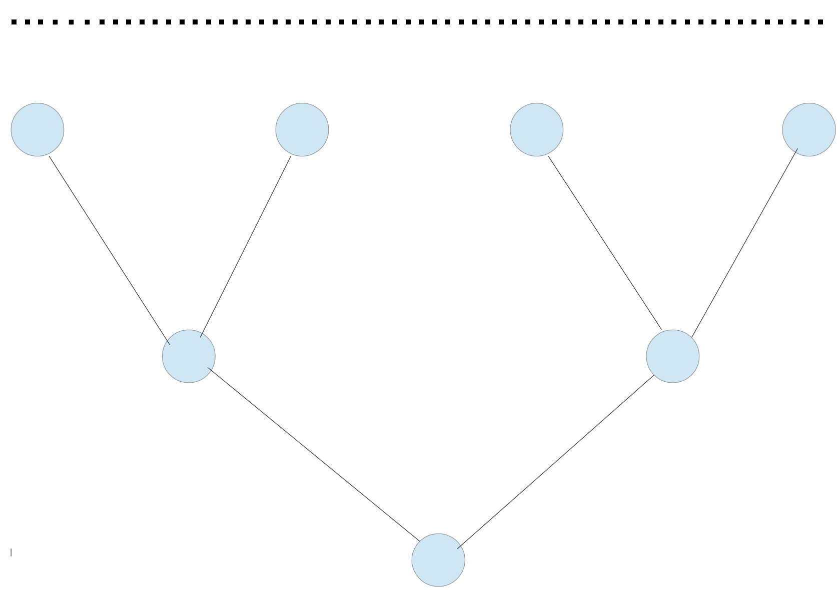
$$\Diamond\Box p \rightarrow \Box\Diamond p$$



We do not know predicate logics of this kind.
The only exception is

<u>Proposition 1</u> $\mathbf{ML}(\mathscr{K}((\mathbf{R}^n)_{\_}\ , \preceq\ )) = \mathbf{QS4}$

The proof is based on the construction from Goldblatt – Shehtman.
Consider the (transitive reflexive) binary tree IT_2:



1. There is a p-morphism (a validity-preserving map)

$$((\mathbf{R}^n)\_\ , \preceq) \twoheadrightarrow \mathrm{IT}_2.$$

2. This readily implies:

$$\mathbf{ML}(\mathscr{K}((\mathbf{R}^n)\_\ , \preceq)) \subseteq \mathbf{ML}(\mathscr{K}\mathrm{IT}_2)$$

3. Also

$$\mathbf{ML}(\mathscr{K}\mathrm{IT}_2) = \mathbf{QS4}$$

(essentially A.Dragalin, 1973; see book09)

4. $\preceq$ is a partial order, so

$\mathbf{ML}(\mathscr{K}((\mathbf{R}^n)\_\ , \preceq))$ contains **QS4.**

5. Thus $\mathbf{ML}(\mathscr{K}((\mathbf{R}^n)\_\ , \preceq)) = \mathbf{QS4}$

Remark The proposition (and the proof) transfers to the spaces over **Q**.

Our plan

1. To approach the predicate logics of chronological necessity, we begin with the logic of 1-density.
2. The proof extends to the logic of 2-density and (probably) to 2-density+confluence.
3. Further steps are left for the future.

The 1-density axiom

Ad:= $\Box\Box p \to \Box p$

The semantic condition: $\forall x,y\ (xRy \Rightarrow \exists z\,(xRz\ \&\ zRy))$

In propositional logic this axiom does not make a problem: the logics **K+**Ad, **K4+**Ad and many others are Kripke-complete.

This follows from the properties of canonical Kripke models.

The *canonical model* of a modal propositional logic L.

$M_L := (W_L, R_L, \theta)$, where

- W_L is the set of all L-complete (maximal consistent) propositional theories
- uR_Lv iff $\Box^- u \subseteq v$

$\Box^- u := \{ A \mid \Box A \in u\}$

- $\theta(p,u)=1$ iff $p \in u$

(for any proposition letter p)

So every L-consistent formula is satisfiable.

Def A modal proposional logic is *canonical* if $(W_L, R_L) \models L$.

So *every canonical logic is complete*.

Canonicity of K4Ad

In many cases canonicity follows from Sahlqvist theorem. For the particular axioms of transitivity and density canonicity is checked by hand.

For density:

If $u\ R_L\ v$, then $\Box^{-}u \cup \{\Diamond A \mid A \in v\}$ is L-consistent (this follows by density). Extend this theory to a maximal one. This gives w such that uR_LwR_Lv.

Predicate canonical models

$VM_L=(VP_L, R_L, D_L, \xi_L)$, where

- VP_L is the set of all *small L-places*:

L-complete theories with the Henkin property:

for any A(x) there is a constant c such that

$(\exists x\, A(x) \to A(c)) \in u$

and with infinitely many spare constants

- uR_Lv iff $\Box^{-}u \subseteq v$
- $D_L(u)$ is the set of constants occurring in u
- $\xi_L(P(\mathbf{a}),u)=1$ iff $P(\mathbf{a}) \in u$

(for any predicate letter P and a list of constants **a**)

[We fix a countable set of all possible constants S*]

<u>Canonical model theorem</u>

$$M_L, u \models A \text{ iff } A \in u$$

(for any closed formula A in the language of u)

The corresponding notion of canonicity: L is canonical if it is valid in the canonical frame.

Fact **QK4** is canonical

But density makes a problem, and probably **QK4Ad** is not canonical.

Technically:

The theory $\Box^{-}u \cup \{\Diamond A \mid A \in v\}$ is consistent, but if we extend it an L-place w, we get new constants (from the Henkin condition),

so we cannot guarantee that $\Box^{-}w \subseteq v$.

To prove completeness we need a careful step-by-step construction. It gives a fragment of some canonical model.

The whole construction is easier to describe in game-theoretic terms.

Such a method orginates from classical model theory (Wifrid Hodges), afterwards it was applied by Ian Hodkinson and Robin Hirsch (“Relation algebras by games”, 2002) and in modal logic by Agi Kurucz (book “Many-dimensional modal logics” by Gabbay, Kurucz, Wolter, Zakharyaschev, 2003).

Def A (finite) *network* is a monotonic (homomorphic) map h from an irreflexive transitive finite tree to L-places.

Put

$h \leq h'$ iff dom(h) is a subframe of dom(h') and $h(u) \subseteq h'(u)$ for $u \in$ dom(h).

A *defect* in h is a pair (u,A) such that $\Diamond A \in h(u)$.

h *eliminates* (u,A) if $A \in v$ for some v such that uRv.

Def The *selection game* $SG_L(\Gamma)$ for an L-place Γ is played by two players: **A** and **E**. The rules:

1. Every position h_n before the (n+1)th move of **A** is a network.
2. The initial network h maps an irreflexive node to Γ.
3. The (n+1)th move of **A** can be of two kinds:
   (i) a defect in h_n ,

(ii) an insert queries: a pairs of succesors (u,v) in h_n .

4. **E** should reply with a network $h_{n+1} \geq h_n$ such that

(i) for a defect move of **A** h_{n+1} should eliminate this defect,

(ii) for an insert query (u,v) of **A** v should no longer be a successor of u in dom(h_{n+1}) .

5. **E** wins if the play continues infinitely or if **A** cannot move (this may happen at the very beginning if the initial Γ is an endpoint in the canonical model VM_L).

<u>Main Lemma 1</u> **E** has a winning strategy in the selection game.

Every infinite play generates a *limit network*

$h_\omega = \cup_n h_n$ (the values $h_\omega(u)$ can be 'large')

and the *limit Kripke model* $M(h_\omega)$ over $\mathrm{dom}(h_\omega)$ such that

$M(h_\omega),u \vDash A$ iff $A \in h_\omega(u)$

for any atomic sentence A in the language of $h_\omega(u)$.

Main Lemma 2 If Γ is not an endpoint in VM_L, then there exists a play generating a limit network h_ω over a dense transitive frame such that

$M(h_\omega),u \vDash A$ iff $A \in h_\omega(u)$

for any sentence A in the language of $h_\omega(u)$.

The idea of the proof: **A** should choose new defects at odd moves and new inserts at even moves. A countable enumeration allows to count all possible defects and inserts. Once **E** wins the play, h_ω has no defects, and its domain has no successor pairs.

Theorem **QK4Ad** is strongly Kripke-complete: every consistent theory is satisfied in a model over a dense transitive frame.

Remark The method is applicable to **QK4.3Ad**, thus giving the result by Giovanna Corsi (1993):

$$\mathbf{ML}(\mathscr{K}(\mathbf{Q}, <)) = \mathbf{QD4.3}\mathrm{Ad}$$