Ordered Domain–Range Semigroups

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Operations on binary relations

Let U be a set. We define operations on elements of $\wp(U \times U)$.

Domain

$$\mathsf{D}(X) = \{(u, u) \mid (u, v) \in X \text{ for some } v \in U\}$$

Range

$$\mathsf{R}(X) = \{(v, v) \mid (u, v) \in X \text{ for some } u \in U\}$$

Composition

$$X;Y = \{(u, v) \mid (u, w) \in X \text{ and } (w, v) \in Y \text{ for some } w \in U\}$$

Identity

$$1' = \{(u, u) \mid u \in U\}$$

Antidomain

$$\mathsf{A}(X) = \{(u, u) \mid u \in U, (u, v) \notin X \text{ for any } v\}$$

Converse

$$X^{\smile} = \{(u, v) \mid (v, u) \in X\}$$

for every $X, Y \subseteq U \times U$.

Domain–range semigroups

Representable domain–range semigroups

A *representable domain–range semigroup* is a subalgebra of

$$(\wp(U \times U), \mathsf{D}, \mathsf{R}, ;)$$

With motivation in software verification:

Jipsen and Struth

Is the class $\mathbb{R}(\mathsf{D}, \mathsf{R}, ;)$ of representable domain–range semigroups finitely axiomatizable?

[Hirsch and M, JLAP 2011]

Let τ be a similarity type such that $\{\mathsf{D}, ;\} \subseteq \tau \subseteq \{\mathsf{D}, \mathsf{R}, \mathsf{A}, ;, 1', 0\}$. The class $\mathbb{R}(\tau)$ of representable τ-algebras is not finitely axiomatizable in first-order logic.

Adding a semilattice structure?

Adding join?

[Hirsch and M, JLAP 2011] using [Andréka 1988]

Let τ be a similarity type such that $\{;, +\} \subseteq \tau \subseteq \{\mathsf{D}, \mathsf{R}, \mathsf{A}, ;, +, ^\smile, ^*, 1', 0, 1\}$. The class $\mathbb{R}(\tau)$ of representable τ-algebras is not finitely axiomatizable in first-order logic.

Adding meet?

[Hirsch and M, AU 2007]

The class $\mathbb{R}(;, \cdot, 1')$ is not finitely axiomatizable in first-order logic.

An ultraproduct construction of non-representable algebras, where $1'$ is an atom. Thus we can augment these algebras with D, R. Thus $\mathbb{R}(\mathsf{D}, \mathsf{R}, ;, \cdot)$ is not finitely axiomatizable.

Adding a lattice structure?

[Andréka, AU 1991]

Let τ be a similarity type such that $\{;, +, \cdot\} \subseteq \tau \subseteq \{;, +, \cdot, -, ^\smile, ^*, 1', 0, 1\}$. The class $\mathbb{R}(\tau)$ of representable τ-algebras is not finitely axiomatizable in first-order logic.

Another ultraproduct construction. Observe that we can define $\mathsf{D}(x) = (x\,;x^\smile) \cdot 1'$, $\mathsf{R}(x) = (x^\smile\,; x) \cdot 1'$ and $\mathsf{A}(x) = -\,\mathsf{D}(x) \cdot 1'$. Thus $\mathbb{R}(\mathsf{D}, \mathsf{R}, \mathsf{A}, ;, +, \cdot, \ldots)$ is not finitely axiomatizable.

Axiomatizing the equational theory

Recall that *antidomain* is defined as

$$\mathsf{A}(X) = \{(u,u) \mid (u,v) \notin X \text{ for any } v\}$$

Observe that $\mathsf{D}(x) = \mathsf{A}(\mathsf{A}(x))$.

[Hollenberg, JOLLI 1997]

The varieties $\mathbb{V}(\mathsf{A},;)$ and $\mathbb{V}(\mathsf{A},;,+)$ generated by $\mathbb{R}(\mathsf{A},;)$ and $\mathbb{R}(\mathsf{A},;,+)$, respectively, are finitely axiomatizable.

Our main result is:

Jackson and M

The variety $\mathbb{V}(\mathsf{D},\mathsf{R},;,+)$ generated by $\mathbb{R}(\mathsf{D},\mathsf{R},;,+)$ is finitely axiomatizable.

Ordered domain–range semigroups

Define $x \leq y$ by $x + y = y$.
The axioms Ax:

$(D1)$	$\mathrm{D}(x) \,;\, x = x$	$(R1)$	$x \,;\, \mathrm{R}(x) = x$
$(D2)$	$\mathrm{D}(x \,;\, y) = \mathrm{D}(x \,;\, \mathrm{D}(y))$	$(R2)$	$\mathrm{R}(x \,;\, y) = \mathrm{R}(\mathrm{R}(x) \,;\, y)$
$(D3)$	$\mathrm{D}(\mathrm{D}(x) \,;\, y) = \mathrm{D}(x) \,;\, \mathrm{D}(y)$	$(R3)$	$\mathrm{R}(x \,;\, \mathrm{R}(y)) = \mathrm{R}(x) \,;\, \mathrm{R}(y)$
$(D4)$	$\mathrm{D}(x) \,;\, \mathrm{D}(y) = \mathrm{D}(y) \,;\, \mathrm{D}(x)$	$(R4)$	$\mathrm{R}(x) \,;\, \mathrm{R}(y) = \mathrm{R}(y) \,;\, \mathrm{R}(x)$
$(D5)$	$\mathrm{D}(\mathrm{R}(x)) = \mathrm{R}(x)$	$(R5)$	$\mathrm{R}(\mathrm{D}(x)) = \mathrm{D}(x)$
$(D6)$	$\mathrm{D}(x) \,;\, y \leq y$	$(R6)$	$x \,;\, \mathrm{R}(y) \leq x$

together with associativity of ; and $+$, idempotency of $+$ and additivity of ;, D, R.

Eliminating join

Assume

$$\mathbb{V}(\mathsf{D}, \mathsf{R}, ;, +) \models s \leq t$$

and we need $Ax \vdash s \leq t$, for all terms s, t.
Using additivity of the operations we have that

$$\mathbb{V}(\mathsf{D}, \mathsf{R}, ;, +) \models s_1 + \ldots + s_n = s \leq t = t_1 + \ldots + t_m$$

for some join-free terms $s_1, \ldots, s_n, t_1, \ldots, t_m$.
It is not difficult to show that this happens iff for every i there is j such that

$$\mathbb{V}(\mathsf{D}, \mathsf{R}, ;, +) \models s_i \leq t_j$$

Thus it is enough to show $Ax \vdash s_i \leq t_j$ for join-free terms.

Domain elements (in the free algebra)

Claim

Let $\mathfrak{A}$ be a model of Ax.

1. The algebra $(\mathrm{D}(A), ;)$ of domain elements is a (lower) semilattice and the semilattice ordering coincides with $\leq$.
2. For every $a \in A$, $\mathrm{D}(a)$ (resp. $\mathrm{R}(a)$) is the minimal element d in $\mathrm{D}(A)$ such that $d \,;\, a = a$ (resp. $a \,;\, d = a$).

Let $\mathfrak{F}_{Var} = (F_{Var}, ;, \mathrm{D}, \mathrm{R}, +)$ be the free algebra of the variety defined by Ax freely generated by a set Var of variables.

Claim

Let r, s, t be join-free terms such that $\mathfrak{F}_{Var} \models \mathrm{D}(r) \leq s \,;\, t$. Then $\mathfrak{F}_{Var} \models \mathrm{D}(r) \leq s = \mathrm{D}(s)$ and $\mathfrak{F}_{Var} \models \mathrm{D}(r) \leq t = \mathrm{D}(t)$.

Claim

Let s, t be join-free terms such that $\mathfrak{F}_{Var} \models s \leq \mathrm{D}(t)$. Then $\mathfrak{F}_{Var} \models s = \mathrm{D}(s)$.

Creating a representable algebra witnessing $Ax \not\vdash s \leq t$

Let T^-_{Var} be the set of join-free terms and $s, t \in T^-_{Var}$. We assume that $Ax \not\vdash s \leq t$ and we will construct a representable algebra $\mathfrak{A} \in \mathbb{R}(\mathrm{D}, \mathrm{R}, ;, +)$ witnessing $s \not\leq t$: $\mathfrak{A} \not\models s \leq t$.
Let F^-_{Var} be the equivalence classes of join-free terms (elements of $\mathfrak{F}_{Var}$).
We will define a labelled, directed graph G_ω as the union of a chain of labelled, directed graphs $G_n = (U_n, \ell_n, E_n)$ for $n \in \omega$, where

- U_n is the set of nodes,
- $\ell_n \colon U_n \times U_n \to \wp(F^-_{Var})$ is a labelling of edges,
- $E_n = \{(u, v) \in U_n \times U_n \mid \ell_n(u, v) \neq \emptyset\}$

Coherence

We will make sure that the following *coherence conditions* are maintained during the construction:

- GenC E_n is a reflexive, transitive and antisymmetric relation on U_n.
- PriC For every $(u, v) \in E_n$, $\ell_n(u, v)$ is a principal upset: $\ell_n(u, v) = a^{\uparrow} = \{x \in F^-_{Var} \mid a \leq x\}$ for some $a \in F^-_{Var}$.
- CompC For all $(u, v), (u, w), (w, v) \in U_n \times U_n$ and $a, b \in F^-_{Var}$, if $a \in \ell_n(u, w)$ and $b \in \ell_n(w, v)$, then $a \,;\, b \in \ell_n(u, v)$.
- DomC For all $(u, v) \in U_n \times U_n$ and $a \in F^-_{Var}$, if $\ell_n(u, v) = a^{\uparrow}$, then $\ell_n(u, u) = \mathrm{D}(a)^{\uparrow}$.
- RanC For all $(u, v) \in U_n \times U_n$ and $a \in F^-_{Var}$, if $\ell_n(u, v) = a^{\uparrow}$, then $\ell_n(v, v) = \mathrm{R}(a)^{\uparrow}$.
- IdeC For all $(u, v) \in U_n \times U_n$, $u = v$ iff $\ell_n(u, v) = \mathrm{D}(a)^{\uparrow}$ for some $a \in F^-_{Var}$.

Saturation

The construction will terminate in ω steps, yielding $G_\omega = (U_\omega, \ell_\omega, E_\omega)$ where $U_\omega = \bigcup_n U_n$, $\ell_\omega = \bigcup_n \ell_n$ and $E_\omega = \bigcup_n E_n$.
By the end of the construction we will achieve the following *saturation conditions*:

CompS For all $(u, v) \in U_\omega \times U_\omega$ and $a, b \in F^-_{Var}$, if $a \,;\, b \in \ell_\omega(u, v)$, then $a \in \ell_\omega(u, w)$ and $b \in \ell_\omega(w, v)$ for some $w \in U_\omega$.

DomS For all $(u, u) \in U_\omega \times U_\omega$ and $a \in F^-_{Var}$, if $\mathrm{D}(a) \in \ell_\omega(u, u)$, then $a \in \ell_\omega(u, w)$ for some $w \in U_\omega$.

RanS For all $(u, u) \in U_\omega \times U_\omega$ and $a \in F^-_{Var}$, if $\mathrm{R}(a) \in \ell_\omega(u, u)$, then $a \in \ell_\omega(w, u)$ for some $w \in U_\omega$.

Initial step

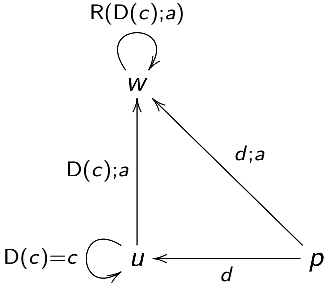
In the 0th step of the step-by-step construction we define $G_0 = (U_0, \ell_0, W_0)$ by creating an edge for every element of F^{-}_{Var}. We define U_0 by choosing elements $u_a, v_a, \ldots \in \omega$ so that $\{u_a, v_a\} \cap \{u_b, v_b\} = \emptyset$ for distinct a, b, and $u_a = v_a$ iff $\mathrm{D}(a) = a$ (i.e., a is a domain element of $\mathfrak{F}_{Var}$). We can assume that $|\omega \setminus U_0| = \omega$. We define

$$\ell_0(u_a, v_a) = a^{\uparrow}$$
$$\ell_0(u_a, u_a) = \mathrm{D}(a)^{\uparrow}$$
$$\ell_0(v_a, v_a) = \mathrm{R}(a)^{\uparrow}$$

and we label all other edges by $\emptyset$.

Step for domain

Our aim is to extend G_m to create an edge (u, w) witnessing a, provided $\mathrm{D}(a) \in \ell_m(u, u) = c^{\uparrow}$.



Domain step

We assume that we have a loop (u, u) labelled by the upset of a domain element $c = \mathrm{D}(c) \leq a$ such that $\mathrm{D}(c) \mathbin{;} a$ is not a domain element, but we may miss an edge (u, w) witnessing a.
We choose $w \in \omega \setminus U_m$, extend ℓ_m by

$$\ell_{m+1}(u, w) = (\mathrm{D}(c) \mathbin{;} a)^{\uparrow}$$
$$\ell_{m+1}(w, w) = (\mathrm{R}(\mathrm{D}(c) \mathbin{;} a))^{\uparrow}$$

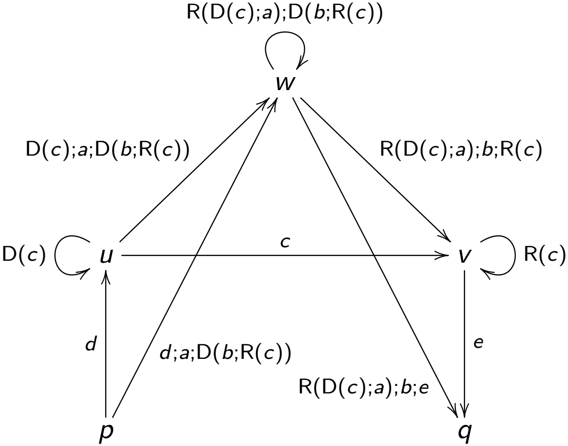
and for every $(p, u) \in E_m$ with $\ell_m(p, u) = d^{\uparrow}$ (some $d \in F^{-}_{Var}$)

$$\ell_{m+1}(p, w) = (d \mathbin{;} a)^{\uparrow}$$

All other edges involving the point w have empty labels.

Step for composition

Our aim is to extend G_m to create edges (u, w) and (w, v) witnessing a and b, provided $a\,;b \in \ell_m(u, v) = c^{\uparrow}$.



Composition step

We assume that

(CC1) $u \neq v$,

(CC2) $\mathsf{D}(c) ; a ; \mathsf{D}(b ; \mathsf{R}(c)) \neq \mathsf{D}(\mathsf{D}(c) ; a ; \mathsf{D}(b ; \mathsf{R}(c)))$,

(CC3) $\mathsf{R}(\mathsf{D}(c) ; a) ; b ; \mathsf{R}(c) \neq \mathsf{R}(\mathsf{R}(\mathsf{D}(c) ; a) ; b ; \mathsf{R}(c))$,

otherwise we define $G_{m+1} = G_m$. If (CC1)–(CC3) hold, then we choose $w \in \omega \setminus U_m$, extend ℓ_m by

$$\ell_{m+1}(u, w) = (\mathsf{D}(c) ; a ; \mathsf{D}(b ; \mathsf{R}(c)))^{\uparrow}$$

$$\ell_{m+1}(w, v) = (\mathsf{R}(\mathsf{D}(c) ; a) ; b ; \mathsf{R}(c))^{\uparrow}$$

$$\ell_{m+1}(w, w) = (\mathsf{R}(\mathsf{D}(c) ; a) ; \mathsf{D}(b ; \mathsf{R}(c)))^{\uparrow}$$

and for $(p, u), (v, q) \in E_m$ with $\ell_m(p, u) = d^{\uparrow}$ and $\ell_m(v, q) = e^{\uparrow}$ (some $d, e \in F^{-}_{Var}$)

$$\ell_{m+1}(p, w) = (d ; a ; \mathsf{D}(b ; \mathsf{R}(c)))^{\uparrow}$$

$$\ell_{m+1}(w, q) = (\mathsf{R}(\mathsf{D}(c) ; a) ; b ; e)^{\uparrow}$$

All other edges involving w will have empty labels.

In the limit

Lemma

G_ω is coherent and saturated.

Coherence of G_ω follows from the coherence of each G_m (easy but tedious).
Saturation of G_ω follows from the fact that we constructed the required witness edges (if they were not present yet in G_m).
Next we define a valuation $\flat$ of variables. Let for term r, its equivalence class in F_{Var} be denoted by $\overline{r}$. We let

$$x^\flat = \{(u, v) \in U_\omega \times U_\omega : \overline{x} \in \ell_\omega(u, v)\}$$

for every variable $x \in Var$.

Truth lemma

Let $\mathfrak{A} = (A, \mathrm{D}, \mathrm{R}, ;, +)$ be the subalgebra of the full algebra $(\wp(U_\omega \times U_\omega), \mathrm{D}, \mathrm{R}, ;, +)$ generated by $\{x^\flat : x \in \mathit{Var}\}$. Clearly $\mathfrak{A}$ is representable.

Lemma

For every join-free term r and $(u, v) \in U_\omega \times U_\omega$,

$$(u, v) \in r^\flat \text{ iff } \overline{r} \in \ell_\omega(u, v)$$

where $r^\flat$ is the interpretation of r in $\mathfrak{A}$ under the valuation $\flat$.

By coherence and saturation of G_ω.

Recall that we assumed that $\mathfrak{F}_{\mathit{Var}} \not\models s \leq t$. In the initial step of the construction we created the edge $(u_{\overline{s}}, v_{\overline{s}})$ such that $\ell_0(u_{\overline{s}}, v_{\overline{s}}) = \overline{s}^{\uparrow}$. Thus $\overline{s} \in \ell_\omega(u_{\overline{s}}, v_{\overline{s}})$ and $\overline{t} \notin \ell_\omega(u_{\overline{s}}, v_{\overline{s}})$. Hence, by Lemma, $(u_{\overline{s}}, v_{\overline{s}}) \in s^\flat$ and $(u_{\overline{s}}, v_{\overline{s}}) \notin t^\flat$. That is, $\mathfrak{A} \not\models s \leq t$, as desired.

Open problems

Adding meet and/or antidomain.

Open problems

Are the varieties generated by

- $\mathbb{R}(\mathsf{D}, \mathsf{R}, \mathsf{A}, ;, +)$
- $\mathbb{R}(\mathsf{D}, \mathsf{R}, ;, +, \cdot)$
- $\mathbb{R}(\mathsf{D}, \mathsf{R}, \mathsf{A}, ;, +, \cdot)$

finitely axiomatizable?