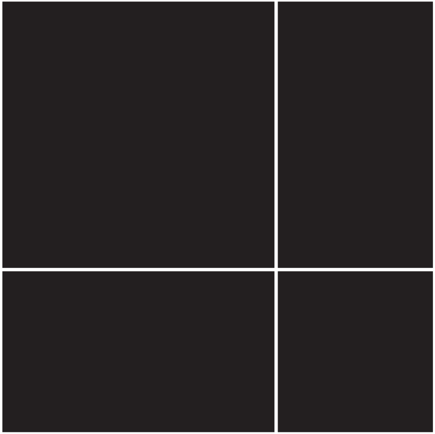
some "no hole" spacetime properties are unstable

john byron manchak

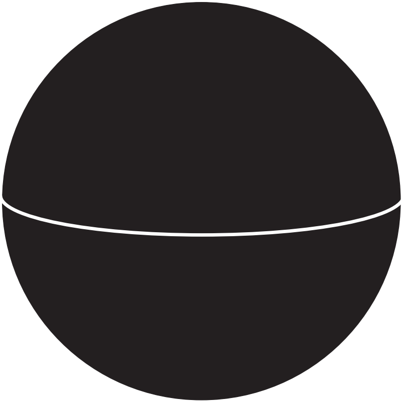
i. preliminaries

a **spacetime** is a pair (M, g).

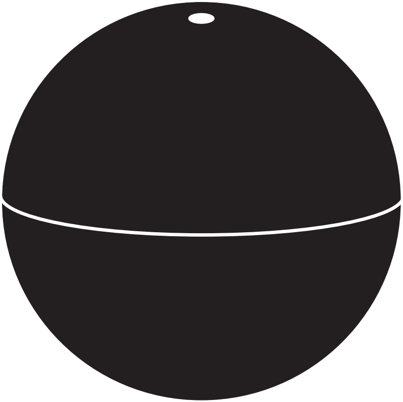
M is the spacetime **manifold**.



the plane

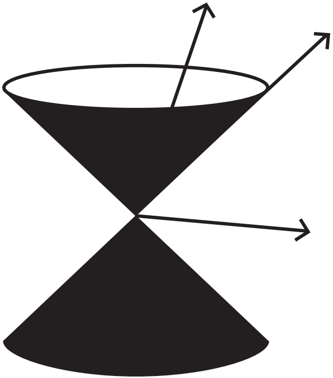


the sphere

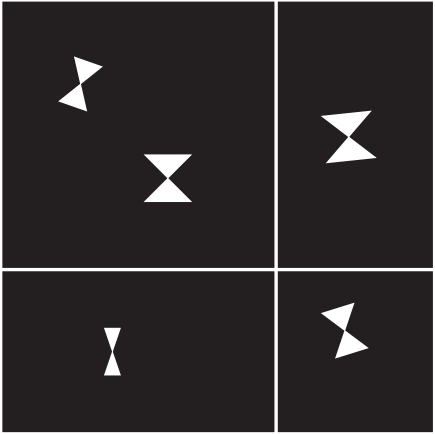


the sphere with point removed

g is the spacetime **metric**.



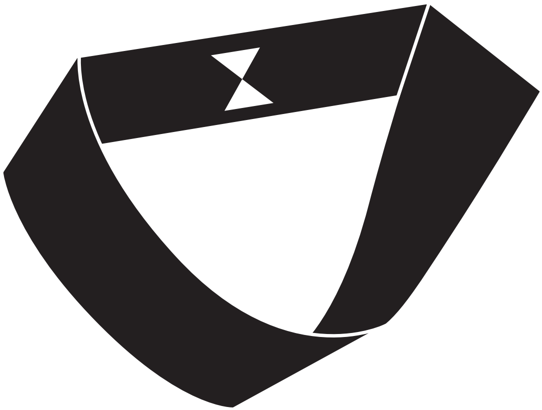
timelike, null, and spacelike vectors



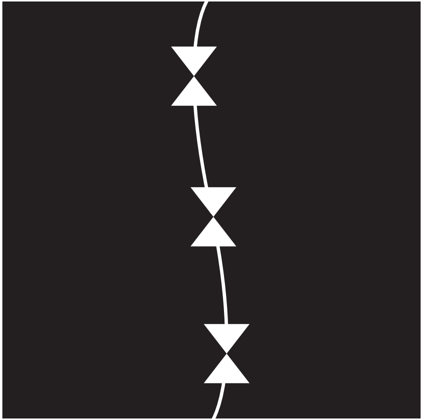
a spacetime

a spacetime is **flat** if its metric structure does not vary over the manifold.

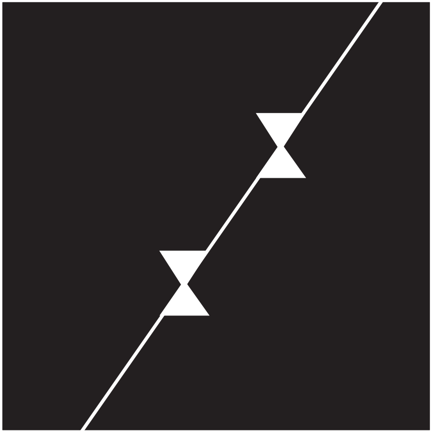
a spacetime is **time-orientable** if, ranging over the entire manifold, we can label the lobes of each light-cone as “past” and “future” in a way that involves no discontinuities.



a spacetime which is not time-orientable a curve is **timelike** if all of its tangent vectors are timelike.



a timelike curve

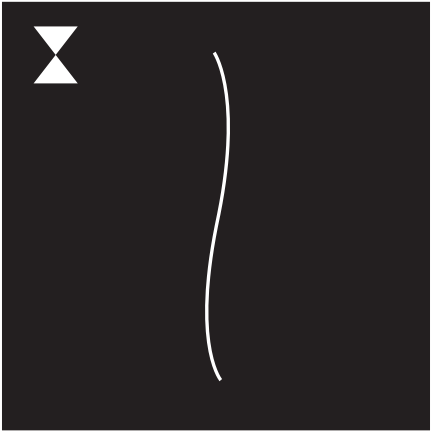


a null curve

a curve is **causal** if it is timelike or null.

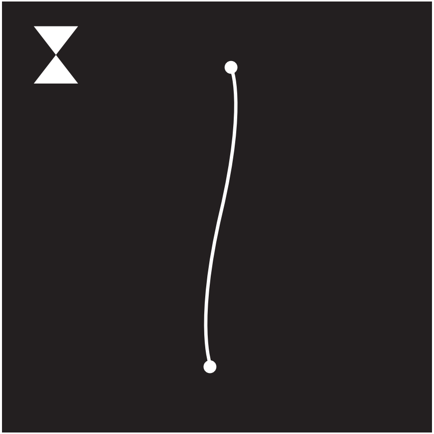
a causal curve is **future-directed** if all of its tangent vectors are in or on the future lobe of the light cone.

a curve is not **maximal** if it can be smoothly extended into a larger curve.

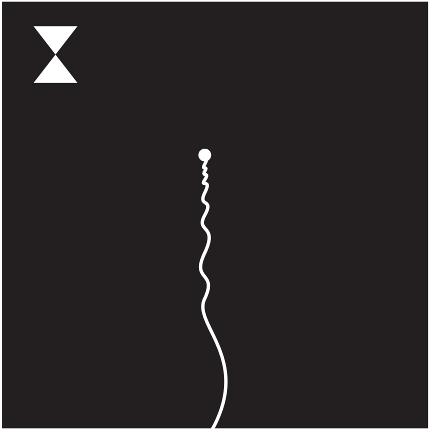


a timelike curve which is not maximal

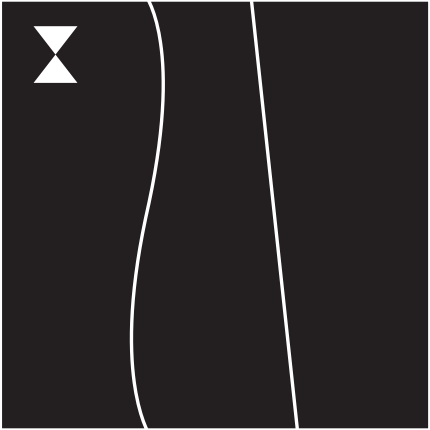
a causal curve may have **past/future endpoints**.



a timelike curve with future and past endpoints



a maximal timelike curve with future endpoint a non-accelerated curve is a **geodesic**.

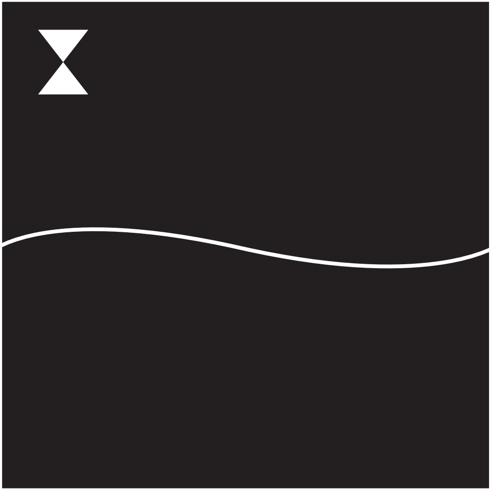


accelerated and geodesic timelike curves a surface is **spacelike** if every curve within the surface is a spacelike curve.

a set is **achronal** if it is not intersected more than once by any timelike curve.

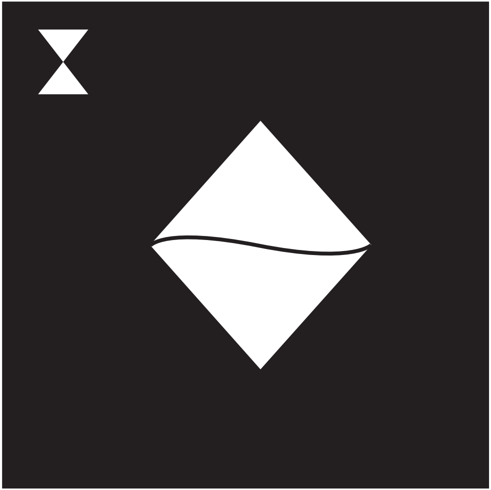
an achronal spacelike surface

a surface is a **slice** if it is spacelike, achronal, closed, and without an ‘edge’.



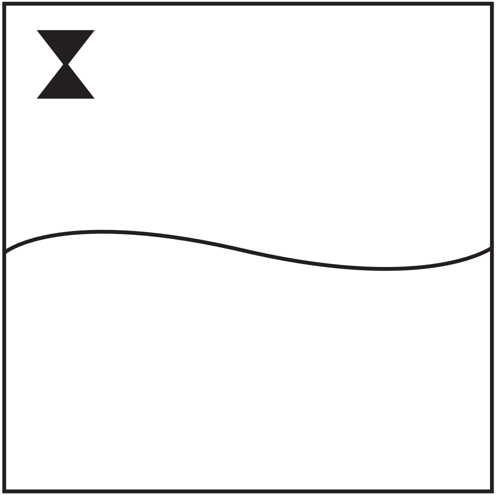
a slice

the **domain of dependence** of S, written $D(S)$, is the set consisting of those points q such that every causal curve without endpoint through q intersects S.

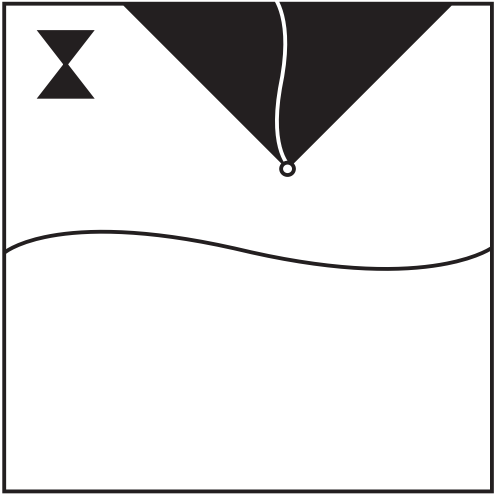


domain of dependence

a spacetime (M, g) is **globally hyperbolic** if it has a slice S such that $D(S) = M$.

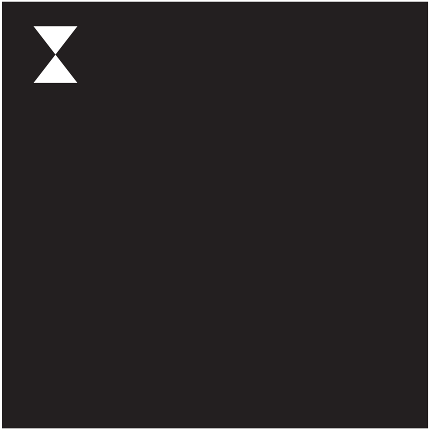


globally hyperbolic spacetime

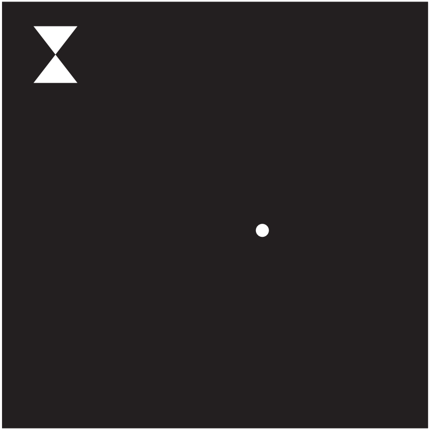


non-globally hyperbolic spacetime

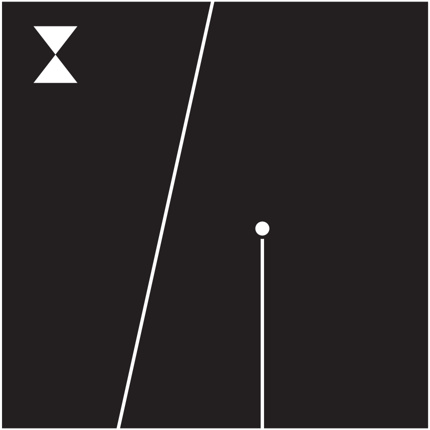
ii. singularities, holes, and extensions a maximal geodesic is **complete** if the parameter time goes from negative infinity to positive infinity.



minkowski spacetime



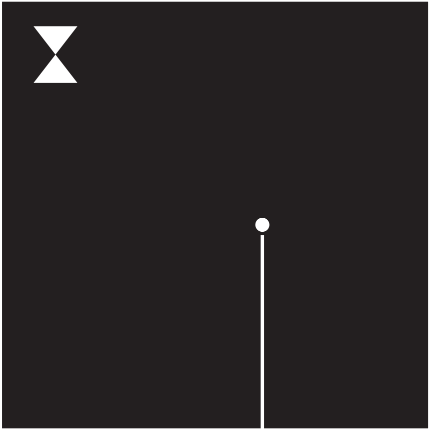
minkowski spacetime with a point removed



complete and incomplete geodesics

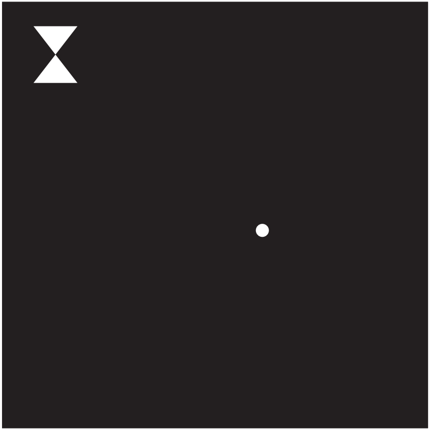
a timelike incomplete geodesic represents a freely falling observer who does not record all possible watch readings.

a causal geodesic without future endpoint is **future-incomplete** if the parameter time does not go to positive infinity.

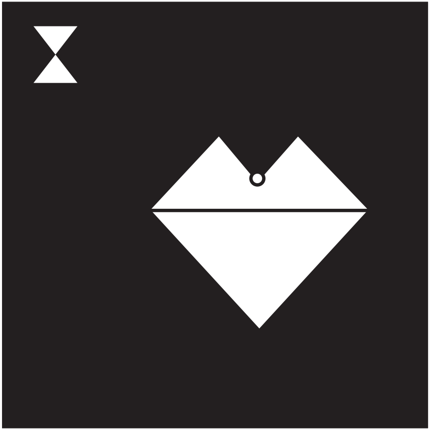


a past-complete but future-incomplete geodesic a spacetime is **geodesically complete** (gc) if it does not contain an incomplete geodesic.

(gc) is a strong condition; it seems that all (or at least some) physically reasonable spacetimes are geodesically incomplete.



minkowski spacetime with a point removed

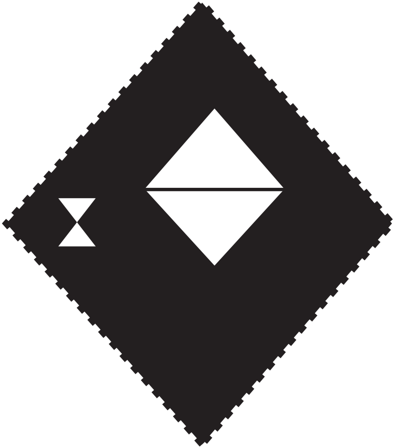


the domain of dependence of a spacelike surface a spacetime is **hole-free** (hf) if, for every spacelike surface S and every metric preserving embedding of its domain of dependence into some other spacetime, the domain of dependance of the image of S is identical to the image of the domain of dependence of S.

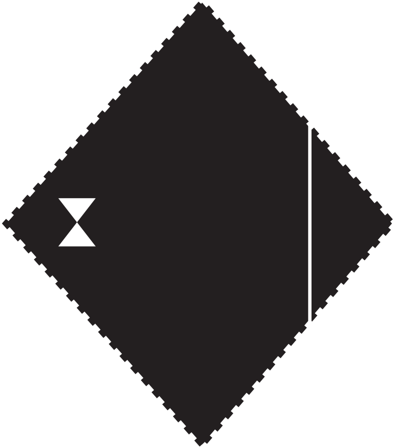
(hf) requires that the domain of dependence of every spacelike surface is "as large as it could have been".

what is the relationship between (gc) and (hf)?

$$(hf) \;\not\Longrightarrow\; (gc)$$



a spacetime which is hole-free...

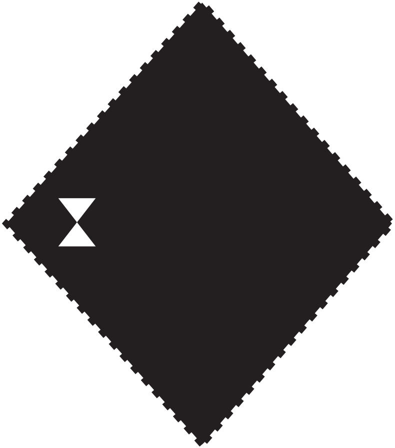


...but geodesically incomplete

$$(\mathrm{gc}) \overset{?}{\Longrightarrow} (\mathrm{hf})$$

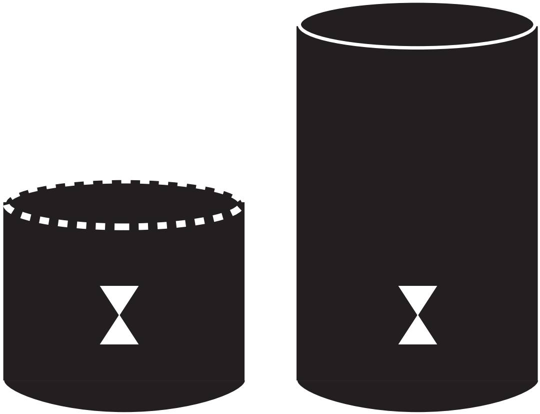
$$(\mathrm{gc}) \Longrightarrow \; ? \; \Longrightarrow (\mathrm{hf})$$

iv. singularities and extensions



a hole-free spacetime

a spacetime is **inextendible** (i) if it cannot be properly embedded, while preserving all metric structure, into another spacetime.

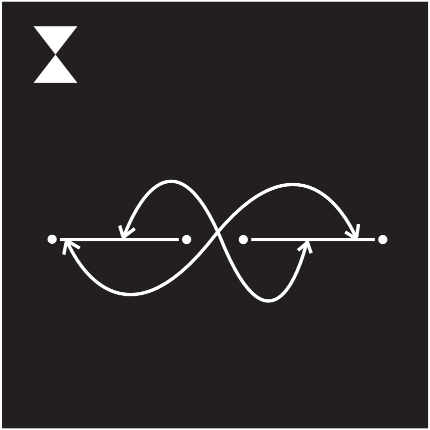


a spacetime and one of its extensions

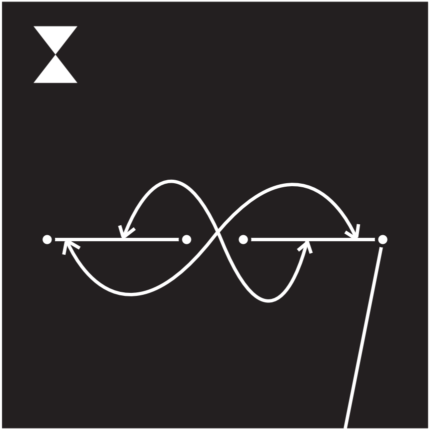
what is the relationship between (i) and (gc)?

$$(gc) \Longrightarrow (i)$$

$$\text{(i)} \quad \not\Longrightarrow \quad \text{(gc)}$$



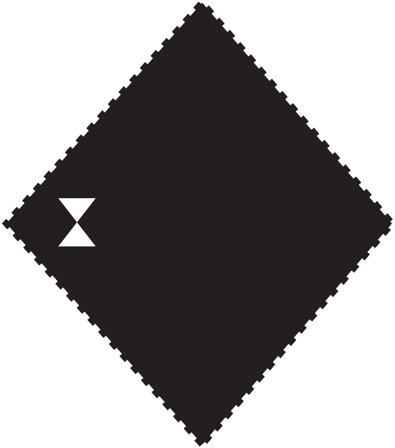
a spacetime which is inextendible...



...but geodesically incomplete

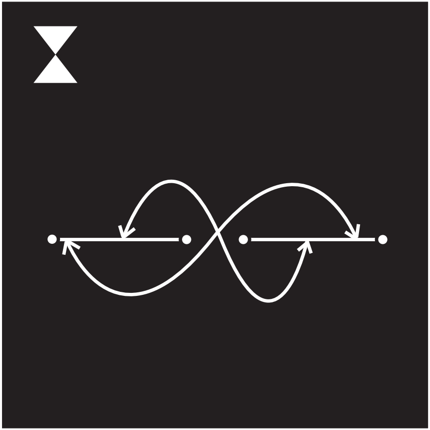
$$\text{(gc)} \Longrightarrow \; ? \; \Longrightarrow \text{(i)}$$

what is the relationship between (hf) and (i)?



a spacetime which is hole-free but extendible

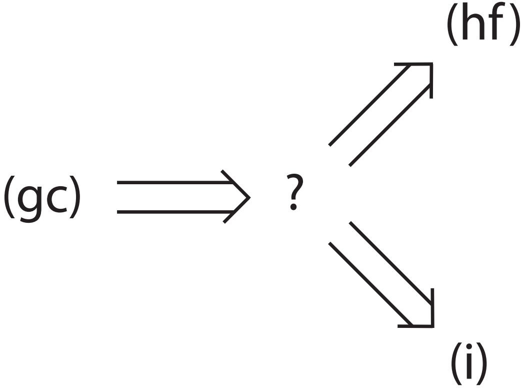
$$\text{(hf)} \not\Longrightarrow \text{(i)}$$



a spacetime which is inextendible but not hole-free

$$\text{(i)} \quad \not\Longrightarrow \quad \text{(hf)}$$

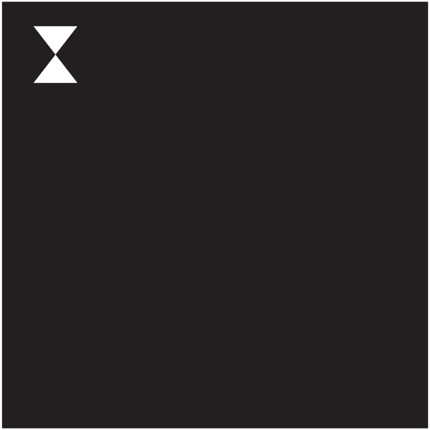
(hf) and (i) rule out two different types of artificial singularities.


(gc)
?
(hf)
(i)

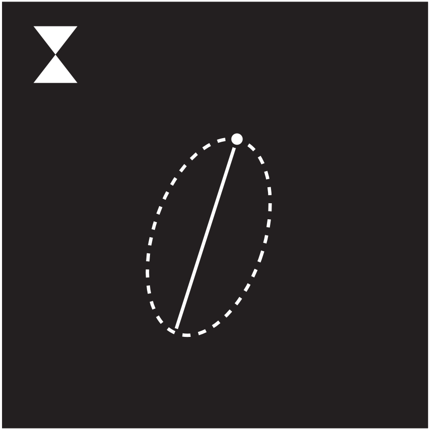
iii. effective completeness

a spacetime is **effectively complete** (ec) if, for every future or past incomplete timelike geodesic, and every open set containing it, there is no metric preserving embedding of the set into some other spacetime such that the image of the curve has future and past endpoints.

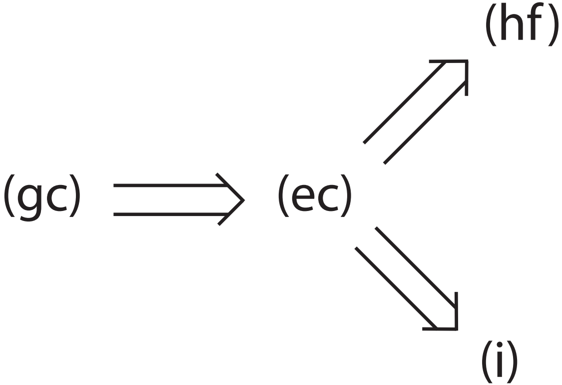
if a spacetime fails to be effectively complete, then there is a freely falling observer who never records some particular watch reading but who “could have” in the sense that nothing in the vicinity precludes it.

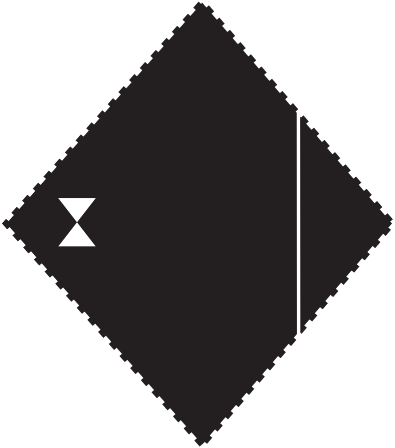


a spacetime which is effectively complete

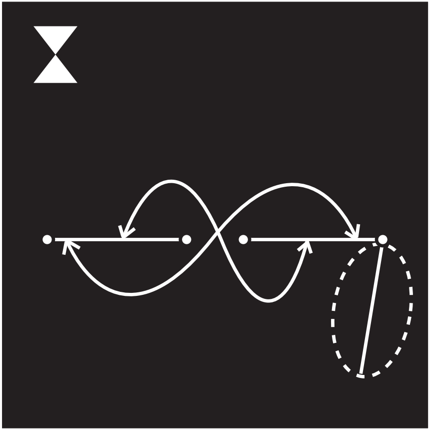


a spacetime which is effectively incomplete

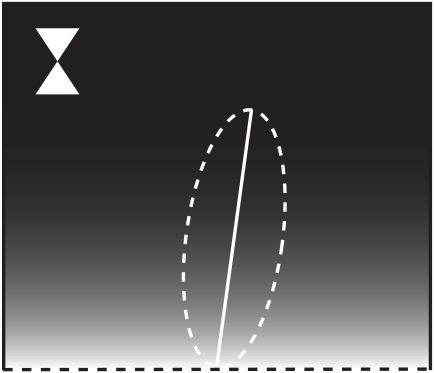

(gc)
(ec)
(hf)
(i)



a spacetime which is hole-free but effectively incomplete



a spacetime which is inextendible but effectively incomplete



a spacetime which is effectively complete but not geodesically complete

iv. instability of effective completeness an $\mathscr{F}$ **neighborhood** of a spacetime (M, g_{ab}) is any set of spacetimes containing the set of all (M, g'_{ab}) such that $\max[h^{am}h^{bn}(g_{ab} - g'_{ab})(g_{mn} - g'_{mn})] < \epsilon$ where h^{ab} is a positive definite metric on M and ϵ is a positive number.

the $\mathscr{F}$ topology is quite fine.

if (M, g_{ab}) is a spacetime and M is noncompact, then the one-parameter family of spacetimes $\{(M, \lambda g_{ab}) : \lambda \in (0, \infty)\}$ does not represent a continuous curve in the $\mathscr{F}$ topology (Geroch 1971).

it seems that the $\mathscr{F}$ topology is too fine to adequately capture, once and for all, what it means to say that one spacetime is “close” to another (Geroch 1971, Fletcher 2016).

but the fact that there are too many open sets in the $\mathscr{F}$ topology makes this topology ideal for proving instability results.

a spacetime property $\mathcal{P}$ is $\mathscr{F}$ **stable** if, for each spacetime (M, g_{ab}) with $\mathcal{P}$, there is a $\mathscr{F}$ neighborhood of (M, g_{ab}) such that every spacetime in the neighborhood also has $\mathcal{P}$.

a spacetime property $\mathcal{P}$ is $\mathscr{F}$ **unstable** if it is not $\mathscr{F}$ stable.

note: if a spacetime property fails to be $\mathscr{F}$ stable, then it will fail to be stable relative to any other topology coarser than $\mathscr{F}$.

which spacetime properties are $\mathscr{F}$ unstable?

proposition (geroch 1970). global hyperbolicity is $\mathscr{F}$ stable.

proposition (williams 1984). geodesic completeness and geodesic inclompletenss are $\mathscr{F}$ unstable. (see: beem and ehrlich 1996.)

given the singularity theorems, the $\mathscr{F}$ instability of geodesic completeness is not too troubling.

on the other hand, the $\mathscr{F}$ instability of geodesic incompleteness has been taken to be quite significant.

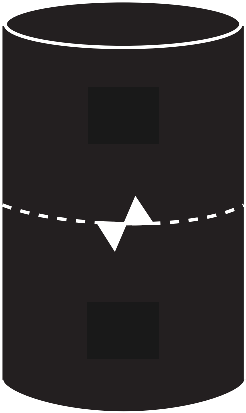
indeed, a great deal of work has been done to show that, if attention is appropriately limited to certain types of nice (i.e. Robertson Walker) spacetimes, the $\mathscr{F}$ stability of geodesic incompleteness can be saved (Beem and Ehrlich 1996).

what about the $\mathscr{F}$ (in)stability of effective completeness, hole-freeness, and inextendibility?

the questions with respect to hole-freeness, and inextendibility are still open. (there is work to do!)

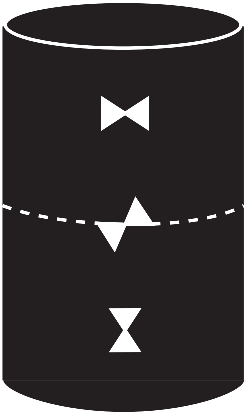
proposition (manchak 2017). effective completeness is $\mathscr{F}$ unstable.

sketch of proof:

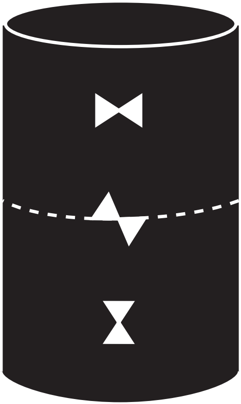


minkowski spacetime “rolled up” along one null direction the spacetime is geodesically complete and therefore effectively complete.

but in every $\mathscr{F}$ open set around the spacetime, there is a spacetime isometric to a Misner-like spacetime.

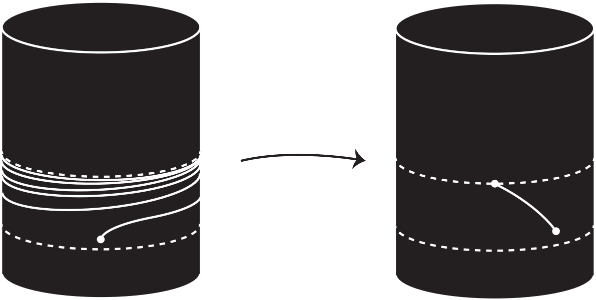


misner spacetime



reverse twisted misner spacetime

misner and reverse twisted misner are not isometric; but the "bottom halves" are isometric.



reverse twist isometry of an open set around a timelike geodesic. in misner, the geodesic has no future endpoint; in reverse twist misner, the image of the geodesic has a future endpoint.

so: miser spacetime is not effectively complete.

upshot (with details filled in): effective completeness is not $\mathscr{F}$ stable.

vii. conclusion

work do to: are there “physically reasonable” properties $\mathcal{P}$ (global hyperbolicity? Robertson Walker?) such that (P & effective completeness) is $\mathscr{F}$ stable?

thank you.