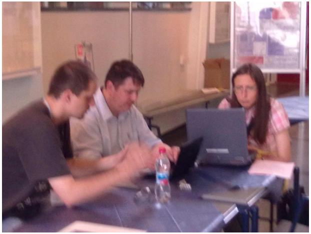
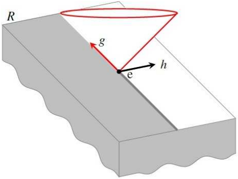
A mathematical logic based approach to isotropy, homogeneity and special principle of relativity

Judit Madarász

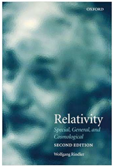
joint research with G. Székely, M. Stannett



Class. Quantum Grav. **31** (2014) 095005

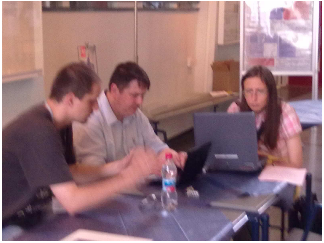


SPR $\not\Longrightarrow$ Isotropy



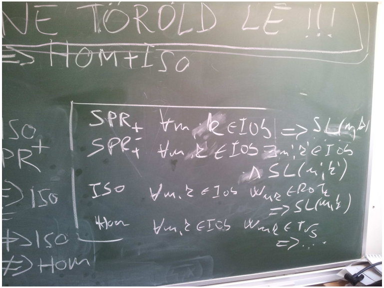
SPR $\Longleftrightarrow$ Isotropy & Homogeneity

SPR $\Longrightarrow$ Isotropy




SPR+


NE TÖRÖLD
CONZ:
SPR+ => HOM+ISO
AxField ⊨
AxTriv
1) "Rindler" ⊨ Hom+Iso
=> SPR+
2) SR ⊨ Hom ⇏ Iso
3) SR ⊨ SPR+ ⇏ Iso
⇏ Hom
Iso
Hom


NE TÖRÖLD LE !!!
=> HOM+ISO
Iso
Hom

$\text{SPR} \not\Longrightarrow \text{Isotropy}$ ⚡ $\text{SPR} \Longrightarrow \text{Isotropy}$

FrameSPR "Inertial frames cannot be distinguished by experiments"

CoordSPR "Inertial coordinate systems cannot be distinguished by experiments"

$\text{FrameSPR} \not\Longrightarrow \text{Isotropy}$ ☺ $\text{CoordSPR} \Longrightarrow \text{Isotropy}$

SPR $\not\Longrightarrow$ Isotropy ⚡ SPR $\Longrightarrow$ Isotropy

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We reconstructed the proof of Dixon and Rindler in FOL:

Isotropy & Homogeneity $\Longleftrightarrow$ CoordSPR

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Isotropy & Homogeneity $\Longleftrightarrow$ CoordSPR

assuming that the world-view transformations between coordinate systems that rest wrt each other and have the same origin are spatial rotations.

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We reconstructed the proof of Dixon and Rindler in FOL:

Isotropy & Homogeneity $\Longleftrightarrow$ CoordSPR

assuming that the world-view transformations between coordinate systems that rest wrt each other and have the same origin are spatial rotations.

Without this assumption:

Isotropy & Homogeneity $\not\Longrightarrow$ FrameSPR

Our results are general:

Our results apply to any language $\mathcal{L}$ that contains our FOL language for kinematics.

E.g. $\mathcal{L}$ can talk about special bodies like photons or electron, masses or energies of bodies, electric field, magnetic field etc.

Set $\mathcal{S}$ of experimental scenarios (set of formulas "describing experiments"):

$\mathcal{S} \subseteq$ "Defined subset of Formulas of $\mathcal{L}$"

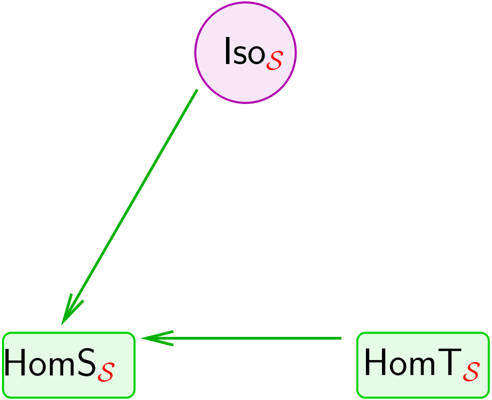
$\mathsf{FrameSPR}_{\mathcal{S}}$ "Inertial frames cannot be distinguished by an experiment in $\mathcal{S}$"

$\mathsf{CoordSPR}_{\mathcal{S}}$, $\mathsf{Iso}_{\mathcal{S}}$ - isotropy, $\mathsf{Hom}_{\mathcal{S}}$ - homogeneity

Our results do not depend on the choice of $\mathcal{S}$.

$\mathsf{Iso}_{\mathcal{S}} \& \mathsf{Hom}_{\mathcal{S}} \implies \mathsf{CoordSPR}_{\mathcal{S}}$ for any $\mathcal{S}$ and $\mathcal{L}$.

Connections between $\mathrm{HomT}_{\mathcal{S}}$-homogeneity of time, $\mathrm{HomS}_{\mathcal{S}}$-homogeneity of space and $\mathrm{Iso}_{\mathcal{S}}$-isotropy



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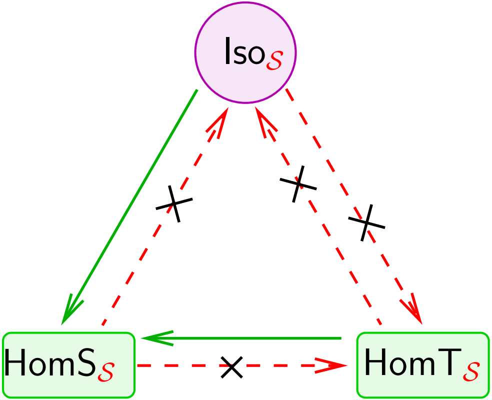

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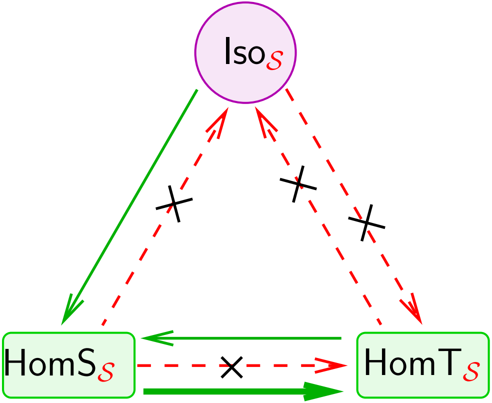
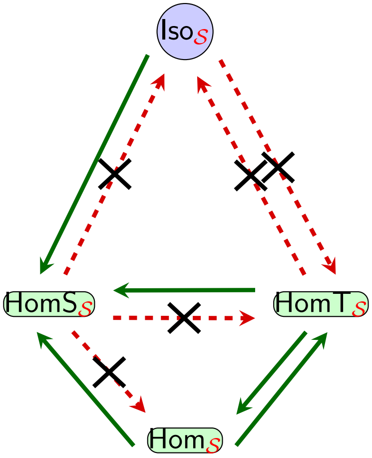
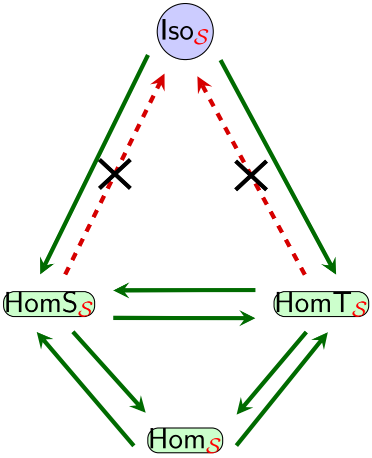


$\exists$ a clock that gets out of synchronism



Classical case



Relativistic case

$$\mathcal{L}_{\text{core}} = \{\text{IOb}, \text{B}, \text{Q}, \qquad\qquad \}$$

IOb $\leftrightsquigarrow$ Inertial Observers (coordinate systems).

B $\leftrightsquigarrow$ Bodies (particles that move)

Q $\leftrightsquigarrow$ Quantities (numbers)

$$\mathcal{L}_{\mathsf{core}} = \{\mathsf{IOb}, \mathsf{B}, \mathsf{Q}, +, \cdot, \leq, \quad \}$$

IOb $\leftrightsquigarrow$ Inertial Observers (coordinate systems).

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$+$, $\cdot$ and $\leq$ $\leftrightsquigarrow$ field operations and ordering

$$\mathcal{L}_{\text{core}} = \{\mathsf{IOb}, \mathsf{B}, \mathsf{Q}, +, \cdot, \leq, \mathsf{W}\}$$

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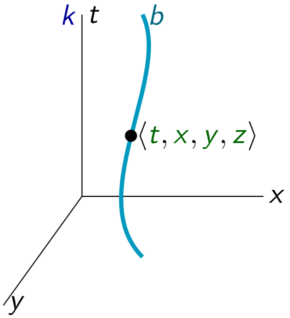
$+$, $\cdot$ and $\leq$ $\leftrightsquigarrow$ field operations and ordering

W $\leftrightsquigarrow$ Worldview (a 6-ary relation of type IObBQQQQ)

$W(k, b, t, x, y, z) \rightsquigarrow$ "observer k coordinatizes body b at spacetime location (coordinate point) $\langle t, x, y, z \rangle$."

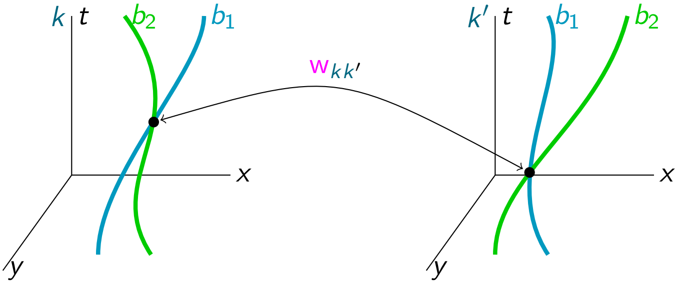


Worldline of body b according to observer k

$$wline_k(b) = \{\langle t, x, y, z\rangle \in Q^4 \ : \ W(k, b, t, x, y, z)\}$$

The **worldview transformation** $\mathsf{w}_{kk'}$ between observers k and k'

$$\mathsf{w}_{kk'}(t,x,y,z:t',x',y',z') \overset{def}{\Longleftrightarrow}$$
$$\forall b\ [\mathsf{W}(k,b,t,x,y,z) \iff \mathsf{W}(k',b,t',x',y',z')].$$



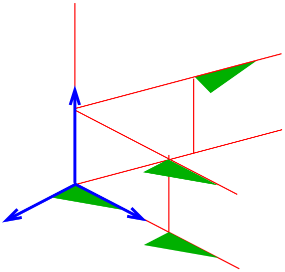
Our results apply to any language $\mathcal{L}$ for which $\mathcal{L}_{\text{core}} \subseteq \mathcal{L}$.

E.g. $\mathcal{L}$ can talk about special bodies like photons or electron, masses or energies of bodies, electric field, magnetic field etc.

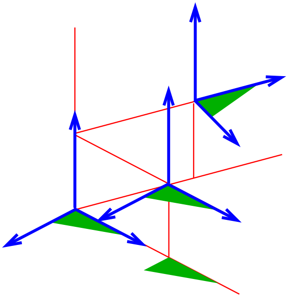
Dixon , 1978 relativity book:

". . . Principle of Uniformity . . . is a formalization of the hypothesis . . . that space and time appear isotropic and homogeneous when viewed in an inertial reference frame."

" Its physical interpretation is that a given experiment will produce the same result wherever and whenever is performed, and whatever the orientation of the apparatus, provided that the circumstances of the experiment are identical in all other respects."



“Two replications of an experiment can be considered as having the same circumstances if there exists two natural coordinate systems for the same inertial reference frame, one of which may be associated with each experiment in such a way that the initial conditions of the two are identical when each is referred to its associated coordinate system. Since natural coordinates are determined up to translation and rotation of the axes, this gives the required homogeneity and isotropy.”



Axioms:

AxEField $(Q, +, \cdot, \leq)$ is an Euclidean ordered field.

Spatial rotations *SRot* of Q^4.

Translations *Tran*, Spatial Translations *STran*, Temporal Translations *TTran* of Q^4

AxTriv Rotations and translations of inertial coordinate systems are also inertial coordinate systems
$\forall k\ \forall T \in SRot \cup Tran\ \exists k'\ w_{kk'} = T.$

AxAftr World-view transformations are affine transformations.

What are experiments? We don't know.

Gergely: Set $\mathcal{S}$ of experimental scenarios:
$\mathcal{S} \subseteq$ "Formulas of $\mathcal{L}$"

$\phi \in \mathcal{S} \implies \phi$ has only one free variable of sort IOb and all the other free variables are of sort Q.

Scenario $\phi(k, \bar{x}) = \phi(k, x_1, \ldots, x_n)$ is a description of an experiment,
where free variables $x_1, \ldots, x_n$ of sort Q are the experimental parameters containing the configuration, progress and outcome of the experiment, and free variable k of sort IOb . . . .

$\psi(k, x_1, x_2, x_3, x_4, x_5) =$

$(\forall p \in Photons)[(x_1, x_2, x_3, x_4) \in wline_k(p) \;\Rightarrow\; speed_k(p) = x_5].$

$\mathfrak{M} \models \psi(h, 0, 0, 0, 0, 1),$

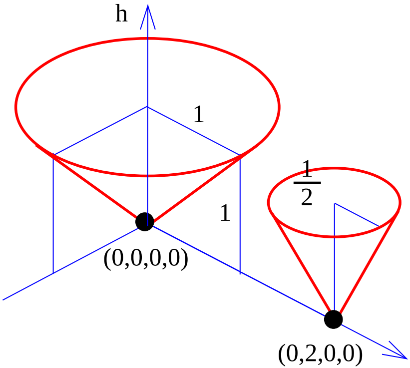
ψ is realizable

$\mathfrak{M} \models \psi(h, 0, 2, 0, 0, \frac{1}{2})$

$\mathfrak{M} \not\models \psi(h, 0, 2, 0, 0, 1)$

ψ is not realizable.

$\mathfrak{M} \not\models \psi(h, 0, 0, 0, 0, 2)$

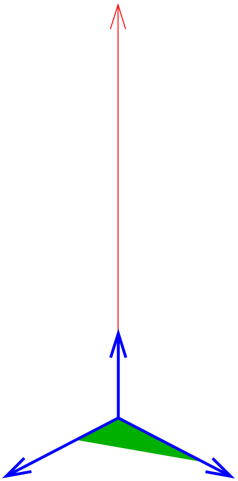


$\text{Agree}\langle k, k', \phi \rangle$: Inertial observers k and k' agree on the realizability of experimental scenario ϕ

$$\text{Agree}\langle k, k', \phi \rangle \quad \forall \bar{x}\ [\phi(k, \bar{x}) \Longleftrightarrow \phi(k', \bar{x})]$$

HomT$_{\mathcal{S}}$ (homogeneity of time)

Temporal translations do not effect the outcomes of experiments.



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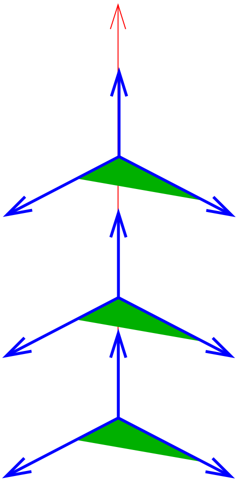

$\text{HomT}_{\mathcal{S}}$ (homogeneity of time)

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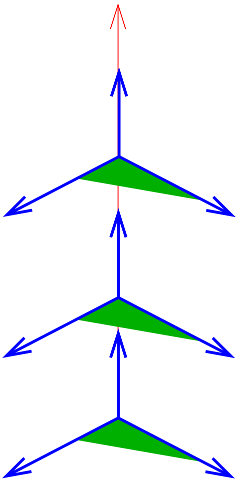
$\text{HomT}_{\mathcal{S}} =$

$\{ \mathsf{w}_{kk'} \in TTran \Rightarrow \text{Agree}\langle k, k', \phi \rangle \ : \ \phi \in \mathcal{S} \}$



$\mathrm{HomS}_{\mathcal{S}}$ (homogeneity of space)

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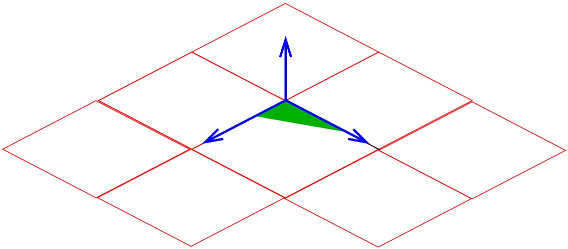

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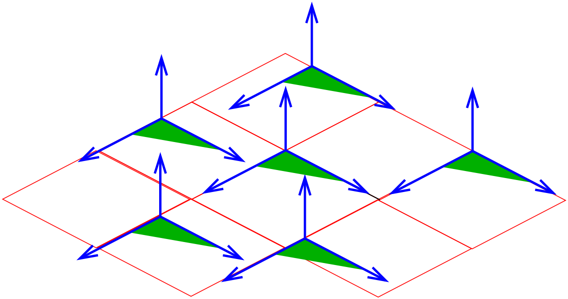

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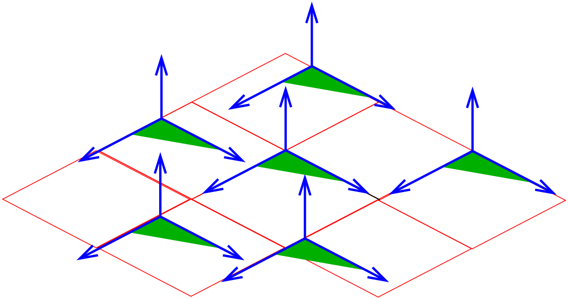

$$\mathrm{HomS}_{\mathcal{S}} = \{\mathsf{w}_{kk'} \in STran \Rightarrow \mathrm{Agree}\langle k, k', \phi\rangle \ : \ \phi \in \mathcal{S}\}.$$

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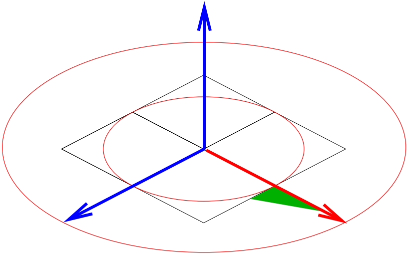

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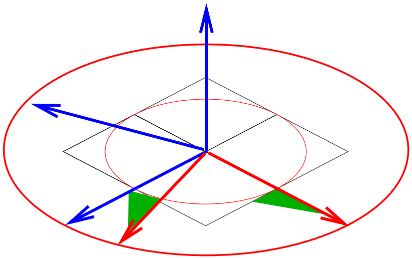

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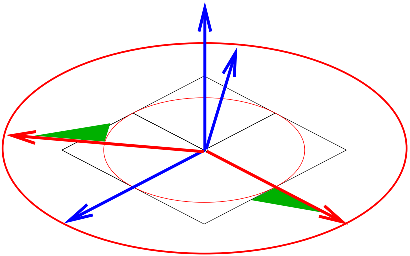

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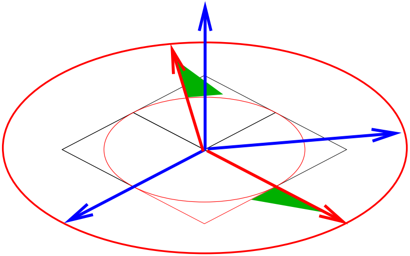

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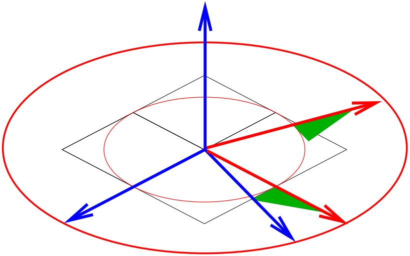

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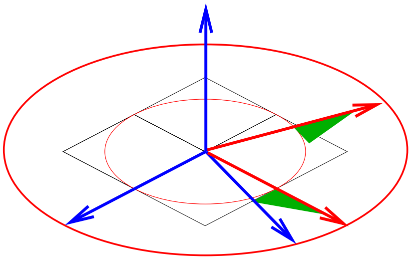
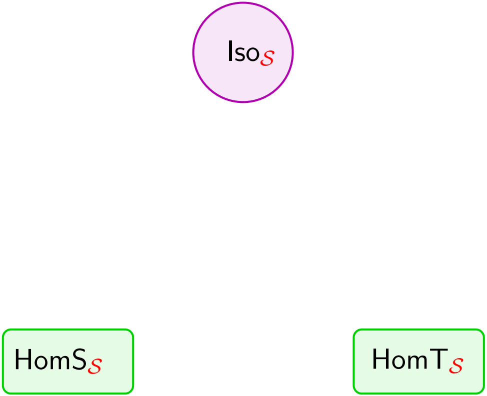


$$\mathsf{Iso}_{\mathcal{S}} = \{ \mathsf{w}_{kk'} \in SRot \Rightarrow \mathsf{Agree}\langle k, k', \phi \rangle \; : \; \phi \in \mathcal{S} \}.$$

AxEField, AxTriv, AxAftr



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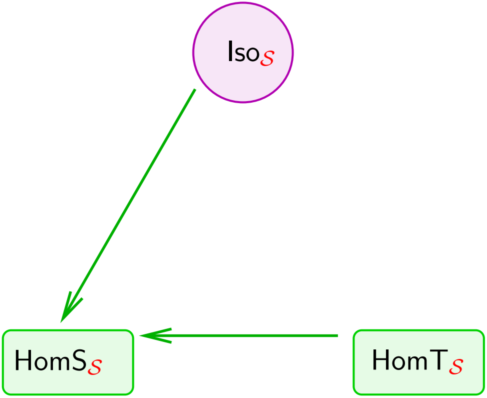

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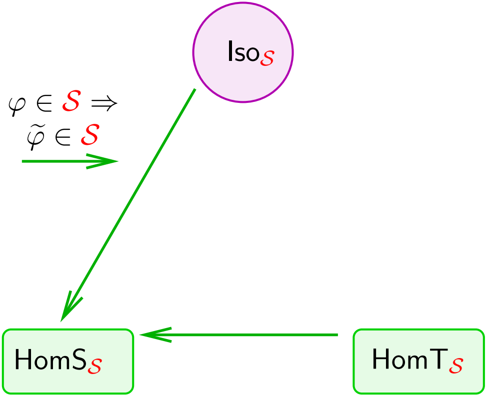
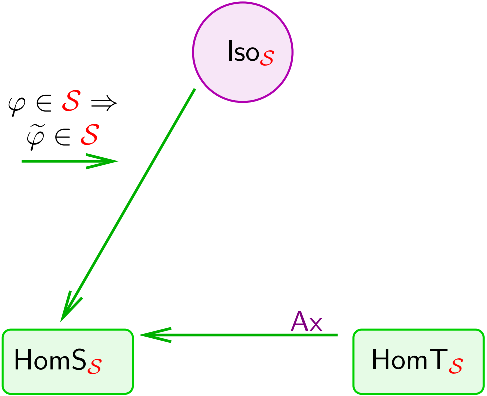


$\widetilde{\varphi}(k, \bar{x}, \bar{X}) =$
$\exists k'[\mathsf{w}_{k'k}$ is determined by $\bar{X} \wedge$
$\varphi(k', \bar{x})]$.

AxEField, AxTriv, AxAftr



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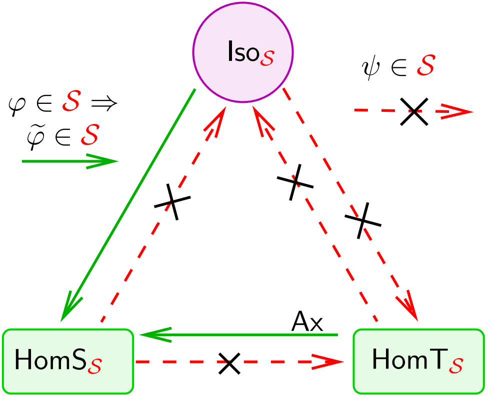
Ax Observers can move in any spatial direction

AxEField, AxTriv, AxAftr



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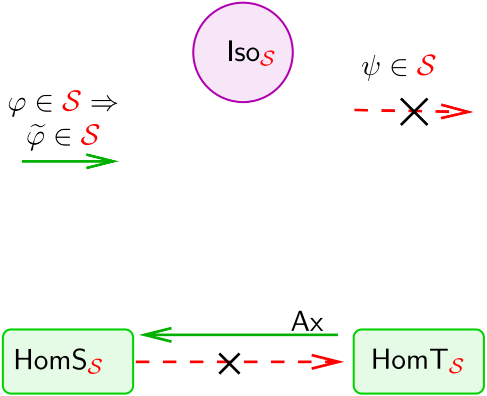
Ax Observers can move in any spatial direction

$\psi(o, \bar{x}, \bar{y}) =$
$\exists b\{\bar{x} + \lambda \cdot \bar{y}\} \subseteq wline_o(b)$.

AxEField, AxTriv, AxAftr



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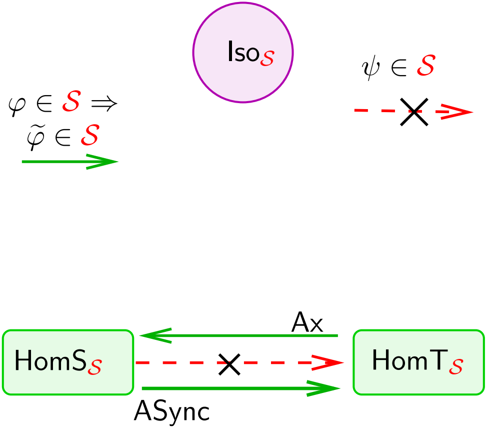

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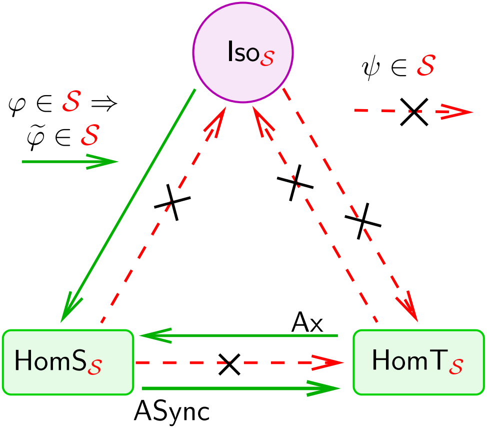
ASync There is a clock that gets out of synchronism

AxEField, AxTriv, AxAftr



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ASync There is a clock that gets out of synchronism

$\mathsf{CoordSPR}_{\mathcal{S}}$

All inertial observers agree on satisfiability of experimental scenarios.

$$\mathsf{CoordSPR}_{\mathcal{S}} = \{\mathsf{Agree}\langle k, k', \phi\rangle \ : \ \phi \in \mathcal{S}\}$$

Theorem:

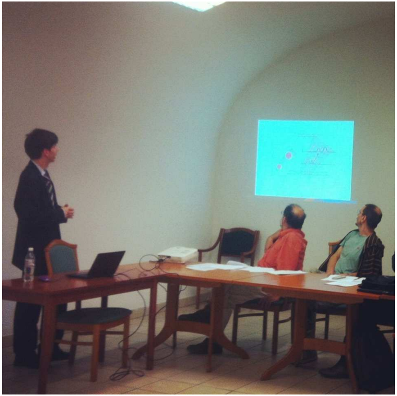
$\mathsf{Hom}_{\mathcal{S}}$ & $\mathsf{Iso}_{\mathcal{S}} \iff \mathsf{CoordSPR}_{\mathcal{S}}$
assuming AxEField, AxTriv, AxAftr, $\varphi \in \mathcal{S} \Rightarrow \widetilde{\varphi} \in \mathcal{S}$ and an axiom saying that world-view transformations between inertial observers that rest wrt each other and have the same origin are spatial rotations.

What are the connections between our formalizations of SPR, Homogeneity, Isotropy with "real SPR", "real Homogeneity", "real Isotropy"?

What are the connections between our set $\mathcal{S}$ of experimental scenarios with "real experiments"?

What are the connections between other formalizations of SPR, HOM?

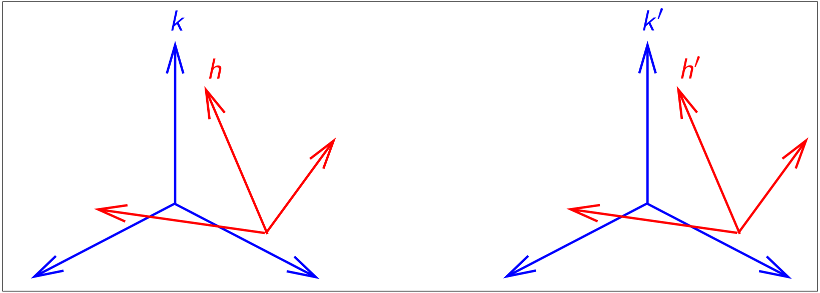
Connections with other approaches:



Lemma:

Assume $\varphi \in \mathcal{S} \Rightarrow \widetilde{\varphi} \in \mathcal{S}$. Then

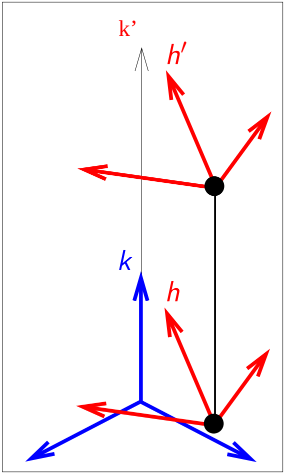
$$[\mathsf{w}_{kh} = \mathsf{w}_{k'h'} \wedge Agree(k, k', \mathcal{S})] \implies Agree(h, h', \mathcal{S}).$$



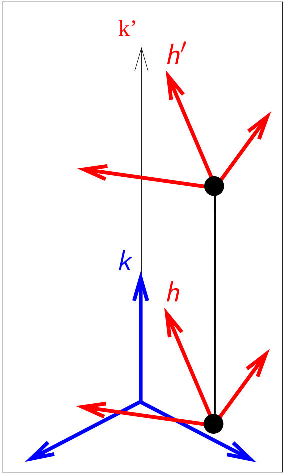
Lemma:

Assume $\mathrm{HomT}_{\mathcal{S}}$, $\varphi \in \mathcal{S} \Rightarrow \widetilde{\varphi} \in \mathcal{S}$. Let k, h, h' be such that $\mathsf{w}_{hh'}$ is a translation and the origins of h and h' are time-like separated in the coordinate system k. Then

$$Agree(h, h', \mathcal{S})$$



Proof:



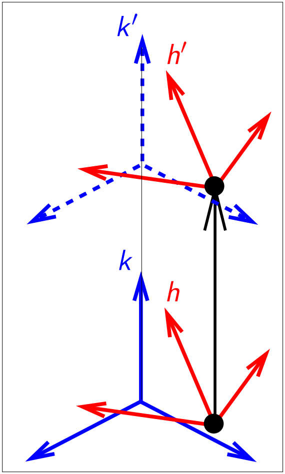
Proof:

Let k' be such that $w_{k'k}$ is a translation by the black arrow.

$w_{kh} = w_{k'h'}$,

$\mathrm{HomT}_{\mathcal{S}} \Longrightarrow Agree(k, k', \mathcal{S})$

$Agree(h, h', \mathcal{S})$ by the previous lemma.



Lemma:

Assume $\mathrm{HomS}_{\mathcal{S}}$, $\varphi \in \mathcal{S} \Rightarrow \widetilde{\varphi} \in \mathcal{S}$. Let k, h, h' be such that $\mathsf{w}_{hh'}$ is a translation and the origins of h and h' are space-like separated in the coordinate system k. Then

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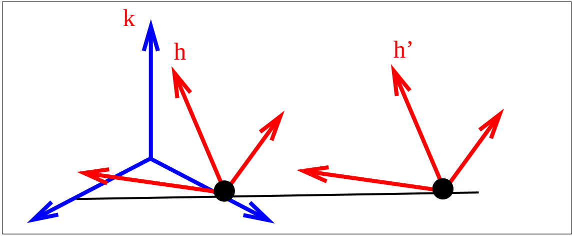

Theorem:

$\mathrm{HomT}_{\mathcal{S}} \Longrightarrow \mathrm{HomS}_{\mathcal{S}}$ if observers can move in any spatial direction and $\varphi \in \mathcal{S} \Rightarrow \widetilde{\varphi} \in \mathcal{S}$

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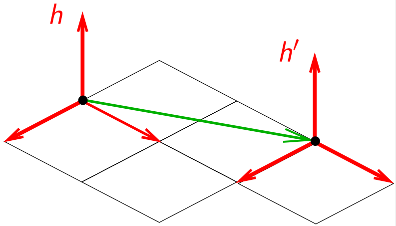

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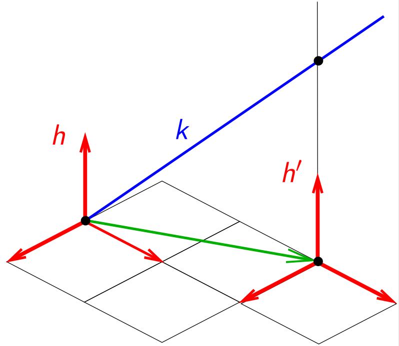

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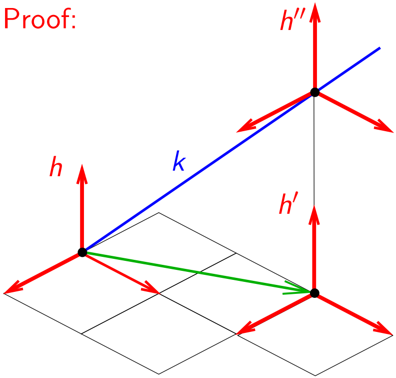

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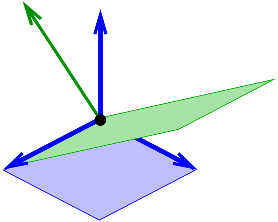

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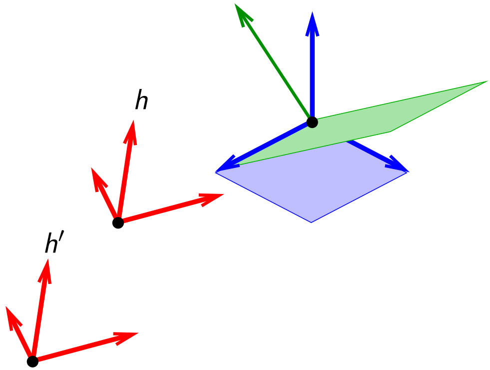

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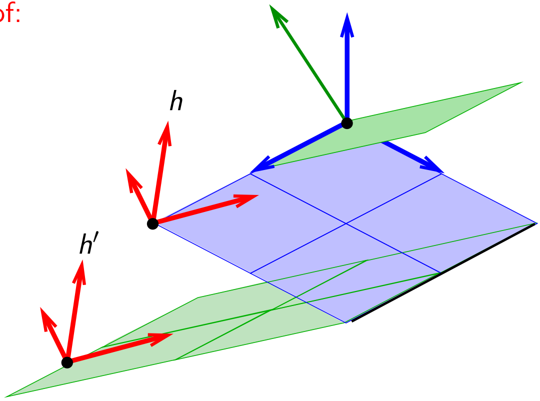

Theorem:

HomS$_{\mathcal{S}}$ $\Longrightarrow$ HomT$_{\mathcal{S}}$ if there is a clock that gets out of synchronism and $\varphi \in \mathcal{S} \Rightarrow \widetilde{\varphi} \in \mathcal{S}$