Are non-Hausdorff space-times physically reasonable?

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Physical reasonability

Physical reasonability conditions may concern various levels of a theory:

1. accommodated as a part of a definition of a manifold (separability conditions, countability conditions, connectedness conditions, compactness conditions, dimension)
2. constraints on energy-momentum tensor (conservation law and various energy conditions)
3. constraints on metric tensor (its signature)
4. constraints on global structure of spacetime (causality conditions, lack of some types of singularities, lack of 'holes')
5. 'cosmological' conditions: isotropy and homogeneity (perhaps approximate, on appropriate scales), further constraints on energy-momentum tensor, asymptotic behaviour of spacetime
6. inextendibility of spacetime

Physical reasonability

Definition (Topological manifold)

An n-dimensional topological manifold is a topological space X that is locally Euclidean of dimension n, that is, every point in X has a neighbourhood in X that is homeomorphic to an open subset of $\mathbb{R}^n$.

Differential manifold — a topological manifold which has an additional differential structure.

All objects and equations of GR are defined on a differential manifold.

Physical reasonability

Additional conditions — examples from GR textbooks:

- Wald (1984) — Hausdorff and paracompact
- Hawking and Ellis (1973) — connected, 4-dimensional, Hausdorff, Lorentzian metric (the last two imply paracompactness)
- Malament (2012) — Hausdorff

Physical reasonability

Separation axioms:

- T_0: Whenever x and y are distinct points in X, there is an open set containing one and not the other.
- T_1: Whenever x and y are distinct points in X, there is a neighbourhood of each not containing the other.
- T_2 (**Hausdorff condition**): Whenever x and y are distinct points in X, there are disjoint open sets U and V in X with $x \in U$ and $y \in V$.
- Regularity: Whenever A is closed in X and $x \notin A$, there are disjoint open sets U and V in X with $x \in U$, $A \subset V$.
- $T_3 =$ regularity $+ T_1$
- Normality: Whenever A and B are disjoint closed sets in X, there are disjoint open sets U and V in X with $A \subset U$, $B \subset V$.
- $T_4 =$ normality $+ T_1$

Physical reasonability

Separation axioms — properties and relations:

- T_1 follows from local Euclidicity.
- $T_4 \Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1 \Rightarrow T_0$
- Neither regularity nor normality implies Hausdorff property.

Physical reasonability

Global vs. local properties:

- local property — for each class of locally isometric spacetimes, either it is possesed by all elements of the class, or by none of them
- global property = not local
- property of being Hausdorff or non-Hausdorff is global
- there exist local counterpart of Hausdorff property, which is satisfied in any locally Euclidean space

Physical reasonability

Some reflections:

- the status of the first group of conditions is a bit different — they do not exclude some solutions of Einstein's equations, but rather specify the structure on which these equations are defined
- basic definitions of topology (like open set) seem to have no physical meaning, so they should be irrelevant to the issue of physical reasonability
- however some topological properties, like Hausdorff condition, may have physical consequences (as we will see)

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Examples of non-Hausdorff manifolds

How to make a non-Hausdorff space from a Hausdorff one?

A subspace of a Hausdorff space is a Hausdorff space. The product of two Hausdorff spaces is also a Hausdorff space.

However, the qoutient space of a Hausdorff space need not be a Hausdorff space.

Examples of non-Hausdorff manifolds

Definition (Quotient space)

If X is a topological space, Y is a set and $g : X \mapsto Y$ is an onto mapping, then the collection τ_g of subsets of Y defined by

$$\tau_g = \{G \subset Y \mid g^{-1}(G) \text{ is open in X}\}$$

is a topology on Y, called the *quotient topology* induced on Y by g. Y is called a *quotient space* of X and g is called a *quotient map*.

The quotient topology induced on Y by g is the largest topology on Y making g continuous.

Examples of non-Hausdorff manifolds

When a quotient space of a given Hausdorff space is Hausdorff?

Theorem

If a quotient map $g : X \mapsto Y$ is closed (for each closed set A in X, $g(A)$ is a closed set in Y) and $g^{-1}(y)$ is compact for each $y \in Y$, then Y is Hausdorff if X is.

Examples of non-Hausdorff manifolds

Another construction which may lead to non-Hausdorff manifolds — gluing together countably many Hausdorff manifolds (Hajicek 1971a):

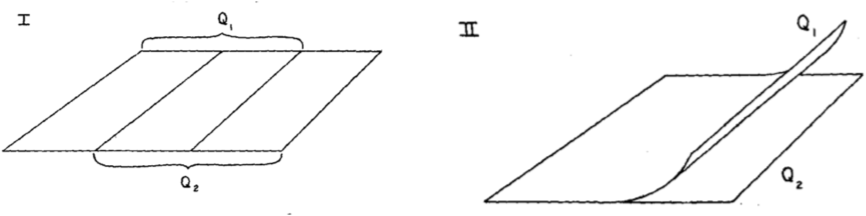
Definition (Gluing)

Let M_1 and M_2 be manifolds. Then $\phi : A \mapsto B, A \subset M_1, B \subset M_2$ is a gluing map if

- A is open,
- ϕ is an isometry (and therefore diffeomorphism).

If all glued manifolds are the same, then this method coincides with the previous one (making a quotient).

Illustration (from Hájíček 1971)



A manifold constructible by gluing together of Hausdorff manifolds admits bifurcate curves of II kind iff the gluing is continuously extendible.

Examples of non-Hausdorff manifolds

- the simplest example (Hawking and Ellis 1973, 2.1)
- extensions of Misner space-time (Hawking and Ellis 1973, 5.8)
- extensions of Taub-NUT space-time (Chruściel and Isenberg 1991, Hajicek 1971a)
- extensions of Gowdy polarized space-time

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Properties of non-Hausdorff manifolds

Mathematical properties (Bourbaki 1966, Munkres 2000, Willard 1970)

Conditions equivalent to Hausdorff condition:

- every net in X (a subset of X indexed by elements of directed set) has a unique limit point
- every filter in X has a unique limit point
- the diagonal $\Delta = \{(x, x) : x \in X\}$ is closed in $X \times X$
- the intersection of the closed neighbourhoods of any point of X consists of that point alone

Properties of non-Hausdorff manifolds

Conditions which follow from Hausdorff condition and may be absent in non-Hausdorff spaces:

- Any continuous map of a topological space X into a Hausdorff space Y is uniquely determined by its values at all points of a dense subset of X.
- If X is a Hausdorff space, then a sequence of points of X converges to at most one point in X.
- Every compact subset of a Hausdorff space is closed.

Every metric space is Hausdorff, so non-Hausdorff spaces are not metrizable.

Properties of non-Hausdorff manifolds

Physical properties (Clarke 1976, Hajicek 1970, 1971a, 1971b)

Clarke and Hajicek define space-time as 4-dimensional manifold, with smooth Lorentzian metric. Hajicek 1971a also adds paracompactness, Hajicek 1971b complete separability (= second countability) and Clarke 1976 connectedness.

Properties of non-Hausdorff manifolds

Definition (Bifurcate curve of the first kind)

A bifurcate curve of the first kind is a pair of curves C, C' in a space-time M, $C, C' : [0,1] \mapsto M$ such that $C = C'$ on $[0, g]$ and $C \neq C'$ on $(g, 1]$ for some $g \in (0,1)$.

Definition (Bifurcate curve of the second kind)

A bifurcate curve of the second kind is a pair of curves C, C' in a space-time M, $C, C' : [0,1] \mapsto M$ such that $C = C'$ on $[0, g)$ and $C \neq C'$ on $[g, 1]$ for some $g \in (0,1)$.

Bifurcate curves of the first kind are present in any space-time.
Only bifurcate curves of the second kind lead to bifurcate geodesics.

Properties of non-Hausdorff manifolds

- Any space-time which is either Hausdorff or non-Hausdorff but without bifurcating curves of the second kind has a maximal extension. (Clarke 1976)
- For any non-Hausdorff manifold, there exist its open covering by $H - submanifolds$, where $H - submanifold$ is an open submanifold, which is Hausdorff and which is not a proper subset of any other open submanifold. (Hajicek 1971b)
- The necessary and sufficient condition for a manifold constructed by gluing together Hausdorff manifolds to admit bifurcate curves of the second kind is that the gluing be continuously extendable. (Hajicek 1971a)
- Thanks to the previous theorem this is a very general result, because every non-Hausdorff manifold may be constructed by gluing together some Hausdorff manifolds.
- A non-Hausdorff space-time either is not strongly causal or admits bifurcate curves of the second kind. (Clarke 1976)

Properties of non-Hausdorff manifolds

Definition (Continuously extendible gluing)

A gluing map $\phi : A \mapsto B, A \subset M_1, B \subset M_2$ is continuously extendible iff there exist A', B', ϕ' such that $A \subset A' \subset M_1, B \subset B' \subset M_2$, $\phi' : A' \mapsto B'$, ϕ' is continuous and $\phi'|_A = \phi$.

Examples of non-Hausdorff space-times which do not admit bifurcate curves of the second kind: extensions of Misner, extensions of Taub-NUT.

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Arguments for and against non-Hausdorff manifolds

In the literature the presence of **bifurcating geodesics** is the main argument invoked **against** non-Hausdorff space-times. The reason is that in such cases the equation of geodesics does not have a unique global solution (although local uniqueness is still satisfied) and that is the breakdown of determinism because geodesics are assumed to be (potential) worldliness of free test particles. However, as we have seen, in many non-Hausdorff space-times there are no bifurcate geodesics and therefore some physicist consider **liberalizations** of the Hausdorff condition. For example, (**Hawking and Ellis** 1973:174) allow for these non-Hausdorff space-times which do not admit bifurcating geodesics; similarly (**Geroch** 1968:465) allows for non-Hausdorff space-times in which every geodesic has a unique extension and every curve has no more than one end point.

Arguments for and against non-Hausdorff manifolds

Earman (2008) — arguments against non-Hausdorff manifolds:

- Mathematical theorems which depend on the Hausdorff condition: every compact set of a topological space is closed and if a sequence of points of a topological space converges, the limit point is unique
- In order to properly formulate local conservation law, energy-momentum tensor should be continuous and differentiable. However, this entails that when energy 'travels' along bifurcate curve, it has to take both branches, because if it went along only one of them, the tensor on another one would be discontinuous. But then global energy conservation would be violated.
- The theorem which guarantees existence and uniqueness of maximal solutions of Einstein's equations (given the appropriate initial data) relies on the Hausdorff condition. The uniqueness result fails if non-Hausdorff branching is allowed – we may attach non-Haussdorffly additional branches at some given moment of time.

Arguments for and against non-Hausdorff manifolds

Earman (2008) — continued:

- The fourth and most philosophical Earman's argument can be summed as follows: both types of branching (on the level of geodesics and of the whole space-time) include a kind of arbitrariness connected with indeterminism. As concerns geodesics branching, he asks rhetorically: "how would such a particle know which branch of a bifurcating geodesic to follow?", suggesting that there is no good answer to this question. As concerns space-time branching, he claims that we need some physical theory that prescribes the dynamics of branching – there should be something that determines which of possible branches are realised. Branching cannot, according to Earman, be regarded as explanatory term; quite the opposite – it requires explanation in other terms.

Arguments for and against non-Hausdorff manifolds

Some of Earman's objections turn out to be harmless if we carefully interpret branching structures as representing possible evolutions, where at most one of branches can be actualised. For example, there is no problem with discontinuity of energy-momentum tensor: in actual reality it is wholly contained in one branch and the discontinuity concerns only branches which are not realised.

The more subtle issue is indeterminism on the level of geodesics (curves followed by free test particles) and space-times which is allowed in some non-Hausdorff cases. There are some principal objections against it: lack of control, lack of factor which determinates the actual evolution (Earman 2008) or breaking "classical causality conception coinciding with determinism" (Hajicek 1971). However, it seems that all of these objections come down to simple rejection of indeterminism, which begs the question.

Arguments for and against non-Hausdorff manifolds

Penrose (1979:592-595):

- considers a model in which „the branching takes place along the future light-cones of the points at which 'observations' (presumably quantum-mechanical) are made"
- according to him this model „(...) is not altogether implausible. It is possible to envisage, for example, that the branching accompanies a kind of retarded collapse of the wavefunction, where on each branch the wavefunction starts out as a different eigenvector of the operator representing the observation"
- possible model of 'Everett-type universe'
- „Such a model could be viewed as an 'objective' description of a world containing some strongly 'subjective' elements. One could envisage different conscious observers threading different routes through the myriads of branches (either by chance, say, or perhaps even by the exercise of some 'free will')."

Arguments for and against non-Hausdorff manifolds

Penrose — critique of this model:

- nobody has shown that Everettian assumptions in fact lead to such a model
- „I feel particularly uncomfortable about my friends having all (presumably) disappeared down different branches of the universe, leaving me with nothing but unconscious zombies to talk to!"
- This model do not solve the problem of the origin of direction of time

Arguments for and against non-Hausdorff manifolds

Summary — possible problems for non-Hausdorff spacetimes:

- strange mathematical properties
- lack of strong causality
- bifurcating curves of the second kind
- lack of maximal extension
- arbitrariness connected with indeterminism
- problems with conservation laws
- Penrose's problem with conscious observers

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