Quantum Logic as *Classical* Logic

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Downloadable *slides* and *paper* [3], respectively

Outline

Introduction
- Mathematical result (what)
- Quantum-mechanical motivation (why)
  - A representative physical experiment...
  - ...beset by an elementary fallacy
  - Orthomodular lattices (OML)
- Algebraic-modal-logical technique (how)
  - Boolean Algebras with Operators (BAO)

Representation Theorem (proof sketch of the result)
- Order-embedding OML into BAO
- Embedding properties of quantum negation
- Lattice-embedding OML into BAO

Conclusion

Theorem (Representation)

Any orthomodular lattice

$$\mathfrak{OML} := \langle \mathrm{H}(\mathcal{S}), 0, \curlywedge, \curlyvee, 1, \boxed{\cdot^{\perp}}, \preccurlyeq \rangle,$$

viewable as arising from a Hilbert-space $\mathrm{H}(\mathcal{S})$ *over some state space* $\mathcal{S}$ *(a set), embeds into a Boolean (powerset) algebra*

$$\mathfrak{BAO} := \langle \mathrm{P}(\mathcal{S}), \emptyset, \cap, \cup, \mathcal{S}, \bar{\cdot}, \langle R \rangle, \subseteq \rangle$$

with an operator $\langle R \rangle$ *via a certain* ***lattice****-embedding* ρ*, that is,* ρ *is a structure-preserving bijection between* $\mathfrak{OML}$ *and*

$$\mathfrak{S} := \langle \{ \rho(H) \mid H \in \mathrm{H}(\mathcal{S}) \}, \rho(0), \cap, \cup, \rho(1), \boxed{\sim := \langle R \rangle \circ \bar{\cdot}}, \subseteq \rangle.$$

A representative physical experiment. . .

Let P, Q, and Q' be statements about three quantum phenomena such that

$$\underbrace{\underbrace{P \text{ is observed to be true}}_{\Box P \text{ is true}} \text{ and } \underbrace{(Q \text{ or } Q') \text{ is observed to be true}}_{\Box(Q \vee Q') \text{ is true}}}_{\Box P \wedge \Box(Q \vee Q') \text{ is true}}.$$

QL≠IL: observing a disjunction does not imply observing one of its disjuncts! The converse implication *is* a valid principle.

Actually,

$$\underbrace{\text{neither } (P \text{ and } Q) \text{ nor } (P \text{ and } Q') \text{ is observed to be true.}}_{\neg\Box(P \wedge Q) \wedge \neg\Box(P \wedge Q') \text{ is true} \quad (\text{or: } \neg(\Box(P \wedge Q) \vee \Box(P \wedge Q')) \text{ is true})}$$

. . . beset by an elementary fallacy

The presentation of the experiment *wrongly* concludes that

(P and (Q or Q')) is true but not ((P and Q) or (P and Q')).

That is,

"(P and (Q or Q'))" and "((P and Q) or (P and Q'))" are not equivalent.

Apparently, the distributivity of classical conjunction and disjunction fails! Whence *wrongly* arises the motivation for special *quantum* conjunction and disjunction.

Wrongly, because the obvious correct conclusion—making explicit *the fact of observing facts*—is that

$$\boxed{(\Box P \land \Box(Q \lor R)) \leftrightarrow (\Box(P \land Q) \lor \Box(P \land R)) \text{ is false,}}$$

which is a normal state of affairs in classical modal logic.

Orthomodular lattices (OML)

- De Morgan (quantum join as meet and complement):

$$H \curlyvee H' = (H^{\perp} \curlywedge H'^{\perp})^{\perp}$$

- orthocomplementarity (quantum complement):
  - involution: $H^{\perp\perp} = H$
  - disjointness: $H \curlywedge H^{\perp} = 0$
  - exhaustiveness: $H \curlyvee H^{\perp} = 1$
  - antitonicity: $H \preccurlyeq H' \Rightarrow H'^{\perp} \preccurlyeq H^{\perp}$
- orthomodularity (OM):

$$\begin{aligned} H \preccurlyeq H' \quad &\text{:iff} \quad H = H \curlywedge H' \\ &\Leftrightarrow \quad H' = H \curlyvee H' \\ &\Rightarrow \quad H' = H \curlyvee (H' \curlywedge H^{\perp}) \qquad \text{(OM)} \end{aligned}$$

Boolean Algebras with Operators (BAO)

BAOs can be viewed as powerset lattices with additional (non-Boolean) operators.

Here, we define

1. the powerset $\mathrm{P}(\mathcal{S})$ of the considered state space $\mathcal{S}$ to be the carrier set of our BAO.
2. the one additional (modal) operator $\langle R \rangle$ for our BAO to be

$$\langle R \rangle(S) \;:=\; \{\, s \in \mathcal{S} \mid \text{there is } s' \in \mathcal{S} \text{ s.t. } s \; R \; s' \text{ and } s' \in S \,\}$$

such that (discovered during the proof):

- $\forall s \forall s'(s \; R \; s' \to s' \; R \; s)$ (symmetry)
- $\underbrace{\forall s \exists s'(s \; R \; s'}_{\text{seriality}} \wedge \forall s''((s' \; R \; s'') \to (s'' = s)))$ (Q-property)

Idea: translate

- *quantum join (disjunction)* of quantum propositions as the quantum complement (negation) of the quantum meet (conjunction) of the quantum complements of those propositions (De Morgan):

$$H \curlyvee H' = (H^{\perp} \curlywedge H'^{\perp})^{\perp}$$

- *quantum complement* as the modal operator applied to the classical complement of the quantum proposition

$$\rho(H^{\perp}) = {\sim}\rho(H) \qquad ({\sim}\text{-homomorphism})$$

where ${\sim} := \langle R \rangle \circ \overline{\cdot}$

- *quantum meet* as classical intersection

$$\rho(H \curlywedge H') = \rho(H) \cap \rho(H') \qquad (\text{meet homomorphism})$$

Proposition

The complete lattice $\mathfrak{BAO}$ is a completion [2, Definition 7.36] of the lattice (and thus partially ordered set) $\mathfrak{OML}$ via the ***order****-embedding ρ, that is,*

$$\text{for all } H, H' \in \mathrm{H}(\mathcal{S}),\ H \preccurlyeq H' \text{ if and only if } \rho(H) \subseteq \rho(H')\,.$$

Proposition (Properties of $\sim$)

1. $\sim\sim\rho(H) = \rho(H)$
2. $\sim(S \cap S') = \sim S \cup \sim S'$ *(thus* $\sim(\rho(H) \cap \rho(H')) = \sim\rho(H) \cup \sim\rho(H')$*)*
3. $\sim(\rho(H) \cup \rho(H')) = \sim\rho(H) \cap \sim\rho(H')$
4. $\rho(H) \cap \sim\rho(H) = \rho(0)$
5. $\rho(H) \cup \sim\rho(H) = \rho(1)$
6. $(H \preccurlyeq H'$ *or* $\rho(H) \subseteq \rho(H'))$ *implies*
   - 6.1 $\sim\rho(H') \subseteq \sim\rho(H)$ *and*
   - 6.2 $\rho(H') = \rho(H) \cup (\rho(H') \cap \sim\rho(H))$
7. $\sim\rho(0) = \rho(1)$
8. $\sim\rho(1) = \rho(0)$

Theorem (Representation)

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Conclusion: QL vs CL, axiomatically speaking

1. $(A \equiv B) \rightarrow_0 ((B \equiv C) \rightarrow_0 (A \equiv C))$
2. $(A \equiv B) \rightarrow_0 (\sim A \equiv \sim B)$
3. $(A \equiv B) \rightarrow_0 ((A \curlywedge C) \equiv (B \curlywedge C))$
4. $(A \curlywedge B) \equiv (B \curlywedge A)$
5. $(A \curlywedge (B \curlywedge C)) \equiv ((A \curlywedge B) \curlywedge C)$
6. $(A \curlywedge (A \curlyvee B)) \equiv A$
7. $(\sim A \curlywedge A) \equiv ((\sim A \curlywedge A) \curlywedge B)$
8. $A \equiv \sim\sim A$
9. $\sim(A \curlyvee B) \equiv (\sim A \curlywedge \sim B)$
10. $(A \equiv B) \equiv (B \equiv A)$
11. $(A \equiv B) \rightarrow_0 (A \rightarrow_0 B)$
12. $(A \rightarrow_0 B) \rightarrow_3 (A \rightarrow_3 (A \rightarrow_3 B))$
13. from A and $A \rightarrow_3 B$ infer B

with the three abbreviations:

$$A \rightarrow_0 B := \sim A \curlyvee B$$

$$A \rightarrow_3 B := (\sim A \curlywedge B) \curlyvee (\sim A \curlywedge \sim B) \curlyvee (A \curlywedge (\sim A \curlyvee B))$$

$$A \equiv B := (A \curlywedge B) \curlyvee (\sim A \curlywedge \sim B)$$

1. the classical propositional axioms plus modus ponens
2. $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$
3. $\Box\Diamond A \leftrightarrow A$, where $\Diamond := \neg\Box\neg$
4. from A infer $\Box A$

This simple *classical* modal logic can be translated to *classical* first-order logic with one relational symbol (R) in a standard way (connection to [1]?).

In conclusion, the logic of quantum mechanics is entirely *classical,* in particular it is *not intuitionistic!*

References

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