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GIP & wGIP

Beyond Gödel's Incompleteness Theorem

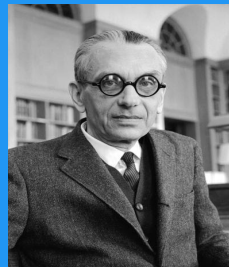


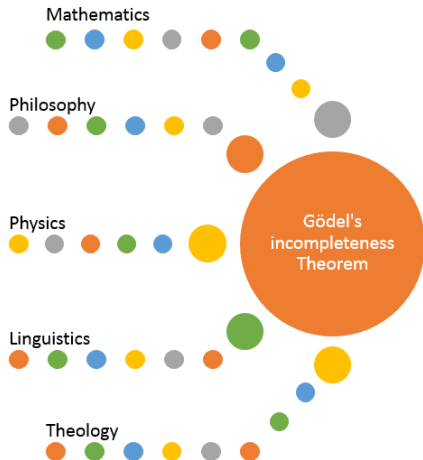
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Introduction

Recent Developments

Open Problems

First Incompleteness Theorem



No effectively axiomatizable first order theory extending elementary arithmetic can be both complete and consistent!

Gödel's Incompleteness Properties



Meaningful Properties

Gödel's Incompleteness Properties

Effectively axiomatizable



GIP



Meaningful Properties

Gödel's Incompleteness Properties

Effectively axiomatizable

Finitely axiomatizable



GIP

wGIP

Meaningful Properties

Origins of GIP and wGIP

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P. Halmos 1950

What is the algebraic counterpart of the first incompleteness theorem?

Origins of GIP and wGIP

P. Halmos 1950

What is the algebraic counterpart of the first incompleteness theorem?

I. Németi 1985

- The atomicity of Lindenbaum-Tarski algebras?
- No! the non-atomicity is a weaker version!
- Two Properties: GIP & wGIP

Basic Facts

GIP \implies wGIP

rec. enum. + dec. \implies failure of GIP

Lindenbaum-Tarski algebras = free algebras

wGIP \iff none atomic free algebras

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Meaningful Properties

Gödel's Incompleteness Properties

Meaningful Properties

Propositional Logic

Gödel's Incompleteness Properties

Meaningful Properties

Propositional Logic

Infinite Language		Finite Language	
GIP	wGIP	GIP	wGIP
X	✓	X	X

Gödel's Incompleteness Properties

Cylindric Algebras

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The atomicity of the free algebra $\mathfrak{Ft}_X CA_n$?

Cylindric Algebras

The atomicity of the free algebra $\mathfrak{Ft}_X CA_n$

$ X \geq \omega$	$ X < \omega \ \& \ n = 2$	$ X < \omega \ \& \ n \geq 3$
✗	✓	✗
D. Pigozzi	L. Henkin	I. Németi

Infinite Generators

D. Pigozzi 1971

Let K be a class of BAO's of a fixed type and let X be an infinite set. The free algebra $\mathfrak{F}_X K$ is atomless.

Cylindric Algebras

Discriminator Varieties

H. Andréka, B. Jónsson and I. Németi 1991

Let K be a variety of BAO's of a fixed type and suppose that K is generated by its finite members. Then

K is discriminator $\implies (\forall \text{ finite } X) \mathfrak{Ft}_X K$ is atomic.

Cylindric Algebras

Gödel's Theorem

Let X be a finite set and let $n \geq 3$:

The free algebra $\mathfrak{Ft}_X CA_n$ is not atomic.

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The free algebra $\mathfrak{Ft}_X RA$ is not atomic.

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The free algebra $\mathfrak{Ft}_X SA$ is not atomic.

Cylindric Algebras

Decidable Classes

Cylindric-like Algebras

Crs_n
 D_n
 P_n
 G_n
 NCA_n
 WCA_n

Relation-like Algebras

RRA_{\emptyset}
 $RRA_{\{r\}}$
 $RRA_{\{s\}}$
 $RRA_{\{r,s\}}$
 NA

Each of them has the f.a.p., but none of them is discriminator.

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Solving The Problem

M. Khaled 2015-2016

None of the finitely generated free algebras of the above classes is atomic

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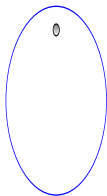
- Disjunctive normal forms.
- Structures with free elements.

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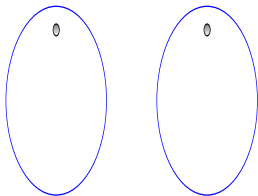


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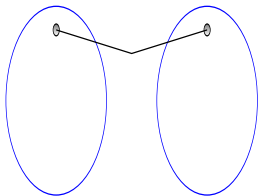


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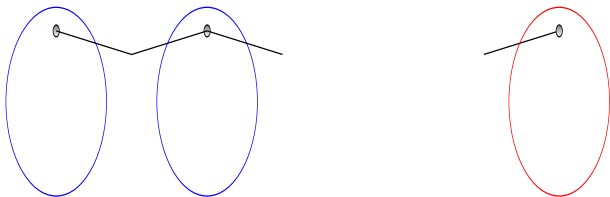


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Non-algebraizable Logics

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Guarded (bounded) versions of FOL:

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Guarded (bounded) versions of FOL:

GF		sGF	
GIP	wGIP	GIP	wGIP
X	X	X	✓

Consequences

$wGIP \not\Rightarrow GIP$ even on finite languages!

$FOL \supseteq GF \supseteq sGF$, where wGIP alternates!

poly. \exists + f.m.p. \implies failure of wGIP!

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M. Khaled 2016

Is there a class of BAO (of fixed similarity type) that is generated by its finite members and all its finitely generated free algebras are atomic, but it is not discriminator?

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T. Aslan & M. Khaled 2017

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T. Aslan & M. Khaled 2017

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T. Aslan & M. Khaled 2017

How “discriminator” can be weakened to replace the implication with equivalence in Andr eka-J onsson-N emeti’s theorem?

Infinite Dimensions

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M. Khaled 2016

Is any of the free algebras $\mathfrak{F}\tau_0 D_\omega$ and $\mathfrak{F}\tau_0 G_\omega$ atomic?

Infinite Dimensions

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Is any of the free algebras $\mathfrak{F}\tau_0 D_\omega$ and $\mathfrak{F}\tau_0 G_\omega$ atomic?

M. Khaled 2016

Let X be a finite set. Is the free algebra $\mathfrak{F}\tau_X P_\omega$ atomic?

More Problems

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I. Németi 1985

Is the equational theory of D_ω decidable?

More Problems

I. Németi 1985

Is the equational theory of D_ω decidable?

M. Khaled 2016

Let n be any ordinal (finite or infinite).

- Is P_n a variety? Is the generated variety finitely schema axiomatizable?
- Is P_n has the finite base property?
- Is the equational theory of P_n decidable?

Undecidable Logics

Undecidable Logics

Z. Gyenis 2012

Are wGIP and GIP equivalent for undecidable logics?

Undecidable Logics

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M. Khaled 2016

Is there any logic that enjoys neither f.m.p. nor wGIP?

Undecidable Logics

Z. Gyenis 2012

Are wGIP and GIP equivalent for undecidable logics?

M. Khaled 2016

Is there any logic that enjoys neither f.m.p. nor wGIP?

M. Khaled 2016

Is there any undecidable logic that does not have GIP? Is there any undecidable logic that does not have wGIP?

Thank You