Structure theorem for a class of group-like residuated chains à la Hahn

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Hahn's Embedding Theorem

PARTIALLY ORDERED ALGEBRAIC SYSTEMS

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1963

5. Hahn's embedding theorem

This section is devoted to the deepest result in the theory of f. o. Abelian groups. This asserts the embeddability of f. o. Abelian groups in the lexicographic product of real groups.

Theorem 16. (HAHN's Embedding Theorem, HAHN [1].) *Every f. o. vector space G over the rational number field is o-isomorphic to a subspace of the lexicographically ordered function space*[20] $W(G)$.

HAHN, H. [1] Über die nichtarchimedischen Grössensysteme, *S.-B. Akad. Wiss. Wien. IIa*, **116** (1907), 601—655.

The original proof of HAHN was extremely long and complicated. Recently, several authors have obtained simpler proofs and generalizations. The proof above is based on an idea of HAUSNER—WENDEL [1]: they proved HAHN's theorem for vector spaces over the real field and CLIFFORD [4] observed that their method works in the general case as well. For other proofs see BANASCHEWSKI [1], GRAVETT [2], RIBENBOIM [2], CONRAD [1], [7]. The last author has extended the theorem to certain p. o. Abelian groups and to even more general systems; he uses decompositions of the given group.

Recently, P. CONRAD, J. HARVEY and CH. HOLLAND proved HAHN's embedding theorem for commutative l. o. groups.

HAUSNER, M.—WENDEL, J. G. [1] Ordered vector spaces, *Proc. Amer. Math. Soc.*, **3** (1952), 977—982.

CLIFFORD, A. H.
— [4] Note on Hahn's theorem on ordered Abelian groups, *Proc. Amer. Math. Soc.*, **5** (1954), 860—863.

BANASCHEWSKI, B. [1] Totalgeordnete Moduln, *Archiv Math.*, **7** (1956), 430—440. — [2] Über die Vervollständigung geordneter Gruppen, *Math. Nachrichten*, **16** (1957), 51—71.

GRAVETT, K. A. H. — [2] Ordered Abelian groups, *Quart. Journ. Math. Oxford*, **7** (1956), 57—63.

RIBENBOIM, P. [2] Sur les groupes totalement ordonnés et l'arithmétique des anneaux de valuation, *Summa Brasil. Math.*, **4** (1958), 1—64. —[3] Sur quelques

CONRAD, P. [1] Embedding theorems for Abelian groups with valuations, *Amer. Journ. Math.*, **75** (1953), 1—29.
— [7] A note on valued linear spaces, *Proc. Amer. Math. Soc.*, **9** (1958), 646—647. — [8]

Comparison

- Hahn's theorem:
- Every totally ordered Abelian group embeds in a lexicographic product of real groups.
- Our embedding theorem:
- Every densely-ordered group-like FL_e-chain, which has finitely many idempotents embeds in a finite partial-lexicographic product of totally ordered Abelian groups.

FL-algebras

An algebra $\mathbf{A} = (A, \wedge, \vee, \cdot, \backslash, /, \mathsf{t}, \mathsf{f})$ is called a *full Lambek algebra* or an *FL-algebra*, if

- $(A, \wedge, \vee)$ is a lattice (i.e., $\wedge$, $\vee$ are commutative, associative and mutually absorptive),
- $(A, \cdot, \mathsf{t})$ is a monoid (i.e., $\cdot$ is associative, with unit element t),
- $x \cdot y \leq z$ iff $y \leq x \backslash z$ iff $x \leq z/y$, for all $x, y, z \in A$,
- f is an arbitrary element of A.

Residuated lattices are exactly the f-free reducts of FL-algebras. So, for an FL-algebra $\mathbf{A} = (A, \wedge, \vee, \cdot, \backslash, /, \mathsf{t}, \mathsf{f})$, the algebra $\mathbf{A}_r = (A, \wedge, \vee, \cdot, \backslash, /, \mathsf{t})$ is a residuated lattice and f is an arbitrary element of A. The maps $\backslash$ and $/$ are called the *left* and *right division*.

- commutative: $x \to y$

Group-like FL_e-algebras

- An FL_e-algebra is a commutative FL-algebra.
- An FL_e-chain is a totally ordered FL_e-algebra.
- An FL_e-algebra is called involutive if $x'' = x$ where $x' = x \to f$ (note that $f' = t$)
- An FL_e-algebra is called group-like if it is involutive and $f = t$

Preliminaries on Residuated Lattices

Residuated Maps

- *Residuation* is a basic concept in mathematics [T. S. Blyth, M. F. Janowitz, Residuation Theory, Pergamon Press, 1972].
- It is very strongly connected with Galois maps [G. K. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. W. Mislove, D. S. Scott, Continuous Lattices and Domains, Encycl. of Math. and its Appl. 93, *Cambridge U. Press*, 2003.]
  and closure operators.

Residuated Monoids

- Residuated lattices have been introduced in the 30s in [Ward, M. and R. P. Dilworth, Residuated lattices, *Transactions of the American Mathematical Society* 45: 335--354, 1939] to investigate ideal theory of commutative rings with unit.

Substructural Logics

- Classical Logic
- Intuitionistic Logic, Super-intuitionistic Logics
- Relevance Logic
- Many-valued Logics
- Mathematical Fuzzy Logics
- Linear LogicLambek Calculus, along with their non-commutative versions

Example Residuated Lattices

- Boolean algebras (Classical Logic)
- Heyting algebras
  [P. T. Johnstone, Stone spaces, *Cambridge University Press*, Cambridge, 1982]
  (e.g., open sets of topological spaces, Intuitionistic Logic)

Example Residuated Lattices

- complemented semigroups
[B. Bosbach, Komplementäre Halbgruppen. Axiomatik und Arithmetik, *Fund. Math.* 64 (1969), 257--287]
- bricks [B. Bosbach, Concerning bricks, *Acta Math. Acad. Sci. Hungar.* 38 (1981), 89-104],
- residuation groupoids [B. Bosbach, Residuation groupoids and lattices, *Studia Sci. Math. Hungar.* 13 (1978), 433-451],
- semiclans [B. Bosbach, Concerning semiclans, *Arch. Math.* 37 (1981), 316--324],
- Bezout monoids [P. N. Ánh, L. Márki and P. Vámos, Divisibility theory in commutative rings: Bezout monoids, *Trans. Amer. Math. Soc.* 364 (2012), 3967-3992],

Example Residuated Lattices

- MV-algebras [Cignoli, R., D'Ottaviano, I.M., Mundici, D.: Algebraic Foundations of Many-Valued Reasoning, Kluwer, Dordrecht, 2000]
- BL-algebras [Hájek, P.: Metamathematics of Fuzzy Logic, Kluwer Academic Publishers, Dordrecht, 1998]
- lattice-ordered groups; a number of other algebraic structures can be rendered as residuated lattices.

A Few Related Representation Theorems

Ordinal Sums

- Every naturally and totally ordered, commutative semigroup is uniquely expressible as the ordinal sum of a totally ordered set of ordinally irreducible such semigroups

  [A. H. Clifford, Naturally totally ordered commutative semigroups, *Amer. J. Math.*, 76 vol. 3 (1954), 631–646.]

The Theory of Compact Semigroups

- Topological semigroups over compact manifolds with connected, regular boundary *B* such that *B* is a subsemigroup: a subclass of compact connected Lie groups and via classifying (I)-semigroups, that is, semigroups on arcs such that one endpoint functions as an identity for the semigroup, and the other functions as a zero.

  [P.S. Mostert, A.L. Shields, On the structure of semigroups on a compact manifold with boundary, *Ann. Math.*, 65 (1957), 117–143.]

The Theory of Compact Semigroups

- (I)-semigroups are ordinal sums of three basic multiplications which an arc may possess.

  The word 'topological' refers to the continuity of the semigroup operation with respect to the topology.

  [P.S. Mostert, A.L. Shields, On the structure of semigroups on a compact manifold with boundary, *Ann. Math.*, 65 (1957), 117–143.]

Structure of GBL-algebras

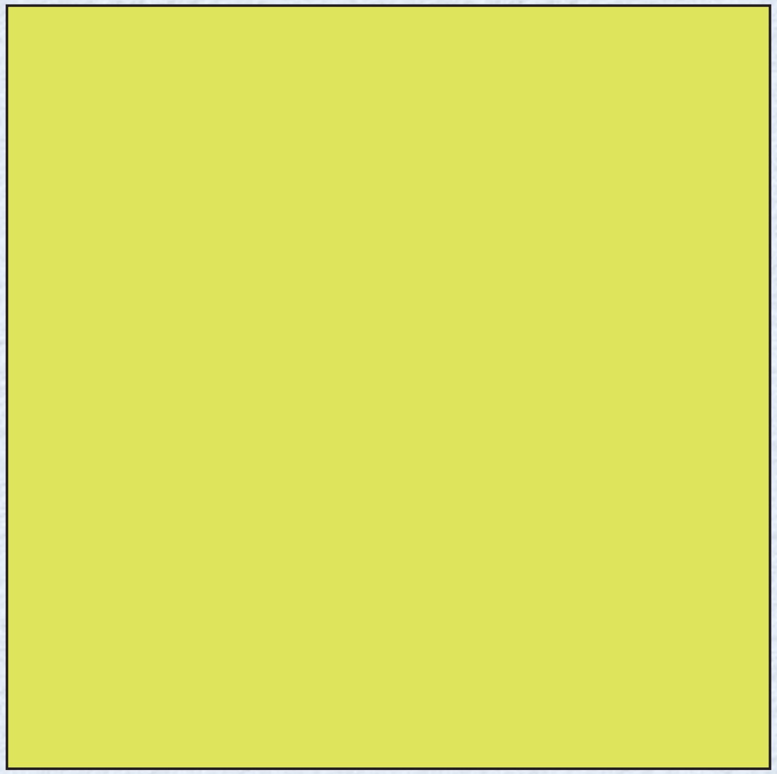
- BL-algebra = naturally ordered + semilinear integral residuated lattice
- BL-algebras are subdirect poset products of MV-chains and product chains.

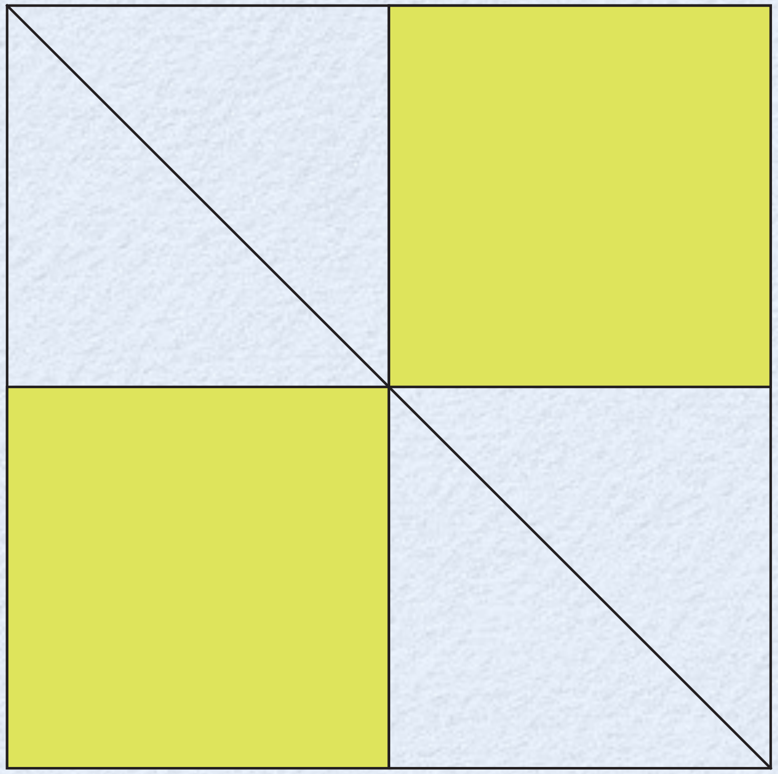
  [P Jipsen, F. Montagna, Embedding theorems for normal GBL-algebras, *Journal of Pure and Applied Algebra*, 214 (2010), 1559–1575.]

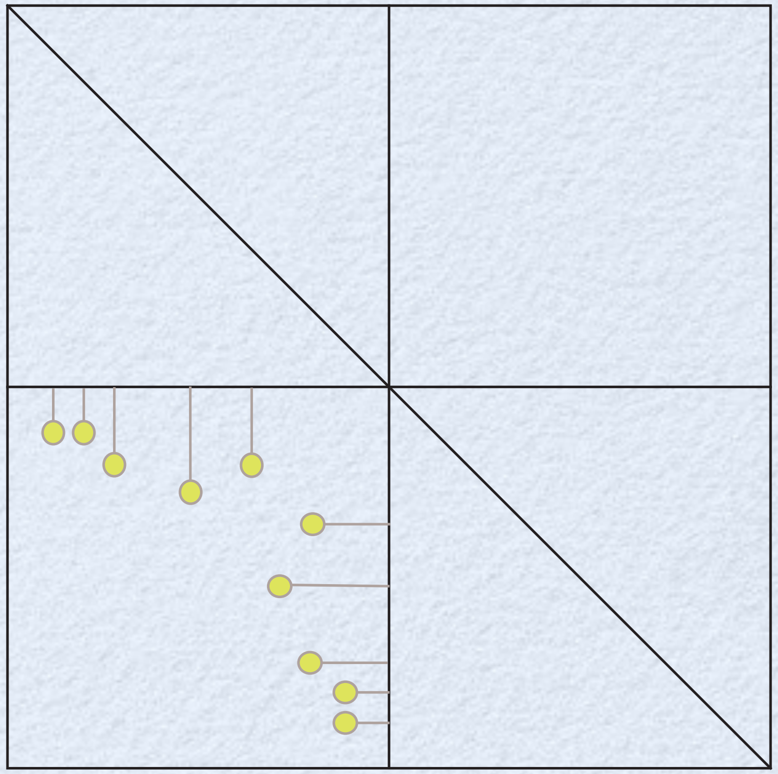
  (A generalization of the Conrad-Harvey-Holland representation)

Weakening the Naturally Ordered Property Entering the Non-integral Case

- [P Jipsen, F. Montagna, Embedding theorems for normal GBL-algebras, ***Journal of Pure and Applied Algebra***, Vol. 214. 1559–1575. (2010)]
- [SJ, F. Montagna, Strongly Involutive Uninorm Algebras ***Journal of Logic and Computation*** Vol. 23 (3), 707-726. (2013)]
- [SJ, F. Montagna, A classification of certain group-like FL_e -chains, ***Synthese*** Vol. 192 (7), 2095-2121. (2015)]







Weakening the Naturally Ordered Property
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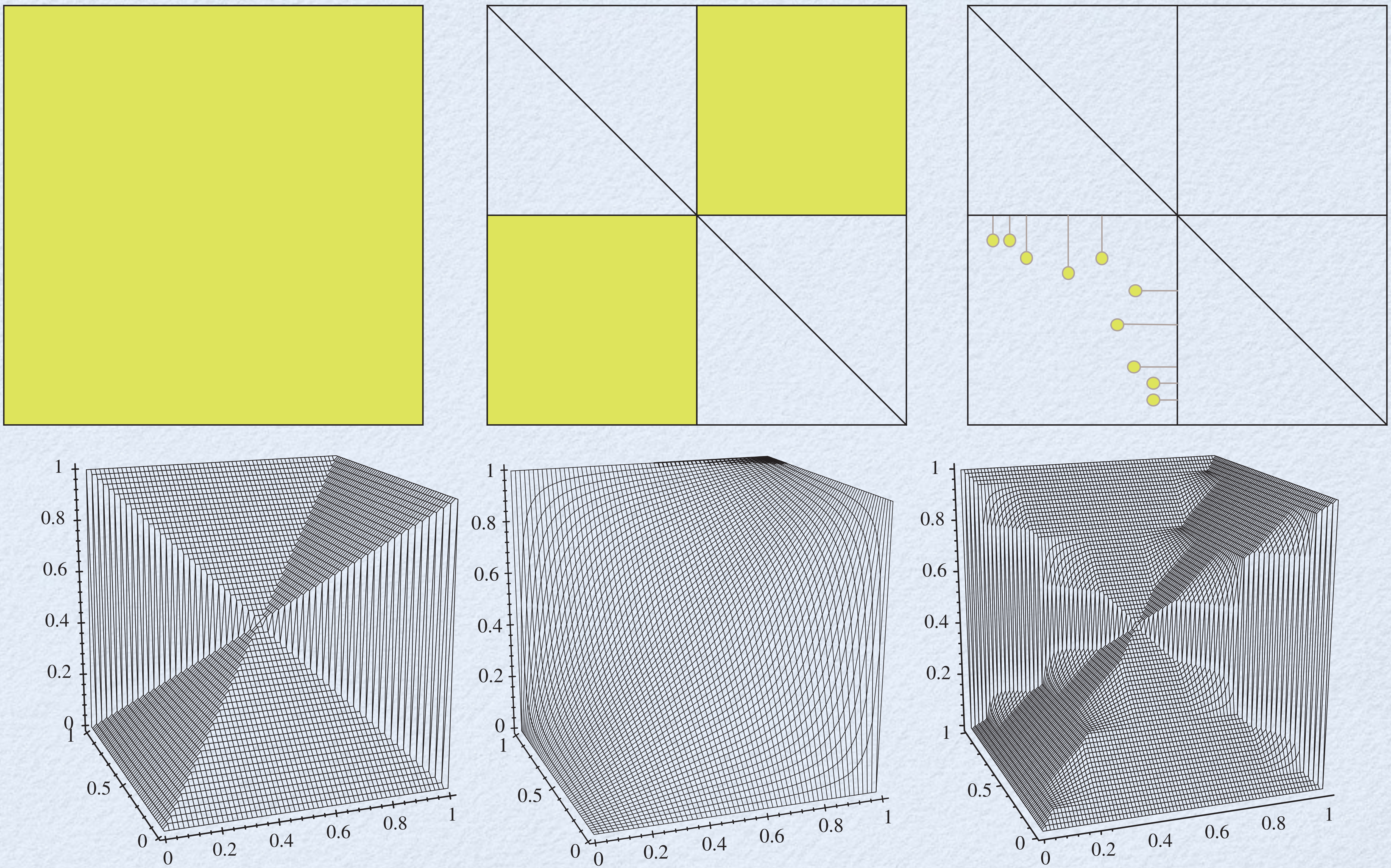
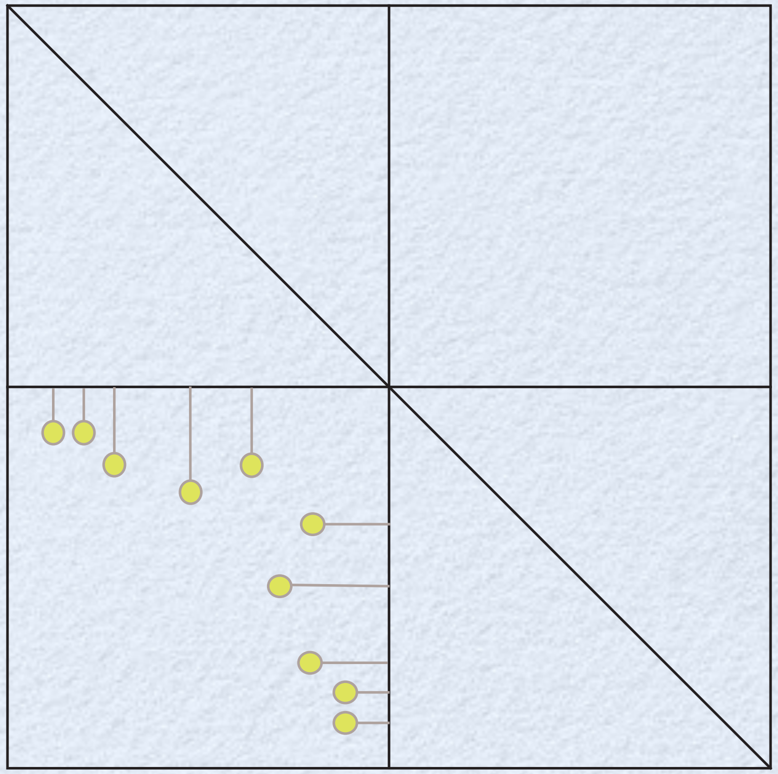


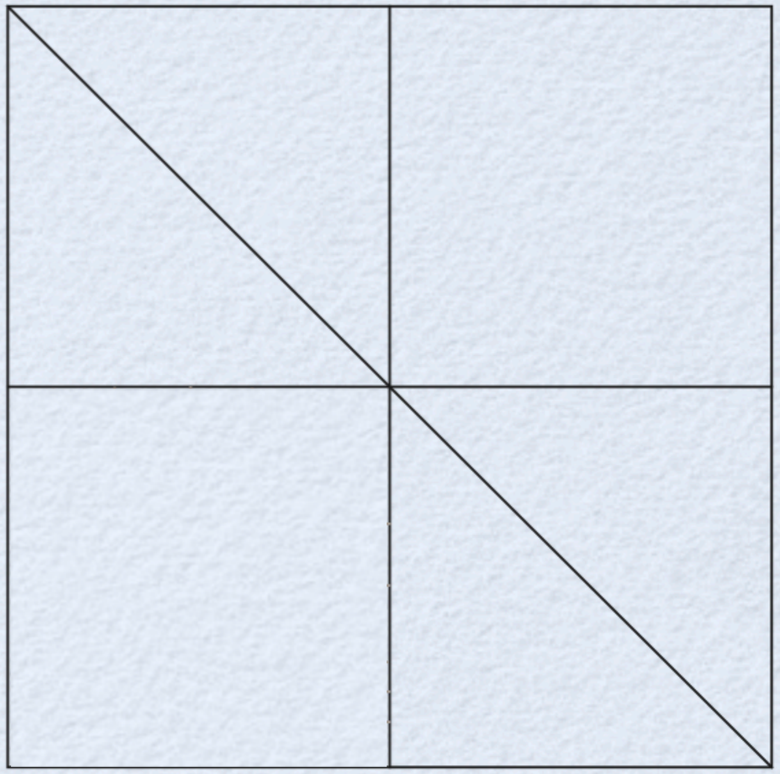
Absorbent Continuous Group-like Commutative Residuated Monoids on Complete and Order-dense Chains

- [SJ, F. Montagna, A classification of certain group-like FL_e -chains, *Synthese* Vol. 192 (7), 2095-2121. (2015)]

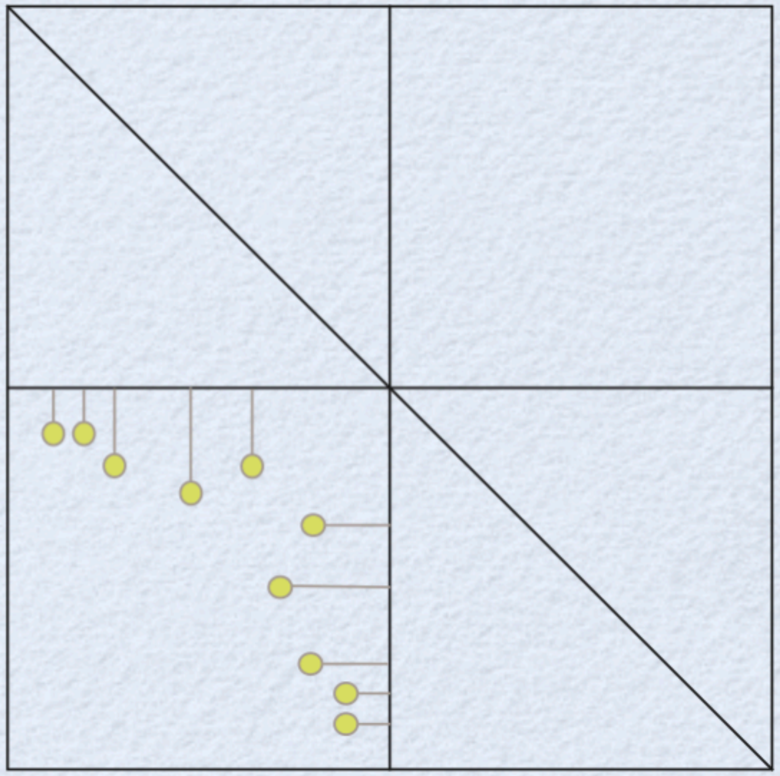


Absorbent Continuous Group-like Commutative Residuated Monoids on Complete and Order-dense Chains

- [SJ,
  Group Representation and Hahn-type Embedding for a Class of Residuated Monoids, (submitted)



- [SJ, F. Montagna,
  A classification of certain group-like FL_e -chains, ***Synthese*** Vol. 192 (7), 2095-2121. (2015)]



About the adjective "group-like" (t=f)

1. Conic representation of group-like FL_e-algebras

- Conic representation: For any conic, IRL

$$x \circledast y = \begin{cases} x \oplus y & \text{if } x, y \in X^+ \\ x \otimes y & \text{if } x, y \in X^- \\ (x \to_\oplus y')' & \text{if } x \in X^+, y \in X^-, \text{ and } x \leq y' \\ (y \to_\otimes x')' & \text{if } x \in X^+, y \in X^-, \text{ and } x \not\leq y' \\ (y \to_\oplus x')' & \text{if } x \in X^-, y \in X^+, \text{ and } x \leq y' \\ (x \to_\otimes y')' & \text{if } x \in X^-, y \in X^+, \text{ and } x \not\leq y' \end{cases}$$

- [S. Jenei, H. Ono, On Involutive FL_e-monoids, *Archive for Mathematical Logic*, (7-8) 719- 738 (2012)
- [S. Jenei, Structural description of a class of involutive uninorms via skew symmetrization, *Journal of Logic and Computation*, 21 vol. 5, 729–737 (2011)

2. Group-like FLe-algebras vs. lattice-ordered groups

SJ

Theorem 2.5. *For a group-like* FL_e*-algebra* $(X, \wedge, \vee, \circledast, \to_\circledast, t, f)$ *the following statements are equivalent:*

(1) *Each element of* X *has inverse given by* $x^{-1} = x'$*, and hence* $(X, \wedge, \vee, \circledast, t)$ *is a lattice-ordered Abelian group,*

(2) $\circledast$ *is cancellative,*

(3) $\tau(x) = t$ *for all* $x \in X$. $\tau(x) = x \to x$

(4) *The only idempotent element in the positive cone of* X *is* t.

3. Representation of group-like FL_e-chains by groups and a Hahn-type embedding

- Coming soon…

Partial-Lexicographic Products

Definition 1. *(Partial-lexicographic products)*
Let $\mathbf{X} = (X, \wedge_X, \vee_X, *, \rightarrow_*, t_X, f_X)$ be a group-like FL_e-algebra and $\mathbf{Y} = (Y, \wedge_Y, \vee_Y, \star, \rightarrow_\star, t_Y, f_Y)$ be an involutive FL_e-algebra, with residual complement $'^*$ and $'^\star$, respectively.

Add a top element $\top$ to Y, and extend $\star$ by $\top \star y = y \star \top = \top$ for $y \in Y \cup \{\top\}$, then add a bottom element $\bot$ to $Y \cup \{\top\}$, and extend $\star$ by $\bot \star y = y \star \bot = \bot$ for $y \in Y \cup \{\bot, \top\}$.

Let $\mathbf{X_1} = (X_1, \wedge_X, \vee_X, *, \rightarrow_*, t_X, f_X)$ be any cancellative subalgebra of $\mathbf{X}$ (by Theorem 1, $\mathbf{X_1}$ is a lattice ordered group). We define

$$\mathbf{X_{\Gamma(X_1,Y^{\bot\top})}} = \left(X_{\Gamma(X_1,Y^{\bot\top})}, \leq, \circledast, \rightarrow_\circledast, (t_X, t_Y), (f_X, f_Y)\right),$$

where

$$X_{\Gamma(X_1,Y^{\bot\top})} = (X_1 \times (Y \cup \{\bot, \top\})) \qquad ,$$

$\leq$ is the restriction of the lexicographic order of $\leq_X$ and $\leq_{Y\cup\{\bot,\top\}}$ to $X_{\Gamma(X_1,Y^{\bot\top})}$, $\circledast$ is defined coordinatewise, and the operation $\rightarrow_\circledast$ is given by $(x_1, y_1) \rightarrow_\circledast (x_2, y_2) = ((x_1, y_1) \circledast (x_2, y_2)')'$ where

$$(x, y)' = \begin{cases} (x'^*, y'^\star) & \text{if } x \in X_1 \\ (x'^*, \bot) & \text{if } x \notin X_1 \end{cases}.$$

Call $\mathbf{X_{\Gamma(X_1,Y^{\bot\top})}}$ the *(type-I) partial-lexicographic product* of X, X_1, and Y, respectively.

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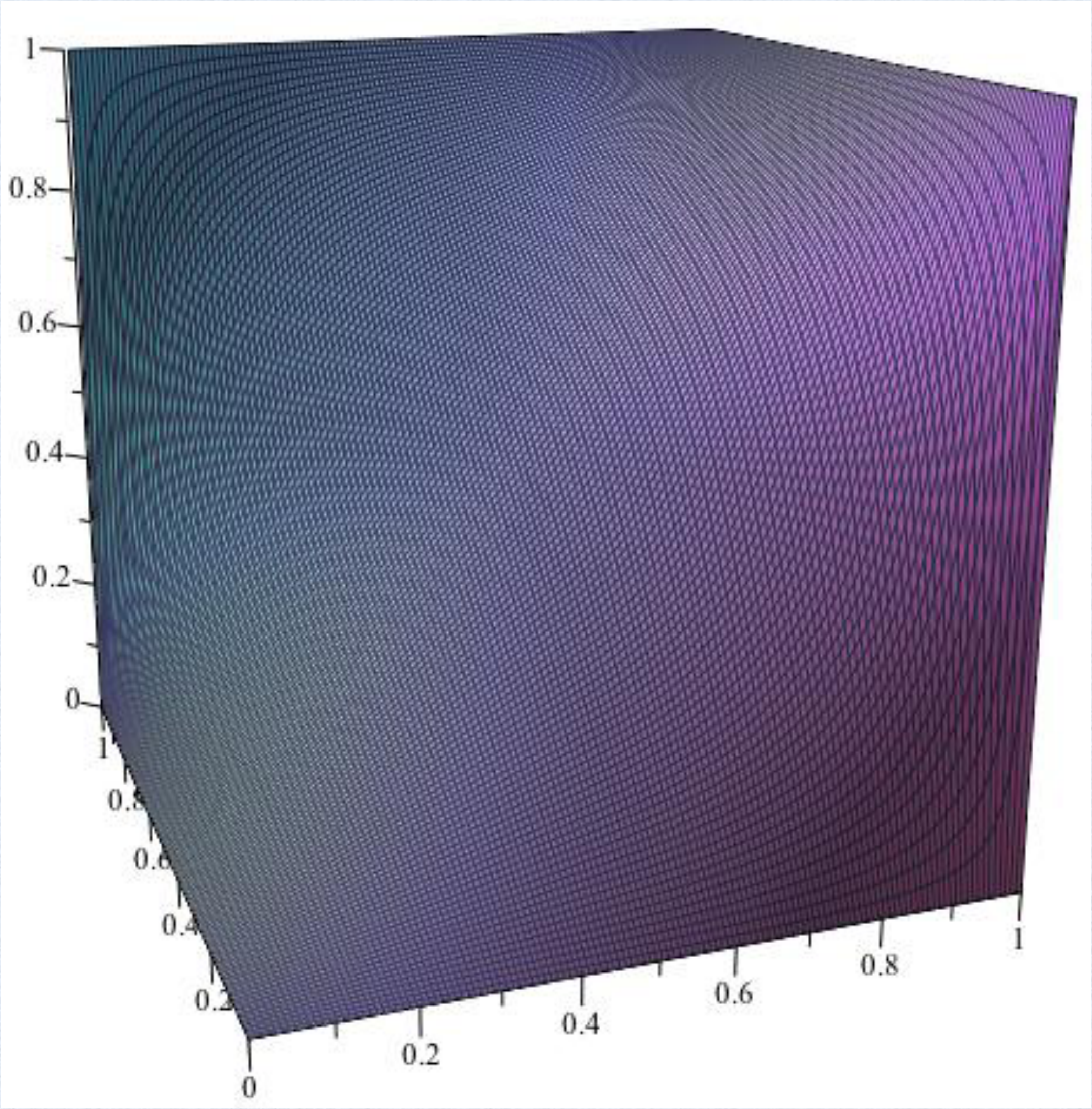

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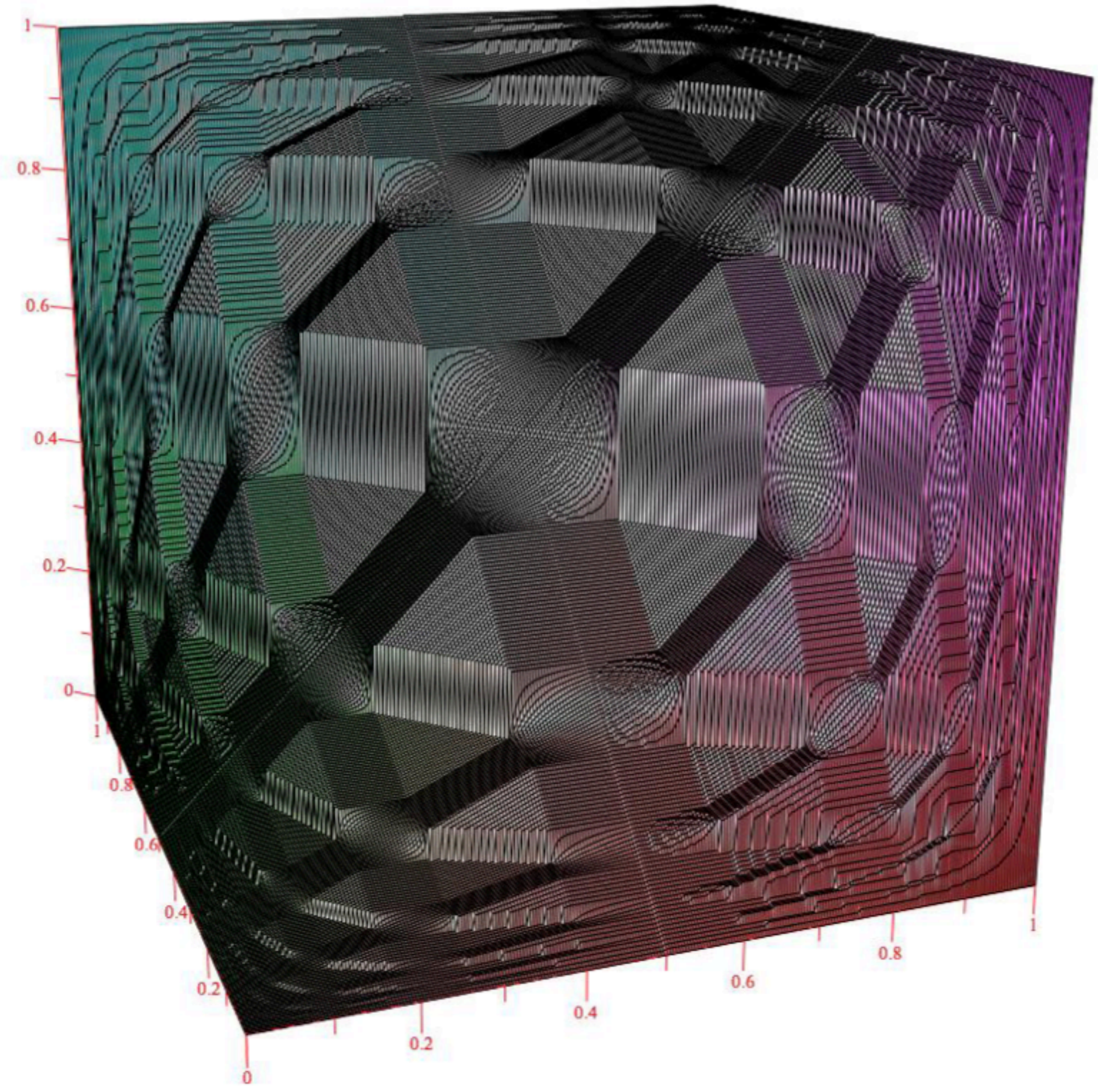

Figure 8: $\mathbf{R}_{\mathbf{\Gamma}(\mathbf{N},\mathbf{R}^{\perp\top})}$

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$$\mathbf{X}_{\mathbf{\Gamma(X_1, Y^{\bot\top})}} = \left(X_{\Gamma(X_1, Y^{\bot\top})}, \leq, ⊛, \to_{⊛}, (t_X, t_Y), (f_X, f_Y)\right),$$

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Call $\mathbf{X}_{\mathbf{\Gamma(X_1, Y^{\bot\top})}}$ the *(type-I) partial-lexicographic product* of X, X_1, and Y, respectively.

Let $\mathbf{X} = (X, \leq_X, *, \to_*, t_X, f_X)$ be a group-like FL_e-chain, $\mathbf{Y} = (Y, \leq_Y, \star, \to_\star, t_Y, f_Y)$ be an involutive FL_e-algebra, with residual complement $'^*$ and $'^\star$, respectively.

Add a top element $\top$ to Y, and extend $\star$ by $\top \star y = y \star \top = \top$ for $y \in Y \cup \{\top\}$.

Further, let $\mathbf{X}_1 = (X_1, \wedge, \vee, *, \to_*, t_X, f_X)$ be a cancellative, discrete, prime[1] subalgebra of $\mathbf{X}$ (by Theorem 1, $\mathbf{X}_1$ is a discrete lattice ordered group). We define

$$\mathbf{X}_{\mathbf{\Gamma(X_1, Y^{\top})}} = \left(X_{\Gamma(X_1, Y^{\top})}, \leq, ⊛, \to_{⊛}, (t_X, t_Y), (f_X, f_Y)\right),$$

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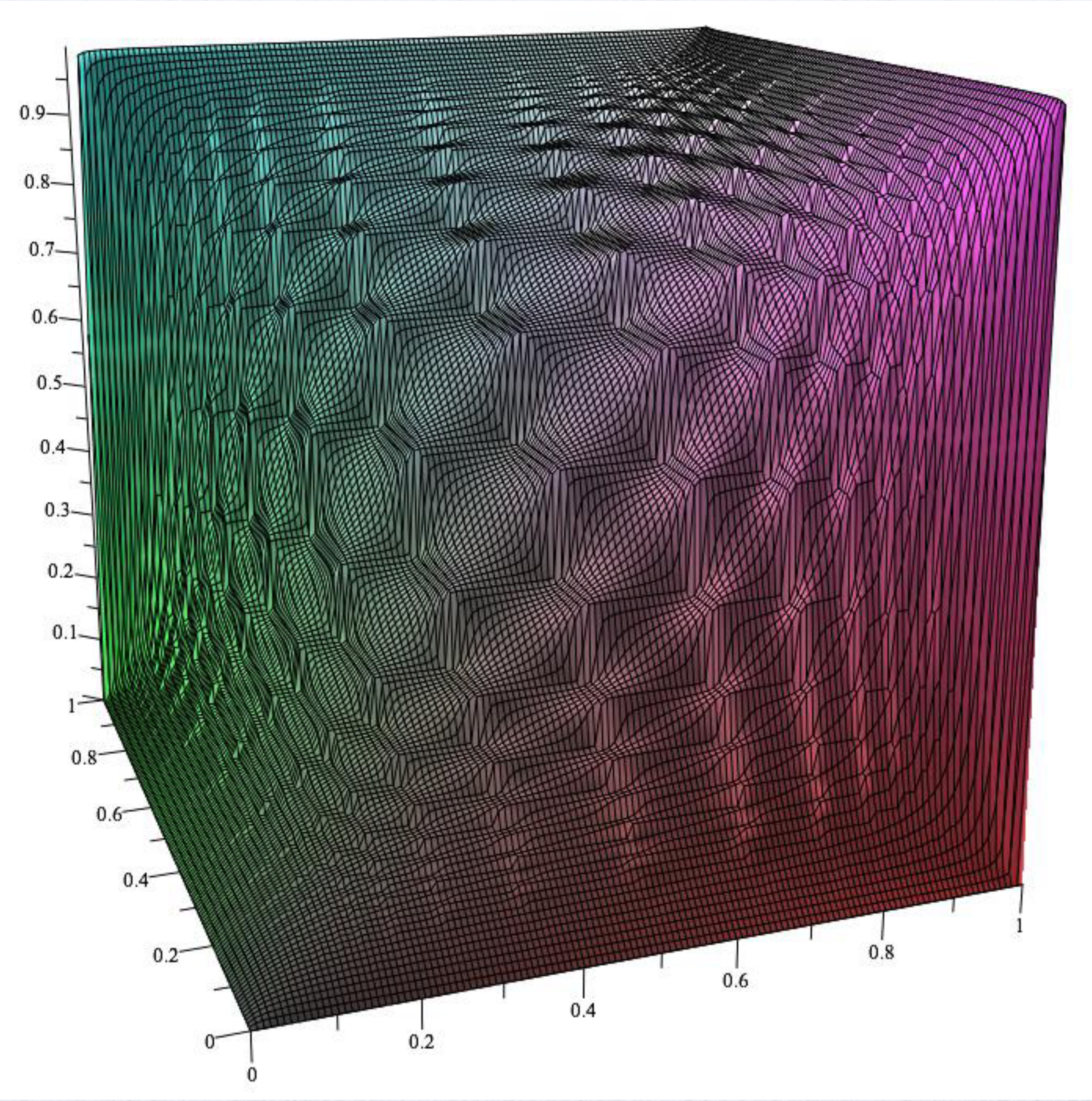
$$(x, y)' = \begin{cases} ((x'^*), \top) & \text{if } x \notin X_1 \text{ and } y = \top \\ (x'^*, y'^\star) & \text{if } x \in X_1 \text{ and } y \in Y \\ ((x'^*)_\downarrow, \top) & \text{if } x \in X_1 \text{ and } y = \top \end{cases}.$$

$$x_\downarrow = \begin{cases} u & \text{if there exists } u < x \text{ such that there is no element in } X \text{ between } u \text{ and } x, \\ x & \text{if for any } u < x \text{ there exists } v \in X \text{ such that } u < v < x. \end{cases}$$

Call $\mathbf{X}_{\mathbf{\Gamma(X_1, Y^{\top})}}$ the *(type-II) partial-lexicographic product* of X, X_1, and Y, respectively.



$\mathbf{N}_{\mathbf{\Gamma}(\mathbf{N},\mathbf{R}^{\top})}$

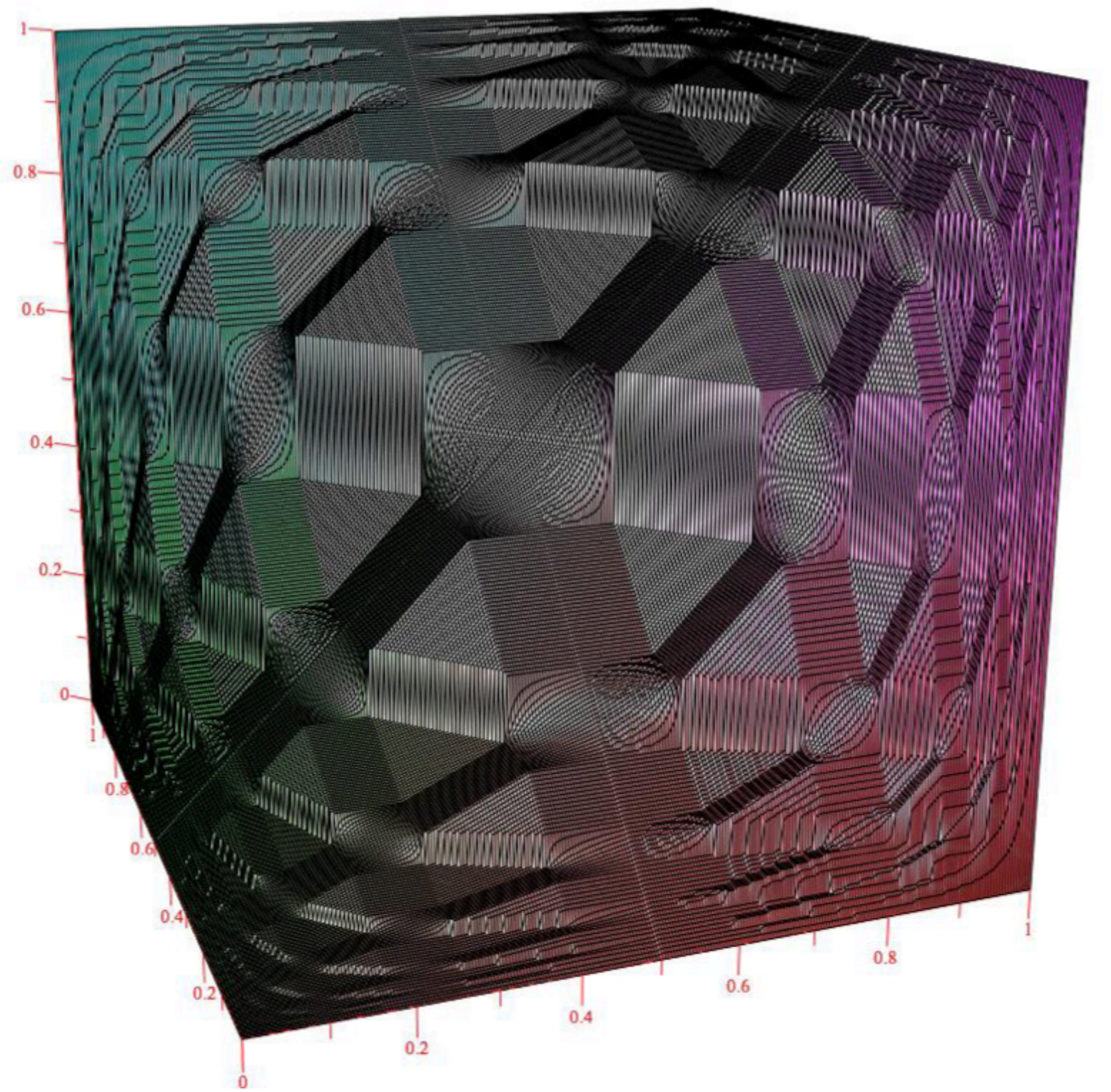
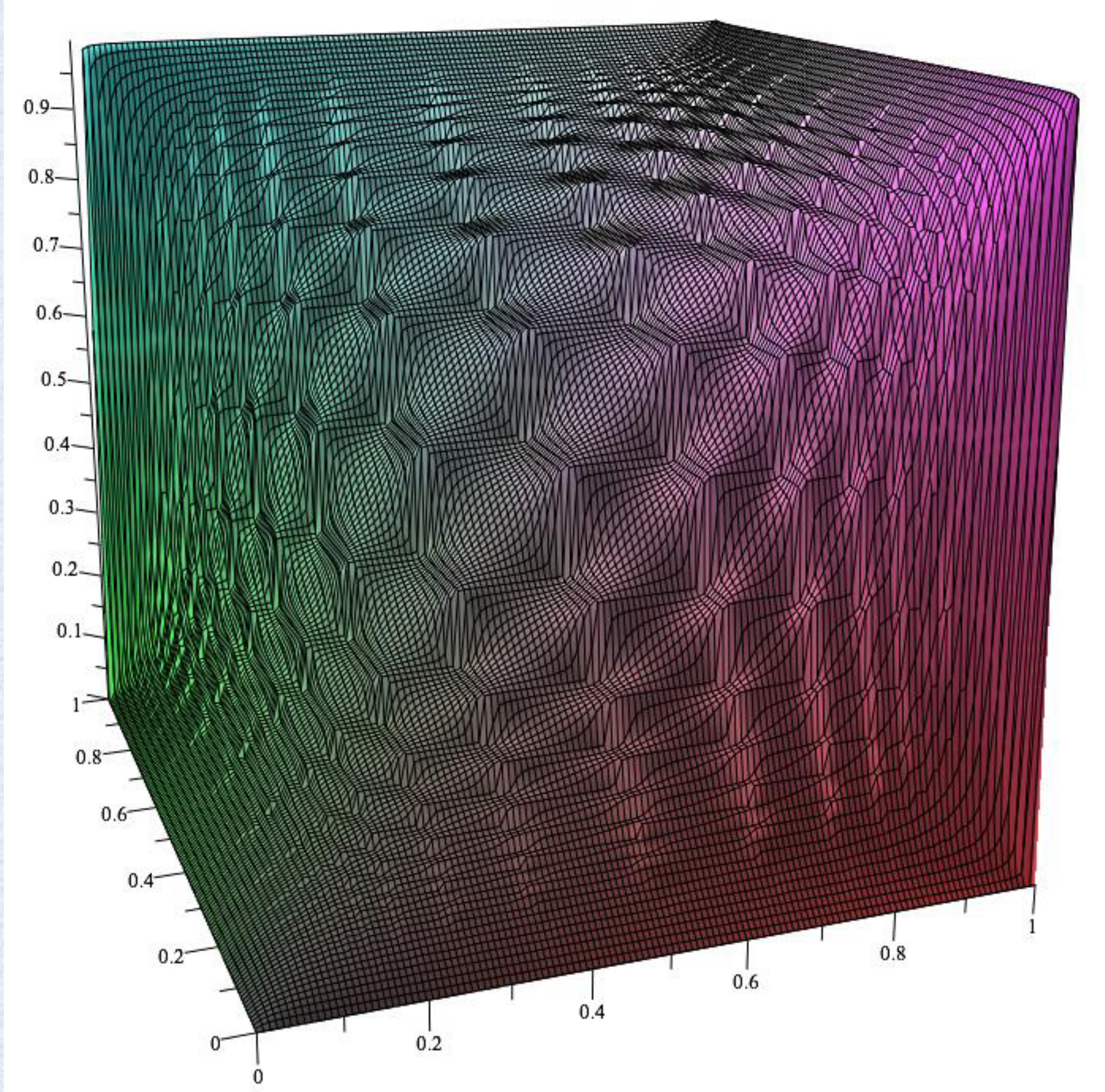


Figure 8: $\mathbf{R}_{\boldsymbol{\Gamma}(\mathbf{N},\mathbf{R}^{\perp\top})}$



$\mathbf{N}_{\boldsymbol{\Gamma}(\mathbf{N},\mathbf{R}^{\top})}$

Theorem 2. $\mathbf{X}_{\mathbf{\Gamma}(\mathbf{X_1},\mathbf{Y}^{\bot\top})}$ *and* $\mathbf{X}_{\mathbf{\Gamma}(\mathbf{X_1},\mathbf{Y}^{\top})}$ *are involutive* FL_e*-algebras. If* $\mathbf{Y}$ *is group-like then also* $\mathbf{X}_{\mathbf{\Gamma}(\mathbf{X_1},\mathbf{Y}^{\bot\top})}$ *and* $\mathbf{X}_{\mathbf{\Gamma}(\mathbf{X_1},\mathbf{Y}^{\top})}$ *are group-like.*

Main Result

Representation by totally ordered Abelian Groups

Theorem 2.21. (Structural representation) *If* $\mathbf{X}$ *is a densely-ordered, group-like* FL_e*-chain, which has only* $n \in \mathbf{N}$ *idempotents in its positive cone then there exist linearly ordered Abelian groups* $\mathbf{G}_i$ $(i \in \{1, 2, \ldots, n\})$, $\mathbf{H}_1 \leq \mathbf{G}_1$, $\mathbf{H}_i \leq \mathbf{\Gamma}(\mathbf{H}_{i-1}, \mathbf{G}_i)$ $(i \in \{2, \ldots, n-1\})$, *and a binary sequence* $\iota \in \{\top\bot, \top\}^{\{2,\ldots,n\}}$ *such that* $\mathbf{X} \simeq \mathbf{X}_n$, *where* $\mathbf{X}_1 := \mathbf{G}_1$ *and* $\mathbf{X}_i := \mathbf{X}_{i-1}{}_{\mathbf{\Gamma}(\mathbf{H}_{i-1}, \mathbf{G}_i{}^{\iota_i})}$ $(i \in \{2, \ldots, n\})$. [32]

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[32] In the spirit of Theorem 2.5 we identify linearly ordered Abelian groups by cancellative, group-like FL_e-chains here; the isomorphism is meant between FL_e-algebras.

Surprising?

- Every *commutative integral monoid on a finite chain* is an *FL_{ew}-chain*.
- It has been shown in [SJ, F Montagna, A Proof of Standard Completeness for Esteva and Godo's Logic MTL, STUDIA LOGICA 70:(2) pp. 183-192. (2002)] that any *FL_{ew}-chain* embeds into a *densely-ordered FL_{ew}-chain*.
- By the rotation construction [SJ, On the structure of rotation-invariant semigroups, ARCHIVE FOR MATHEMATICAL LOGIC 42(5) 489-514. (2003)], any *densely-ordered FL_{ew}-chain* embeds into a *densely-ordered, involutive FL_{ew}-chain*.
- *Densely-ordered, involutive FL_e-chains*, with the t = f condition and with the assumption on the number of idempotent elements results in a strong structural representation, which uses *only* linearly ordered Abelian groups.

Corollary: Embedding

Corollary 2.23. (Hahn-type embedding) *Densely-ordered, group-like FL_e-chains with a finite number of idempotents embed in the finite partial-lexicographic product of lexicographic products of real groups.*

Corollary 2.24. (Lexicographical embedding of the monoid reduct) *The monoid reduct of any densely-ordered, group-like FL_e-chain with a finite number of idempotents embeds in the* lexicographic *product of the 'extended' additive group of the reals*[33]*.*

That is all!