Derivation of the transformation laws for the electrodynamic quantities from electrodynamics without presuming covariance

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The invariance in form of the equations of electrodynamics under Lorentz transformations was shown by Lorentz and Poincaré before Einstein formulated the special theory of relativity. We will now discuss this covariance and consider its consequences. There are two points of view possible. One is to take some experimentally proven fact such as the invariance of electric charge and try to deduce that the equations must be covariant. The other is to demand that the equations be covariant in form and to show that the transformation properties of the various physical quantities, such as field strengths and charge and current, can be satisfactorily chosen to accomplish this. Although the first view is to some the most satisfying, we will adopt the second course. Classical electrodynamics *is* correct, and it can be cast in covariant form. (Jackson: *Classical Electrodynamics*, **1st edition**, 1962, p. 377)

The invariance in form of the equations of electrodynamics under Lorentz transformations was shown by Lorentz and Poincaré before the formulation the special theory of relativity. This invariance of form or *covariance* of the Maxwell and Lorentz force equation implies that the various quantities $\varrho, \mathbf{j}, \mathbf{E}, \mathbf{B}$ that enter these equations transform in well-defined ways under Lorentz transformations. Then the terms of the equations can have consistent behavior under Lorentz transformations. (Jackson: *Classical Electrodynamics*, **3rd edition**, 1999, p. 553)

Assume that the equations of ED are covariant

$\downarrow$

Transformation rules

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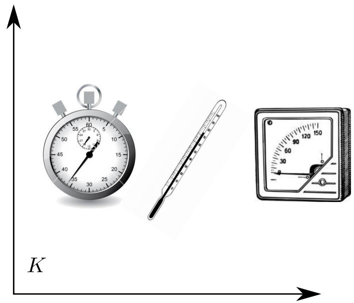
The standard treatment of covariance is a kind of question begging, until we have an independent verification of the transformation rules

Obtain the transformation rules without presuming covariance

$\downarrow$

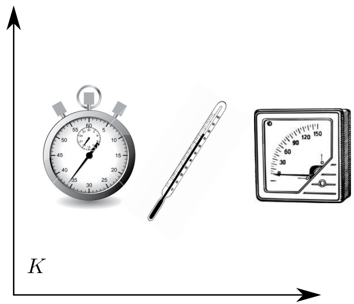
Covariance of the equations of ED

Physical quantities in K

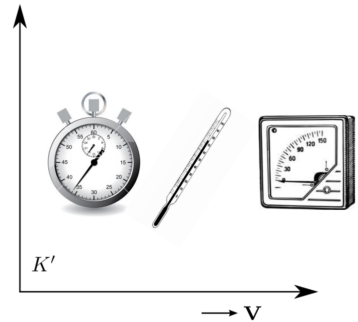


$\mathbf{E}, \mathbf{B}, \varrho, \mathbf{j}, \ldots$

Physical quantities in K'

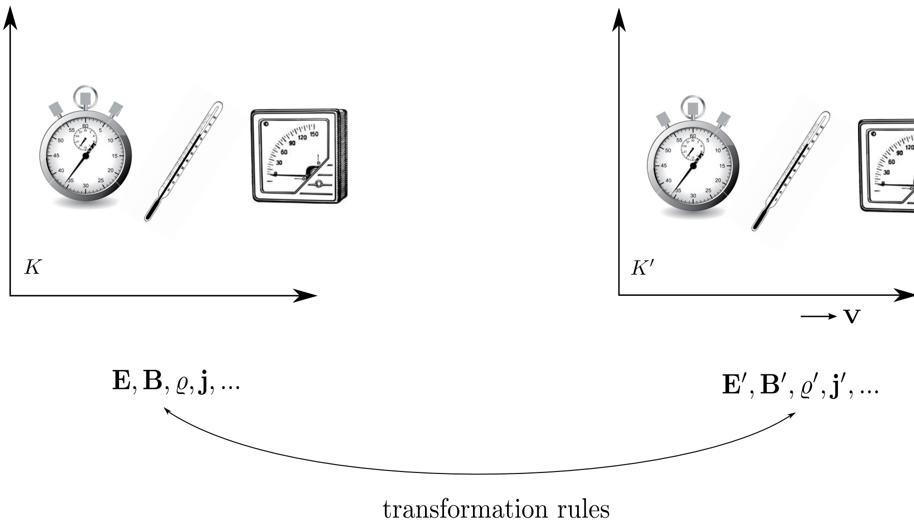


$\mathbf{E}, \mathbf{B}, \varrho, \mathbf{j}, \ldots$

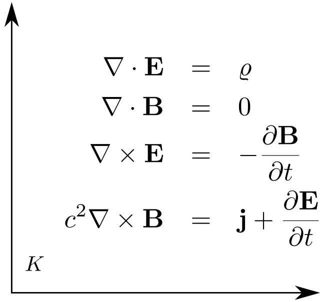


$\mathbf{E}', \mathbf{B}', \varrho', \mathbf{j}', \ldots$

Transformation rules



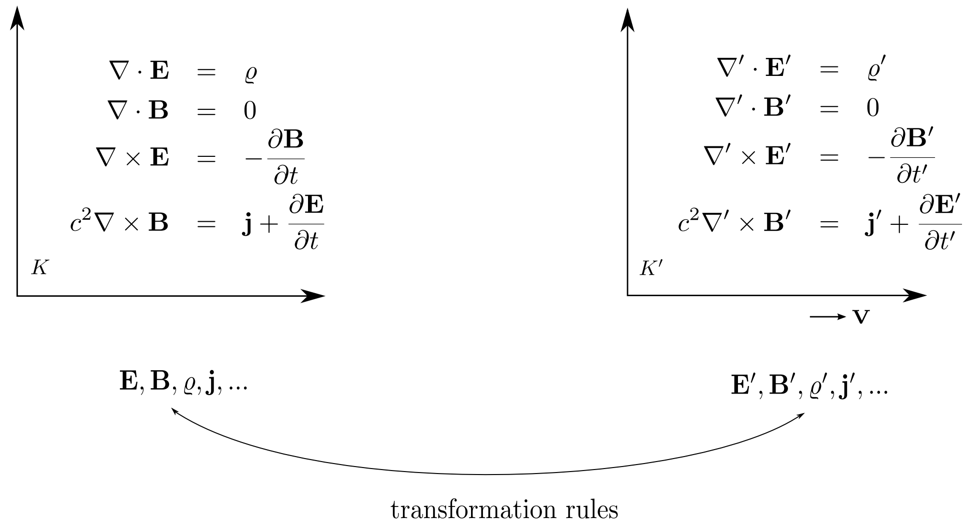
Equations in K



$$
\begin{aligned}
\nabla \cdot \mathbf{E} &= \varrho \\
\nabla \cdot \mathbf{B} &= 0 \\
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\
c^2 \nabla \times \mathbf{B} &= \mathbf{j} + \frac{\partial \mathbf{E}}{\partial t}
\end{aligned}
$$

$\mathbf{E}, \mathbf{B}, \varrho, \mathbf{j}, \ldots$

Covariance



$\nabla \cdot \mathbf{E} = \varrho$
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$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$
$c^2 \nabla \times \mathbf{B} = \mathbf{j} + \frac{\partial \mathbf{E}}{\partial t}$
K
$\nabla' \cdot \mathbf{E}' = \varrho'$
$\nabla' \cdot \mathbf{B}' = 0$
$\nabla' \times \mathbf{E}' = -\frac{\partial \mathbf{B}'}{\partial t'}$
$c^2 \nabla' \times \mathbf{B}' = \mathbf{j}' + \frac{\partial \mathbf{E}'}{\partial t'}$
K'
$\mathbf{v}$
$\mathbf{E}, \mathbf{B}, \varrho, \mathbf{j}, \ldots$
$\mathbf{E}', \mathbf{B}', \varrho', \mathbf{j}', \ldots$
transformation rules

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Obtain the transformation rules without presuming covariance

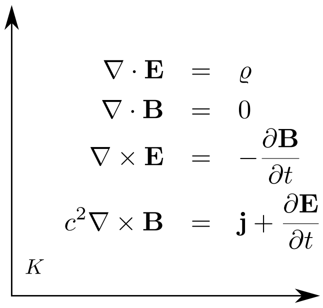
$\downarrow$

Covariance of the equations of ED

Lorentzian pedagogy

"the laws of physics in any *one* reference frame account for all physical phenomena, including the observations of moving observers"
(J.S. Bell: *How to teach special relativity*)

Lorentzian pedagogy



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$$\mathbf{E}, \mathbf{B}, \varrho, \mathbf{j}, \ldots$$

Lorentzian pedagogy

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c^2 \nabla \times \mathbf{B} &= \mathbf{j} + \frac{\partial \mathbf{E}}{\partial t}
\end{aligned}$$

K

$\longrightarrow \mathbf{v}$

$\mathbf{E}, \mathbf{B}, \varrho, \mathbf{j}, \ldots$

$\mathbf{E}', \mathbf{B}', \varrho', \mathbf{j}', \ldots$

Lorentzian pedagogy

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c^2 \nabla \times \mathbf{B} &= \mathbf{j} + \frac{\partial \mathbf{E}}{\partial t}
\end{aligned}
$$

$\longrightarrow \mathbf{v}$

K

$\mathbf{E}, \mathbf{B}, \varrho, \mathbf{j}, \ldots$

$\mathbf{E}', \mathbf{B}', \varrho', \mathbf{j}', \ldots$

transformation rules

Derive the transformation rules
from the equations of ED in one single frame

$\downarrow$

Covariance of the equations of ED

Derive the transformation rules

from the equations of ED in one single frame

$\downarrow$

Covariance of the equations of ED

The Lorentzian pedagogy derivation is a consistency check of the standard treatment of covariance

Lorentzian pedagogy

$$\begin{aligned}
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\end{aligned}$$

K

$\longrightarrow \mathbf{v}$

$\mathbf{E}, \mathbf{B}, \varrho, \mathbf{j}, \ldots$

$\mathbf{E}', \mathbf{B}', \varrho', \mathbf{j}', \ldots$

Sketch of a possible construction

1. Kinematic notions are already defined
2. Choose a standard test particle
3. $\mathbf{E}$:= acceleration of the standard test particle at rest, $\mathbf{B}$:= ...
4. $\varrho := \nabla \cdot \mathbf{E} \quad \mathbf{j} := c^2 \nabla \times \mathbf{B} - \partial_t \mathbf{E}$
5. Charged point-particle:
   (a) $\frac{d}{dt} \frac{\mathbf{v}}{\sqrt{1-\frac{\mathbf{v}^2}{c^2}}} = \pi \left(\mathbf{E} + \mathbf{v} \times \mathbf{B} \right)$
   (b) $\varrho \left(\mathbf{x}, t \right) = \alpha \delta \left(\mathbf{x} - \mathbf{r} \left(t \right) \right) \quad \mathbf{j} \left(\mathbf{x}, t \right) = \alpha \delta \left(\mathbf{x} - \mathbf{r} \left(t \right) \right) \mathbf{v} \left(t \right)$
6. $\mu := \alpha / \pi$

Laws of ED

We have defined: $\mathbf{E}$, $\mathbf{B}$, ϱ, $\mathbf{j}$, charged point-particle, α, π, μ

(L1)

$$\begin{aligned} \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{E} + \partial_t \mathbf{B} &= 0 \end{aligned}$$

(L2) Each particle is a charged point-particle with some π and α.

(L3) If there are n particles in a given space-time region, then the source densities are:

$$\begin{aligned} \varrho(\mathbf{x}, t) &= \sum_{i=1}^{n} \alpha_i \delta(\mathbf{x} - \mathbf{r}_i(t)) \\ \mathbf{j}(\mathbf{x}, t) &= \sum_{i=1}^{n} \alpha_i \delta(\mathbf{x} - \mathbf{r}_i(t)) \mathbf{v}_i(t) \end{aligned}$$

Putting laws (L1)–(L3) together, we have the coupled Maxwell–Lorentz equations:

$$\begin{aligned}
\nabla \cdot \mathbf{E}(\mathbf{x},t) &= \sum_{i=1}^{n} \alpha_i \delta(\mathbf{x}-\mathbf{r}_i(t)) \\
c^2 \nabla \times \mathbf{B}(\mathbf{x},t) - \partial_t \mathbf{E}(\mathbf{x},t) &= \sum_{i=1}^{n} \alpha_i \delta(\mathbf{x}-\mathbf{r}_i(t))\, \mathbf{v}_i(t) \\
\nabla \cdot \mathbf{B}(\mathbf{x},t) &= 0 \\
\nabla \times \mathbf{E}(\mathbf{x},t) + \partial_t \mathbf{B}(\mathbf{x},t) &= 0 \\
\frac{d}{dt} \frac{\mathbf{v}_i(t)}{\sqrt{1-\frac{\mathbf{v}_i(t)^2}{c^2}}} &= \pi_i \left\{ \mathbf{E}(\mathbf{r}_i(t), t) + \mathbf{v}_i(t) \times \mathbf{B}(\mathbf{r}_i(t), t) \right\}
\end{aligned}$$

$$(i = 1, 2, \ldots n)$$

Lorentzian pedagogy

$$
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c^2 \nabla \times \mathbf{B} &= \mathbf{j} + \frac{\partial \mathbf{E}}{\partial t}
\end{aligned}
$$

K

$\longrightarrow \mathbf{v}$

$\mathbf{E}, \mathbf{B}, \varrho, \mathbf{j}, \ldots$

$\mathbf{E}', \mathbf{B}', \varrho', \mathbf{j}', \ldots$

Operational definitions of the ED quantities in K'

1. Kinematic notions are already defined—the measuring rods and clocks are at rest relative to K'

2. Take an identical copy of the standard test particle

3. $\mathbf{E}' :=$ (acceleration)′ of the standard test particle at (rest)′, $\mathbf{B}' := ...$

4. $\varrho' := \nabla \cdot \mathbf{E}' \quad \mathbf{j}' := c^2 \nabla \times \mathbf{B}' - \partial'_t \mathbf{E}'$

5. (Charged point-particle)′:

   (a) $\frac{d}{dt'} \frac{\mathbf{v}'}{\sqrt{1-\frac{\mathbf{v}'^2}{c^2}}} = \pi' \left(\mathbf{E}' + \mathbf{v}' \times \mathbf{B}' \right)$

   (b) $\varrho' \left(\mathbf{x}', t' \right) = \alpha' \delta \left(\mathbf{x}' - \mathbf{r}' \left(t' \right) \right) \quad \mathbf{j}' \left(\mathbf{x}', t' \right) = \alpha' \delta \left(\mathbf{x}' - \mathbf{r}' \left(t' \right) \right) \mathbf{v}' \left(t' \right)$

6. $\mu' := \alpha' / \pi'$

Observations of moving observer—applying the Lorentzian pedagogy

Long long *calculations* based on the "primed" definitions, the laws of ED in K, and the kinematic Lorentz transformations. The typical schema:

1. From the "primed" definition we know the *physical situation during the operational procedure*
2. We can describe this situation in terms of the quantities in K (using *the kinematic* Lorentz transformation)
3. We *calculate* the acceleration of the standard test particle in K, applying the *Lorentz equation of motion in K*
4. We can apply (backward) the Lorentz transformation of acceleration

Theorem 1.

$$
\begin{aligned}
E'_x &= E_x \\
E'_y &= \frac{E_y - VB_z}{\sqrt{1 - \frac{V^2}{c^2}}} \\
E'_z &= \frac{E_z + VB_y}{\sqrt{1 - \frac{V^2}{c^2}}}
\end{aligned}
$$

Theorem 2.

$$
\begin{aligned}
B'_x &= B_x \\
B'_y &= \frac{B_y + \frac{V}{c^2} E_z}{\sqrt{1 - \frac{V^2}{c^2}}} \\
B'_z &= \frac{B_z - \frac{V}{c^2} E_y}{\sqrt{1 - \frac{V^2}{c^2}}}
\end{aligned}
$$

Theorem 3.

$$
\begin{aligned}
\varrho' &= \frac{\varrho - \frac{V}{c^2} j_x}{\sqrt{1 - \frac{V^2}{c^2}}} \\
j_x' &= \frac{j_x - V\varrho}{\sqrt{1 - \frac{V^2}{c^2}}} \\
j_y' &= j_y \\
j_z' &= j_z
\end{aligned}
$$

Theorem 4.

$$\textit{charged point-particle} \Leftrightarrow \textit{(charged point-particle)}'$$

such that $\pi' = \pi$ *and* $\alpha' = \alpha$.

Proof

1. Lorentz equation of motion: Situation in K'—$\mathbf{E}', \mathbf{B}', \mathbf{v}', \mathbf{r}', t' \Rightarrow$ Situation in K—$\mathbf{E}, \mathbf{B}, \mathbf{v}, \mathbf{x}, t \Rightarrow$ Acceleration in K (from the Lorentz eq.) $\Rightarrow$ Acceleration in K', and one finds the (Lorentz eq.)′

2. Densities: trajectory in $K' \overset{kin.LT}{\Rightarrow}$ trajectory in $K \Rightarrow$ densities in K $\overset{densities'\,T}{\Rightarrow}$ densities in K', and one finds:

$$\begin{aligned}
\varrho'\left(\mathbf{x}', t'\right) &= \alpha\delta\left(\mathbf{x}' - \mathbf{r}(t')\right) \\
\mathbf{j}'\left(\mathbf{x}', t'\right) &= \alpha\left(\mathbf{x}' - \mathbf{r}(t')\right)\mathbf{v}(t')
\end{aligned}$$

Theorem 5.

$$\begin{aligned} \nabla \cdot \mathbf{B}' &= 0 \\ \nabla \times \mathbf{E}' + \partial_{t'} \mathbf{B}' &= 0 \end{aligned}$$

Theorem 6. *If there are n particles in a given space-time region, then the (source densities)′ are:*

$$\begin{aligned} \varrho'\left(\mathbf{x}',t'\right) &= \sum_{i=1}^{n}\alpha_i'\delta\left(\mathbf{x}'-\mathbf{r}_i'\left(t'\right)\right) \\ \mathbf{j}'\left(\mathbf{x}',t'\right) &= \sum_{i=1}^{n}\alpha_i'\delta\left(\mathbf{x}'-\mathbf{r}_i'\left(t'\right)\right)\mathbf{v}_i'\left(t'\right) \end{aligned}$$

Putting all these together:

$$
\begin{aligned}
\nabla \cdot \mathbf{E}'\left(\mathbf{x}', t'\right) &= \sum_{i=1}^{n} \alpha_i' \delta\left(\mathbf{x}' - \mathbf{r}_i'\left(t'\right)\right) \\
c^2 \nabla \times \mathbf{B}'\left(\mathbf{x}', t'\right) - \partial_{t'} \mathbf{E}'\left(\mathbf{x}', t'\right) &= \sum_{i=1}^{n} \alpha_i' \delta\left(\mathbf{x}' - \mathbf{r}_i'\left(t'\right)\right) \mathbf{v}_i'\left(t'\right) \\
\nabla \cdot \mathbf{B}'\left(\mathbf{x}', t'\right) &= 0 \\
\nabla \times \mathbf{E}'\left(\mathbf{x}', t'\right) + \partial_{t'} \mathbf{B}'\left(\mathbf{x}', t'\right) &= 0 \\
\frac{d}{dt'} \frac{\mathbf{v}_i'(t')}{\sqrt{1 - \frac{\mathbf{v}_i'(t')^2}{c^2}}} &= \pi_i' \left\{ \mathbf{E}'\left(\mathbf{r}_i'\left(t'\right), t'\right) + \mathbf{v}_i'(t') \times \mathbf{B}'\left(\mathbf{r}_i'\left(t'\right), t'\right) \right\} \\
& \qquad\qquad\qquad (i = 1, 2, \ldots n)
\end{aligned}
$$

What we proved

- The transformation rules derivable from the laws of ED in one single frame are identical with the textbook transformations—derived from the presumption of covariance
- The Maxwell–Lorentz equations are covariant against the transformation rules derivable from the laws of ED in one single frame

In the derivation we essentially made use of:

- the relativistic version of Lorentz's equation
- the kinematic Lorentz transformations
- the operational definitions of the ED quantities

The principle of covariance is *consistent* with ED in one single frame—ED including all the above ingredients