On a meaningful axiomatic joint derivation of the Doppler effect and the Lorentz-FitzGerald Contraction

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The Logic, Relativity and Beyond 3rd conference
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Invariance of the form of a law over changes of units

The mathematical expression of a scientific or geometric law typically does not depend on the units of measurement.

Example: The Pythagorean Theorem.
The square of the hypotenuse of a right triangle is equal to the sum of the squares of the other two sides.

This makes sense because measurement units have no representation in nature.

Any mathematical model or law whose form would be fundamentally altered by a change of units would be a poor representation of the empirical world.

About the meaningfulness axiom

When properly formalized, this

invariance under changes of units

becomes a powerful '*meaningfulness*' axiom.

Basic idea. Combining this meaningfulness axiom with abstract, intuitive, 'gedanken' type properties, such as:

associativity, permutability, bisymmetry, homogeneity, transitivity

or some other conditions in the same vein, enables the derivation of physical or geometrical laws (possibly up to some numerical parameter values).

The four parts of this talk. (Sorry, but no proofs.)

1. Defining meaningfulness (for functions of two ratio scales variables)
2. Two introductory examples of meaningful derivation:
   - The Pythagorean Theorem
   - Beer's law
3. List of similar results $\left\{\begin{array}{l}\text{Parts 1, 2 and 3 summarizes the content of} \\ \text{our book: \textit{On Meaningful Scientific Laws,}} \\ \text{Falmagne and Doble, Springer, 2015}\end{array}\right.$
4. Joint derivation of the relativistic Doppler effect and the Lorentz-FitzGerald Contraction (up to some exponents) from the meaningfulness axiom and an abstract, intuitive axiom

$$[\mathrm{R}] \qquad L(L(\lambda, v), w) = L(\lambda, v \oplus w).$$

L and λ	measure wavelengths,
v and w	measure some speeds,
$\oplus$	is an abstract representation of the relativistic addition of velocities.

Motivating the meaningfulness axiom

The trouble with the equation

$$L(\ell, v) = \ell\sqrt{1 - \left(\frac{v}{c}\right)^2},$$

representing the Lorentz-FitzGerald Contraction, is its ambiguity. Writing $L(70, 3)$ has no empirical meaning if one does not specify, for example, that the pair $(70, 3)$ refers to 70 meters and 3 kilometers per second, respectively.

Such a parenthetical reference is standard in a scientific context, but it is not instrumental for our purpose, which is to express, formally, an invariance with respect to any change in the units.

Motivating the meaningfulness axiom

To rectify the ambiguity, we regard

$$L(\ell, v) \qquad \text{to be a shorthand notation for} \qquad L_{1,1}(\ell, v),$$

in which ℓ and L on the one hand, and v on the other hand, are measured in terms of two particular *initial* or *anchor* units fixed arbitrarily.

Describing the phenomenon in terms of other units means defining a new function

$$L_{\alpha,\beta}, \qquad \text{which is different from} \qquad L = L_{1,1};$$

but, from an empirical standpoint, $L_{\alpha,\beta}$ carries exactly the same information as $L_{1,1}$.

(The first paper with that idea by Falmagne and Narens, 1983).

Motivating the meaningfulness axiom

The connection between L and $L_{\alpha,\beta}$ is actually:

$$\frac{1}{\alpha} L_{\alpha,\beta}(\alpha\ell, \beta v) = L(\ell, v)$$

because we can derive

$$\frac{1}{\alpha} L_{\alpha,\beta}(\alpha\ell, \beta v) = \alpha \left(\frac{1}{\alpha}\right) \ell \sqrt{1 - \left(\frac{\beta v}{\beta c}\right)^2} = \ell \sqrt{1 - \left(\frac{v}{c}\right)^2}. \qquad (\square)$$

This implies, for any α, β, ν and μ in $\mathbb{R}_{++}$,

$$\boxed{\frac{1}{\alpha} L_{\alpha,\beta}(\alpha\ell, \beta v) = \frac{1}{\nu} L_{\nu,\mu}(\nu\ell, \mu v).}$$

This is a special case of the invariance axiom that we were looking for.

Objections?

Looking at the equation

$$\frac{1}{\alpha} L_{\alpha,\beta}(\alpha\ell, \beta v) = \frac{1}{\nu} L_{\nu,\mu}(\nu\ell, \mu v). \qquad (*)$$

one might object that going in that direction would render the scientific or geometric notation very complicated.

But the complication is only temporary. When we have extracted all the useful consequences from the meaningfulness axiom, we can go back to the usual notation.

This is straightforward because

$$\tfrac{1}{\alpha} L_{\alpha,\beta}(\alpha\ell, \beta v) = L_{1,1}(\ell, v) = L(\ell, v).$$

The meaningfulness axiom for two ratio scales variables

This example makes clear that the **concept of meaningfulness** must apply to a **collection** of scientific or geometric functions (we call them *codes*), and not to a particular function.

Definition

Let J_1, J_2, and J_3 be three non-negative, real intervals.
Suppose that

$$\mathcal{F} = \{F_{\alpha,\beta} \,|\, \alpha, \beta \in \mathbb{R}_{++}\} \qquad (\mathbb{R}_{++} =]0, \infty[)$$

is a collection of *codes*, with for the *initial code F*

$$F = F_{1,1} : J_1 \times J_2 \overset{onto}{\longrightarrow} J_3 \,.$$

Each of α and β indexing a code $F_{\alpha,\beta}$ in $\mathcal{F}$ represents a change of the unit of one of the two measurement scales. The intervals J_1, J_2, and J_3 **may be changed** correspondingly.

The meaningfulness axiom: the self transforming case

In some important cases, as in all the examples of this talk,
the unit of the code is the same as that of its first variable.

Continuation

The collection of codes $\mathcal{F}$ is *self transforming meaningful*, or *ST-meaningful,* if for any $(x_1, x_2) \in J_1 \times J_2$ and $(\alpha, \beta), (\mu, \nu) \in \mathbb{R}^2_{++}$:

$$\frac{1}{\alpha} F_{\alpha,\beta}(\alpha x_1, \beta x_2) = \frac{1}{\mu} F_{\mu,\nu}(\mu x_1, \nu x_2) \qquad (**)$$

which yields $\qquad F_{\alpha,\beta}(\alpha x_1, \beta x_2) = \alpha F_{1,1}(x_1, x_2) = \alpha F(x_1, x_2).$

Equation $(**)$ generalizes the Equation $(*)$ that you have seen.
For completeness: In the general case of n ratio scale variables and not self-transforming, Equation $(**)$ becomes

$$\boxed{\frac{1}{\prod_{i=1}^{n} \alpha_i^{\delta_i}} F_{\boldsymbol{\alpha}}(\alpha_1 x_1, \ldots, \alpha_n x_n) = \frac{1}{\prod_{i=1}^{n} \mu_i^{\delta_i}} F_{\boldsymbol{\mu}}(\mu_1 x_1, \ldots, \mu_n x_n).}$$

As an introduction: the Pythagorean Theorem.

In this talk, we use the meaningfulness axiom
together with some some abstract, possibly intuitive axioms,
to derive some scientific or geometrical laws.

These developments can be applied to various abstract axioms.
One example is the *associativity equation:*

$$F(F(x,y),z) = F(x,F(y,z))$$

which can be shown to hold for right triangles by a simple argument.

In this equation, each of the terms

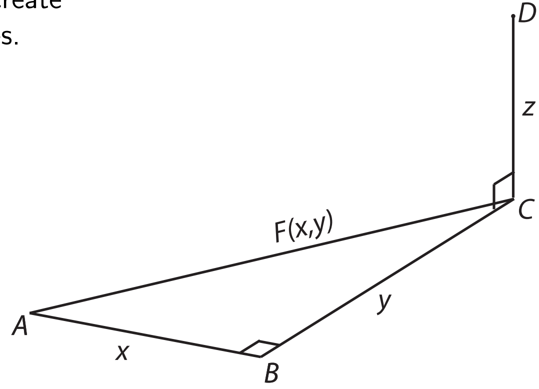
$$F(x,y), \quad F(y,z), \quad F(F(x,y),z) \quad \text{and} \quad F(x,F(y,z)) \qquad (*)$$

denotes the length of the hypothenuse of a right triangle as a function of the length of its two sides.

The Pythagorean Theorem.

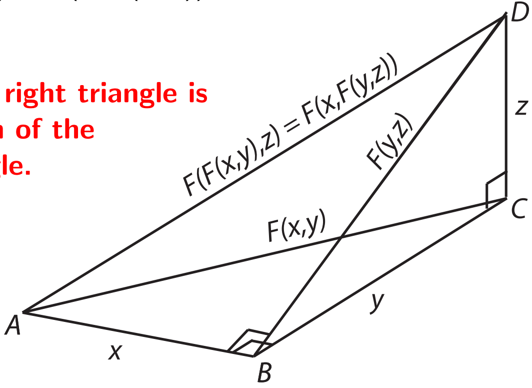
The *associativity equation* $F(F(x,y),z) = F(x,F(y,z))$.

Draw the right triangle $\triangle ABC$ with hypothenuse AC of length $F(x,y)$, with x and y denoting the length of the two sides of the right angle.
Then draw the perpendicular CD of length z.
This will automatically create
three more right triangles.



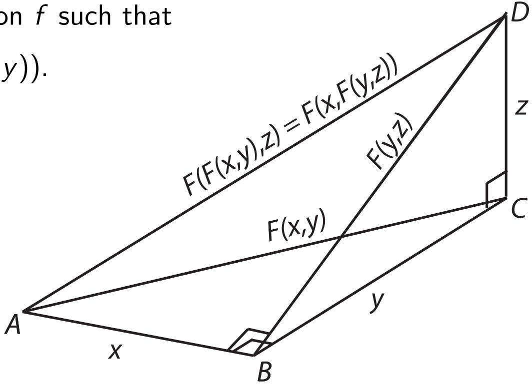
The three new right triangles are $\triangle BCD$, $\triangle ABD$, and $\triangle ACD$. The right triangles, $\triangle ABD$ and $\triangle ACD$, have the same hypothenuse AC, with length $F(F(x,y),z) = F(x,F(y,z))$.

We conclude that:
The hypothenuse of a right triangle is an associative function of the two sides of the triangle.



Using standard functional equations methods (cf. Aczél, 1966), you can show that under some weak general conditions (such as continuity at one point) the associativity equation implies the existence of a function f such that

$$F(x,y) = f^{-1}(f(x)+f(y)).$$



The Pythagorean Theorem.

Injecting now meaningfulness, we get the following

Theorem

Suppose that $\mathcal{F} = \{F_\alpha \,|\, \alpha \in \mathbb{R}_{++}\}$ is a ST-meaningful collection of codes, with $F_\alpha : \mathbb{R}_{++} \times \mathbb{R}_{++} \xrightarrow{onto} \mathbb{R}_{++}$ for all α in $\mathbb{R}_{++}$. If one of these codes is strictly increasing in both variables, symmetric, homogeneous and associative, then any code $F_\alpha \in \mathcal{F}$ must have the form

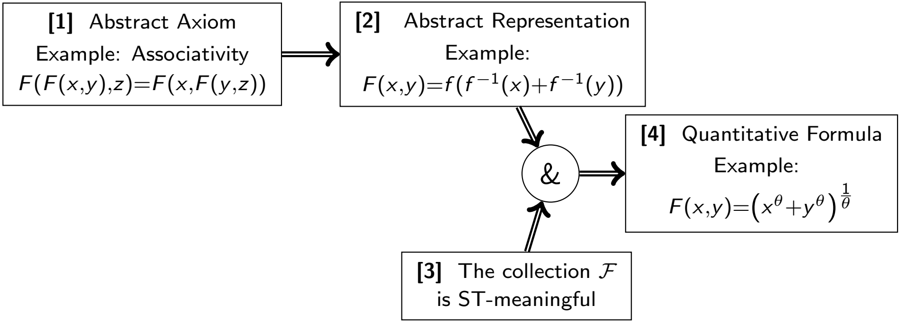
$$F_\alpha(x, y) = \left(x^\theta + y^\theta\right)^{\frac{1}{\theta}} = F(x, y)\,,$$

for some constant $\theta \in \mathbb{R}_{++}$.

For a proof, see Falmagne and Doble, 2015, Theorem 7.1.1, page 85.
Using a simple geometric argument, you can show that $\theta = 2$, which gives us another proof of the Pythagorean Theorem[1].

[1] To be added to the 367 proofs in Elisha Scott Loomis book.

Proof Schema: $([1] \implies [2])\ \&\ [3]) \implies [4]$



Proofs schema: An abstract axiom yields an abstract representation. The latter, paired with a meaningfulness condition leads, via functional equation arguments, to one or a couple of potential scientific laws specified up to the value(s) of numerical parameter(s).

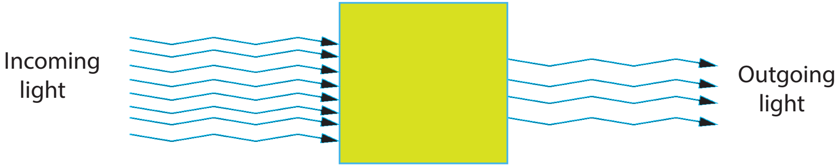
The abstract axiom(s) should be **intuitively cogent**, or **easy to prove**.

Another example: The Translation Equation for Beer's law

Beer's law is the equation

$$F(x, y) = x\,\mathrm{e}^{-\frac{y}{c}}$$

which describes the attenuation of light resulting from the properties of the material through which the light is traveling. The constant c measures the concentration of the material.



Following the guidelines of the Proof Schema, we first formulate the abstract axiom.

Another example: The Translation Equation for Beer's law

Definition

Let J and J' be two non-negative real intervals. A code $F : J \times J' \to J$ is *translatable*, (cf. Aczél, 1966, page 245) if

$$F(F(x,y),z) = F(x, y+z) \qquad (x \in J,\ y, z, y+z \in J'). \tag{1}$$

Functional equation lemma in Aczel's book

Let $F : J \times J' \to H$ be a code such that J and J' are half-open interval ($J' =]d, \infty[$ for some $d \in \mathbb{R}_+$, and for some $a \in \mathbb{R}_+$, either $J =]a, b]$ for some $b \in \mathbb{R}_{++}$ or $J =]a, \infty[$), with $F(x,y)$ strictly decreasing in y.

Then, the code $F : J \times J' \to H$ is translatable if and only if there exists a function f satisfying the equation

$$F(x,y) = f(f^{-1}(x) + y).$$

Another example: The Translation Equation for Beer's law

Injecting now the meaningfulness condition, we obtain:

Theorem

Let $\mathcal{F} = \{F_{\mu,\nu} \,|\, \mu, \nu \in \mathbb{R}_{++}\}$ *be a ST-meaningful collection of codes, with* $F_{\mu,\nu} : \mathbb{R}_{++} \times \mathbb{R}_{++} \xrightarrow{onto} \mathbb{R}_{++}$. *Suppose that one of these codes, say the code* $F_{\mu,\nu}$, *is:*

- *strictly decreasing in the second variable,*
- *translatable, and*
- *left homogeneous of degree one, that is: for any a in* $\mathbb{R}_{++}$, *we have* $F_{\mu,\nu}(ax, y) = aF_{\mu,\nu}(x, y)$.

Then there is a constant $c > 0$ *such that the initial code F has the form*

$$F(x, y) = x\,\mathrm{e}^{-\frac{y}{c}};$$

so for any code $F_{\alpha,\beta} \in \mathcal{F}$, *we have*

$$F_{\alpha,\beta}(x, y) = x\,\mathrm{e}^{-\frac{y}{\beta c}}.$$

Examples of results for other abstract conditions

Name and formula of abstract axiom	Abstract representation[1]: $\exists$ functions f, m, g, etc.	Resulting meaningful scientific laws[2]
Associativity $F(F(x,y),z)=F(x,F(y,z))$	$F(x,y)=f(f^{-1}(x)+f^{-1}(y))$	$F(x,y)=(y^{\theta}+x^{\theta})^{\frac{1}{\theta}}$
Translatability $F(F(x,y),z)=F(x,y+z)$	$F(x,y)=f(f^{-1}(x)+y)$	Beer's Law $F(x,y)=x\mathrm{e}^{-\frac{y}{c}}$
Quasi-permutability $F(G(x,y),z)=F(G(x,z),y)$	$F(x,y)=m(f(x)+g(y))$	$F(x,y)=(x^{\eta}+\lambda y^{\eta}+\theta)^{\frac{1}{\eta}}$ or $F(x,y)=\phi x y^{\gamma}$ or $(x^{\eta}+y^{\eta})^{\frac{1}{\eta}}$
Bisymmetry $F(F(x,y),F(z,w))=F(F(x,z),F(y,w))$	$F(x,y)=f\big((1-q)f^{-1}(x)+qf^{-1}(y)\big)$	$F(x,y)=\big(1-q)x^{\theta}+qy^{\theta}\big)^{\frac{1}{\theta}}$ or $F(x,y)=x^{1-q}y^{q}$

[1] Aczél, J. *Lectures on Functional Equations and their Applications*, Academic Press, 1966, for the functional equations results in column 2.

[2] Falmagne, J.-Cl. and Doble, C.W., *On Meaningful Scientific Laws*, Springer, 2015.

Meaningful joint derivation of the Lorentz-FitzGerald Contraction and the relativistic Doppler Function

In the previous examples, we uncovered the abstract axioms and their abstract representations by looking in the functional equations literature—e.g. in Janos Aczél's famous book.

In the case of the Doppler Effect or the Lorentz-FitzGerald Contraction however, we had to invent one or more abstract conditions that would seem to fit the situation (cf. Falmagne and Doignon, 2010). This is how I came up with the two abstract axioms:

$$\text{[R]} \qquad L(L(\lambda, v), w) = L(\lambda, v \oplus w),$$

$$\text{[M]} \qquad L(\lambda, v) \leq L(\lambda', v') \iff L(\lambda, v \oplus w) \leq L(\lambda', v' \oplus w).$$

Moreover, we did not have the abstract representations of these abstract axioms. We had to obtain such a representation.

Five conditions on LFD Functions and abstract LFD-pairs

LFD is an acronym for Lorentz-FitzGerald-Doppler

Some natural, basic conditions on L and $\oplus$.

Definition

Let $L : \mathbb{R}_{++} \times [0, c[\to \mathbb{R}_{++}$ be a *code*, with $c > 0$ a standing for the speed of light. The code L is a *LFD Function* if there is a binary operator $\oplus : [0, c[\times [0, c[\to [0, c[$ such that the pair $(L, \oplus)$ satisfies the following five conditions:

1. The function L is strictly increasing in the first variable, strictly decreasing in the second variable, continuous in both variables, and we have

$$L(\lambda, v) \leq L(\lambda', v') \iff L(a\lambda, v) \leq L(a\lambda', v').$$

for all $\lambda, \lambda' \in \mathbb{R}_{+}$ and $v, v' \in [0, c]$, and for any $a > 0$.
In words: The order does not change when the magnitude of the wave is multiplied by a positive constant.

Basic conditions on LFD Functions and abstract LFD-pairs

Continuation

2. $L(\lambda, 0) = \lambda$ *for all* $\lambda \in \mathbb{R}_+$. **No effect if** $v = 0$
3. $\lim_{v \to c} L(\lambda, v) = 0$. **At the limit, the object disappears.**
4. The operation $\oplus$ is continuous, strictly increasing in both variables, and has 0 as an identity element.
5. **Either** Axiom [R] or Axiom [M] below is satisfied:

[R] $L(L(\lambda, v), w) = L(\lambda, v \oplus w) \qquad (\lambda > 0, \; \textit{and } v, \; w \in [0, c[).$

[M] $L(\lambda, v) \le L(\lambda', v') \iff L(\lambda, v \oplus w) \le L(\lambda', v' \oplus w)$

$$(\lambda, \lambda' > 0, \; \textit{and } v, \; v', \; w \in [0, c[).$$

When these five conditions are satisfied, the pair $(L, \oplus)$ is called an *abstract LFD-pair.* All five conditions are natural ones for both the Lorentz-FitzGerald Contraction and the Doppler effect.

Representation Theorem for abstract LFD-pairs

Theorem

Suppose that $(L, \oplus)$ is an abstract LFD-pair. Then the following equivalences hold:

$$[\mathrm{R}] \iff ([\mathrm{DE}^\dagger] \;\&\; [\mathrm{AV}^\dagger]) \iff [\mathrm{M}],$$

with for some strictly increasing function u and some positive constant ξ:

$$[\mathrm{DE}^\dagger] \qquad L(\lambda, v) = \lambda \left(\frac{c - u(v)}{c + u(v)} \right)^{\xi}$$

$$[\mathrm{AV}^\dagger] \qquad v \oplus w = u^{-1} \left(\frac{u(v) + u(w)}{1 + \frac{u(v)u(w)}{c^2}} \right).$$

For the proof, see Falmagne, J.-Cl. and Doignon, J.-P., *Aequationes Mathematicae*, 80: 85-99, 2010.

Propagation lemma for LFD functions

Lemma

Suppose that one ordered pair from a meaningful LFD-system $(\mathcal{L}, \mathcal{O})$ is an abstract LFD-pair, that is, it satisfies Conditions 1-5 of the definition of an abstract LFD-pair. Then any ordered pair $(L_{\alpha,\beta}, \oplus_\beta)$, with $L_{\alpha,\beta} \in \mathcal{L}$ and $\oplus_\beta \in \mathcal{O}$, is also such an abstract LFD-pair.

So, meaningfulness enables the propagation of all five conditions to all the ordered pairs $(L_{\alpha,\beta}, \oplus_\beta)$ in a system $(\mathcal{L}, \mathcal{O})$.

We omit the simple proof of that lemma.

Representation Theorem for the LFD-system

Theorem

Suppose that one ordered pair $(L_{\mu,\nu}, \oplus_\nu)$ *from a meaningful LFD-system* $(\mathcal{L}, \mathcal{O})$ *is an abstract LFD-pair, that is,* $(L_{\mu,\nu}, \oplus_\nu)$ *satisfies Conditions* 1-5 *of the definition of an abstract LFD-pair.*

1. *If* $L_{\mu,\nu}(\lambda, v)$ *does not vary with* ν*, then the function u in Axioms* [DE†] and [AV†] *is the identity function. So these axioms become for the initial code L, for some positive constant* ξ*:*

[D] $$L(\lambda, v) = \lambda \left(\frac{c - v}{c + v} \right)^{\xi}$$

and for the operation $\oplus$*:*

[AV] $$v \oplus w = \frac{v + w}{1 + \frac{vw}{c^2}},$$

the standard representation for the relativistic addition of velocities.

Representation Theorem for the LFD-system

Continuation

2. Suppose that $L_{\mu,\nu}(\lambda, v)$ varies with ν. Then different forms of Axioms $[\mathrm{DE}^{\dagger}]$ and $[\mathrm{AV}^{\dagger}]$ are possible, which are, for the initial code:

$$[\overrightarrow{\mathrm{LF}}] \qquad L(\lambda, v) = \lambda \left(1 - \left(\frac{v}{c}\right)^{\psi}\right)^{\xi}$$

for some positive constants ξ and ψ. (So, with $\xi = \frac{1}{2}$ and $\psi = 2$, $[\overrightarrow{\mathrm{LF}}]$ becomes $[\mathrm{LF}]$, the Lorentz-FitzGerald Contraction.)

For the initial $\oplus$ operation, this implies that:

$$[\overrightarrow{\mathrm{AV}}] \qquad v \oplus w = c\left(\left(\frac{v}{c}\right)^{\psi} - \left(\frac{v}{c}\right)^{\psi}\left(\frac{w}{c}\right)^{\psi} + \left(\frac{w}{c}\right)^{\psi}\right)^{\frac{1}{\psi}}.$$

Conjecture

If $L_{\mu,\nu}(\lambda, v)$ varies with ν, then $[\overrightarrow{\mathrm{LF}}]$ and $[\overrightarrow{\mathrm{AV}}]$ are the only possible forms of Axioms $[\mathrm{DE}^{\dagger}]$ and $[\mathrm{AV}^{\dagger}]$ for L and $\oplus$.

Final comments

Note that, with $\psi = 2$,

$$\overrightarrow{[\mathrm{AV}]} \qquad v \oplus w = c\left(\left(\frac{v}{c}\right)^{\psi} - \left(\frac{v}{c}\right)^{\psi}\left(\frac{w}{c}\right)^{\psi} + \left(\frac{w}{c}\right)^{\psi}\right)^{\frac{1}{\psi}}.$$

specializes into:

$$[\mathrm{AV}^{\star}] \qquad v \oplus w = c\sqrt{\left(\frac{v}{c}\right)^2 + \left(\frac{w}{c}\right)^2 - \left(\frac{v}{c}\right)^2\left(\frac{w}{c}\right)^2}.$$

an equation which arises in the case of perpendicular motions[a].

In fact, in Corollary 9 of Falmagne and Doignon (2010), we proved the implication

$$[\mathrm{LF}] \implies ([\mathrm{R}] \iff [\mathrm{AV}^{\star}] \iff [\mathrm{M}]).$$

So, we already knew a while ago that

[LF] is consistent with [R] and [M] and inconsistent with [AV].

[a] Cf. Ungar, American Journal of Physics, 1991, Equation (8).

But what does perpendicular motion have to do with the [LF] Equation?

Final comments

The last theorem also implies that the Lorentz-FitzGerald Contraction Equation is inconsistent with the standard Formula [AV] representing the relativistic addition of velocities.

One of the results in my paper with Doignon in Aequationes Mathematica (2010) is the implication

$$[\mathrm{AV}] \implies ([\mathrm{R}] \iff [\mathrm{DE}] \iff [\mathrm{M}]).$$

Accordingly, if the standard formula [AV] for the relativistic addition of velocities is assumed, then [LF] is also inconsistent with either of [R] or [M].

Final comments

The results presented here suggest a systematic investigation of abstract conditions that seem intuitively consonant to some physical or geometrical situations. Pairing such conditions with

1. their abstract functional equations representations
2. and then, their meaningful representations

in the style of the examples of this talk might, in the long term, generate an extensive catalogue of possible meaningful scientific laws.

Were such a catalogue to exist, it could be consulted by scientists searching for mathematical formalizations of phenomena about which they have some intuition.

Thank you!

SKETCH OF PROOF. We first show that, if one of the codes in the collection $\mathcal{F}$ is translatable, then by the meaningfulness condition, the translatability condition *propagates* to all the codes in the collection.
Without loss of generality, we suppose that the initial code $F = F_{1,1}$ is translatable.
Successively, we have for any code $F_{\alpha,\beta}$ in $\mathcal{F}$:

$$\begin{aligned}
F_{\alpha,\beta}(F_{\alpha,\beta}(x,y),z) &= \alpha F\left(\frac{F_{\alpha,\beta}(x,y)}{\alpha}, \frac{z}{\beta}\right) && \text{(by ST-meaningfulness)} \\
&= \alpha F\left(F\left(\frac{x}{\alpha}, \frac{y}{\beta}\right), \frac{z}{\beta}\right) && \text{(by ST-meaningfulness)} \\
&= \alpha F\left(\frac{x}{\alpha}, \frac{y}{\beta} + \frac{z}{\beta}\right) && \text{(by the translatability of } F\text{)} \\
&= F_{\alpha,\beta}(x, y+z) && \text{(by ST-meaningfulness)}\,.
\end{aligned}$$

So, $F_{\alpha,\beta}$ is translatable.

By meaningfulness, we can also show that left homogeneity of degree one propagates to all the codes in the collection $\mathcal{F}$. (We omit this part of the proof.)

Because $F_{\alpha,\beta}$ is translatable, Lemma **??** implies that there exists a strictly decreasing function $f_{\alpha,\beta} : \mathbb{R}_{++} \to \mathbb{R}_{++}$ such that

$$\begin{aligned} F_{\alpha,\beta}(ax, y) &= f_{\alpha,\beta}(f_{\alpha,\beta}^{-1}(ax) + y) \\ &= af_{\alpha,\beta}(f_{\alpha,\beta}^{-1}(x) + y) = aF_{\alpha,\beta}(x, y) \quad \left(\begin{array}{c}\text{by left homogeneity} \\ \text{of } F_{\alpha,\beta}\end{array}\right). \end{aligned}$$

Set $f_{\alpha,\beta}^{-1}(x) = w$, and so $f_{\alpha,\beta}(w) = x$. Applying $f_{\alpha,\beta}^{-1}$ on both sides of the second equation above, we get

$$(f_{\alpha,\beta}^{-1} \circ af_{\alpha,\beta})(w) + y = (f_{\alpha,\beta}^{-1} \circ af_{\alpha,\beta})(w + y),$$

or with $g_{a,\alpha,\beta} = (f_{\alpha,\beta}^{-1} \circ af_{\alpha,\beta})$,

$$g_{a,\alpha,\beta}(w) + y = g_{a,\alpha,\beta}(w + y),$$

a Pexider equation in the variables w and y.

So,the function $g_{a,\alpha,\beta}$ is of the form

$$g_{a,\alpha,\beta}(w) = w + B(a,\alpha,\beta).$$

for some function $B(a,\alpha,\beta)$ which must be decreasing in a. Rewriting the last equation in terms of the function $f_{\alpha,\beta}$ yields

$$(f^{-1}_{\alpha,\beta} \circ af_{\alpha,\beta})(w) = w + B(a,\alpha,\beta)$$

or equivalently, with $x = f_{\alpha,\beta}(w)$, we get

$$f^{-1}_{\alpha,\beta}(ax) = f^{-1}_{\alpha,\beta}(x) + B(a,\alpha,\beta),$$

another Pexider equation (c.f. Aczél, page 141) that is, an equation of the form: $h(ax) = h(x) + g(a)$. By functional equations arguments, the equation

$$f^{-1}_{\alpha,\beta}(ax) = f^{-1}_{\alpha,\beta}(x) + B(a,\alpha,\beta),$$

implies for some constants $k(\alpha,\beta) > 0$ and $b(\alpha,\beta)$,

$$f^{-1}_{\alpha,\beta}(x) = -k(\alpha,\beta)\ln x + b(\alpha,\beta)$$

which gives us, with $t = f_{\alpha,\beta}^{-1}(x)$,

$$f_{\alpha,\beta}(t) = \mathrm{e}^{\frac{t-b(\alpha,\beta)}{-k(\alpha,\beta)}}.$$

So, we get

$$F_{\alpha,\beta}(x,y) = f_{\alpha,\beta}(f_{\alpha,\beta}^{-1}(x) + y) = x\,\mathrm{e}^{-\frac{y}{k(\alpha,\beta)}}$$

after some manipulation. By the left homogeneity of $F_{\alpha,\beta}$ and the ST-meaningfulness of the family $\mathcal{F}$, we must have

$$\frac{1}{\alpha}F_{\alpha,\beta}(\alpha x, \beta y) = F_{\alpha,\beta}(x, \beta y) = x\,e^{-\frac{\beta y}{k(\alpha,\beta)}} = F(x,y).$$

The last equation shows that $\frac{\beta}{k(\alpha,\beta)}$ does not depend upon α or β.
Defining $c = \frac{k(\alpha,\beta)}{\beta}$, we finally obtain $F(x,y) = x\,\mathrm{e}^{-\frac{y}{c}}$.
Accordingly, we obtain for any code $F_{\mu,\nu} \in \mathcal{F}$, using left homogeneity of degree 1 in the second equation below

$$F_{\mu,\nu}(x,y) = \mu F\left(\frac{x}{\mu}, \frac{y}{\nu}\right) = F\left(x, \frac{y}{\nu}\right) = x\,\mathrm{e}^{-\frac{y}{\nu c}}.$$

□

Propagation Lemma.*Suppose that one ordered pair* $(L_{\mu,\nu}, \oplus_\nu)$ *from a meaningful Doppler-system* $(\mathcal{L}, \mathcal{O})$ *is an abstract Doppler-pair, that is,* $(L_{\mu,\nu}, \oplus_\nu)$ *satisfies Conditions* 1-5 *of the definition of an abstract Doppler-pair. Then any ordered pair* $(L_{\alpha,\beta}, \oplus_\beta)$*, with* $L_{\alpha,\beta} \in \mathcal{L}$ *and* $\oplus_\beta \in \mathcal{O}$*, is also such an abstract Doppler-pair.*

Proof. Without loss of generality, we can assume that the ordered pair $(L, \oplus)$ of initial code L is an abstract Doppler-pair, and so satisfies Conditions 1-5. By meaningfulness, we have:
$L_{\alpha,\beta}(\lambda, v) = \alpha L\left(\frac{\lambda}{\alpha}, \frac{v}{\beta}\right)$ and $v \oplus_\beta w = \beta\left(\frac{v}{\beta} \oplus \frac{w}{\beta}\right)$. Conditions 1 to 4 readily follow. Condition 1 holds because, successively:

$$\begin{aligned} L_{\alpha,\beta}(\lambda) \leq L_{\alpha,\beta}(\lambda', v') &\iff \alpha L\left(a\,\frac{\lambda}{\alpha}, \frac{v}{\beta}\right) \leq \alpha L\left(a\,\frac{\lambda'}{\alpha}, \frac{v'}{\beta}\right) \\ &\iff \alpha L\left(a\,\frac{\lambda}{\alpha}, \frac{v}{\beta}\right) \leq \alpha L\left(a\,\frac{\lambda'}{\alpha}, \frac{v'}{\beta}\right) \\ &\iff L_{\alpha,\beta}(a\lambda, v) \leq L_{\alpha,\beta}(a\lambda', v'). \end{aligned}$$

For Condition 3, we have $\lim_{v\to c} L_{\alpha,\beta}(\lambda, v) = \alpha \lim_{\frac{v}{\beta}\to\frac{c}{\beta}} L\left(\frac{\lambda}{\alpha}, \frac{v}{\beta}\right) = 0$.

We omit the proofs of Conditions 2 and 4 which are straightforward consequences of ST-meaningfulness.

We turn to Condition 5. Since Axioms [R] and [M] are equivalent, it suffices to prove that the ordered pair $(L_{\alpha,\beta}, \oplus_\beta)$ satisfies Axiom [R]. By the ST-meaningfulness of $\mathcal{L}$,

$$L_{\alpha,\beta}(L_{\alpha,\beta}(\lambda, v), w) = \alpha L\left(\frac{L_{\alpha,\beta}(\lambda, v)}{\alpha}, \frac{w}{\beta}\right) = \alpha L\left(\frac{\alpha L\left(\frac{\lambda}{\alpha}, \frac{v}{\beta}\right)}{\alpha}, \frac{w}{\beta}\right).$$

Canceling the α's in the fraction inside the parentheses in the r.h.s. gives

$$\begin{aligned} L_{\alpha,\beta}(L_{\alpha,\beta}(\lambda, v), w) &= \alpha L\left(L\left(\frac{\lambda}{\alpha}, \frac{v}{\beta}\right), \frac{w}{\beta}\right) = \alpha L\left(\frac{\lambda}{\alpha}, \frac{v}{\beta} \oplus \frac{w}{\beta}\right) \\ &= \alpha L\left(\frac{\lambda}{\alpha}, \frac{1}{\beta}(v \oplus_\beta w)\right) \\ &= L_{\alpha,\beta}\left(\lambda, v \oplus_\beta w\right). \end{aligned}$$

$\square$

Representation Theorem

Suppose that one ordered pair $(L_{\mu,\nu}, \oplus_\nu)$ *from a meaningful Doppler-system* $(\mathcal{L}, \mathcal{O})$ *is an abstract Doppler-pair, that is,* $(L_{\mu,\nu}, \oplus_\nu)$ *satisfies Conditions* 1-5. *Suppose also that* $L_{\mu,\nu}$ *does not vary with* ν. *Then, Axioms* [DE†] and [AV†] become for the initial code L:

$$[\mathrm{DE}] \quad L(\lambda, v) = \lambda \left(\frac{c - v}{c + v}\right)^{\xi} \quad (\text{with } \lambda \in \mathbb{R}_+,\ v \in [0, c[\text{ and } \xi \in \mathbb{R}_{++})$$

$$[\mathrm{AV}] \quad v \oplus w = \frac{v + w}{1 + \frac{vw}{c^2}} \quad (\text{with } v, w \in [0, c[\,).$$

Without loss of generality, we can assume that $(L, \oplus)$ is an abstract Doppler-pair, with L the initial code of the meaningful Doppler-system $(\mathcal{L}, \mathcal{O})$; that is, $(L, \oplus)$ satisfies Conditions 1-5. By ST-meaningulness, we have for any code $L_{\alpha,\beta}$:

$$L_{\alpha,\beta}(\lambda, v) = \alpha L\left(\frac{\lambda}{\alpha}, \frac{v}{\beta}\right) = \alpha\left(\frac{\lambda}{\alpha}\right)\left(\frac{\frac{c}{\beta} - u\left(\frac{v}{\beta}\right)}{\frac{c}{\beta} + u\left(\frac{v}{\beta}\right)}\right)^{\xi} \quad \left(\text{with } \frac{v}{\beta} \in \left[0, \frac{c}{\beta}\right[\right)$$

This implies

$$L_{\alpha,\beta}(\lambda, v) = \lambda\left(\frac{\frac{c}{\beta} - u\left(\frac{v}{\beta}\right)}{\frac{c}{\beta} + u\left(\frac{v}{\beta}\right)}\right)^{\xi} \qquad \left(\text{with } \frac{v}{\beta} \in \left[0, \frac{c}{\beta}\right[\right).$$

$L_{\alpha,\beta}(\lambda, v)$ cannot depend upon β. As the ratio

$$\frac{\frac{c}{\beta} - u\left(\frac{v}{\beta}\right)}{\frac{c}{\beta} + u\left(\frac{v}{\beta}\right)} \quad \left\{\begin{matrix}\text{is a function of } v \text{ only,}\\ \text{independent of } \beta,\\ \text{we must have}\end{matrix}\right\} \quad g(v) = \frac{\frac{c}{\beta} - u\left(\frac{v}{\beta}\right)}{\frac{c}{\beta} + u\left(\frac{v}{\beta}\right)}$$

for some function $g : [0, c[\to\ [0, c[$. The solution of the equation

$$g(v) = \frac{\frac{c}{\beta} - u\left(\frac{v}{\beta}\right)}{\frac{c}{\beta} + u\left(\frac{v}{\beta}\right)}$$

for the strictly increasing continuous function u is :
$u(v) = \theta v$ for all $v \in]0, c[$ with $\theta > 0$. Using the representation $[\mathrm{DE}^\dagger]$:

$$L(\lambda, v) = \lambda \left(\frac{c - u(v)}{c + u(v)}\right)^{\xi} = \lambda \left(\frac{c - \theta v}{c + \theta v}\right)^{\xi}.$$

But the code L must satisfy Condition 3, which requires that $\lim_{v \to c} L(\lambda, v) = 0$. This implies

$$\lim_{v \to c} \lambda \left(\frac{c - \theta v}{c + \theta v}\right)^{\xi} = \lambda \left(\frac{c - \theta c}{c + \theta c}\right)^{\xi} = \lambda \left(\frac{1 - \theta}{1 + \theta}\right)^{\xi} = 0 \quad \text{which holds only if } \theta = 1 \,.$$

We conclude that the function u of must be the identity function: $u(v) = v$. □

Looking in Aczél's book, we can find many candidate functional equation results which, combined with meaningfulness, would generate tentative scientific or geometric laws. Here are some examples.

The Transitivity Axiom

$$F(F(x,z),F(y,z)) = F(x,y)$$

The representation equation is:

$$F(x,y) = f^{-1}(f(x) - f(y)) \qquad \text{(with } f : \mathbb{R} \to]a,b[\,.\text{)}$$

Condition: $F\ ]a,b[\times]a,b[\to]a,b[$ with $F(x,y)$ continuous and strictly monotonic in y.
(Aczél, 1966, Page 277)

Several Functions of Several Variables. 1.

$$F(x,z) = G(x,y) + H(y,z).$$

The representation equations are:

$$F(x,z) = h(z) - f(x),\ \ G(x,y) = g(y) - f(x),\ \ H(y,z) = h(z) - g(y).$$

(Aczél, 1966, page 303)

Several Functions of Several Variables. 2.

$$F(G(x,y), H(u,v)) = K(M(x,u), N(y,v))$$

The representation equations are (Aczél, 1966, page 315):

$$\begin{aligned} F(x,y) &= k(f(x)+g(y)), & G(x,y) &= f^{-1}(p(x)+q(y)), \\ H(x,y) &= g^{-1}(r(x)+s(y)), & K(x,y) &= k(l(x)+m(y)), \\ M(x,y) &= l^{-1}(p(x)+r(y)), & N(x,y) &= m^{-1}(q(x)+s(y)). \end{aligned}$$

Several Functions of Several Variables. 3.

$$F(F(x,y),z) = F(x, K(y,z))$$

The representation equations are:

$$F(x,y) = f(f^{-1}(x) + k^{-1}(y)) \qquad K(x,y) = k(k^{-1}(x) + k^{-1}(y))$$

(Aczél, 1966, page 316).

Several Functions of Several Variables. 4.

This one is similar to Case 3, but involves four unknown functions.

$$F(G(x,y),z) = H(x, K(y,z)).$$

The representation equations are (Aczél, 1966, page 329):

$$F(x,y) = h(f(x) + g(y)), \qquad G(x,y) = f^{-1}(k(x) + m(y))$$
$$H(x,y) = h(k(x) + l(y)) \qquad K(x,y) = l^{-1}(m(x) + g(y)).$$

Several Functions of Several Variables. 5.

Generalized bisymmetry.

$$F(G(x,y),H(u,v)) = K(M(x,u),N(y,v)).$$

The representation equations are (Aczél, 1966, page 332) :

$$\begin{aligned} F(x,y) &= k(f(x)+g(y)) & G(x,y) &= f^{-1}(p(x)+q(y)) \\ H(x,y) &= g^{-1}(r(x)+s(y)) & K(x,y) &= k(l(x)+m(y)) \\ M(x,y) &= l^{-1}(p(x)+r(y)) & N(x,y) &= m^{-1}(q(x)+s(y)). \end{aligned}$$

Several Functions of Several Variables. 6.

Generalized distributivity.

$$F(G(x,y),z) = H(J(x,z),K(y,z))$$

The representation equations are (Aczél, 1966, page 335):

$$F(x,y) = p(f(y)g^{-1}(x) + \alpha(y) + \beta(y)), \quad G(x,y) = g(h(x) + k(y))$$
$$H(x,y) = p(m(x) + n(y)), \quad J(x,y) = m^{-1}(f(y)h(x) + \alpha(y))$$
$$K(x,y) = n^{-1}(f(y)k(x) + \beta(y)).$$

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