Atom-canonicity in varieties of relation and cylindric algebras with applications to omitting types in multi-modal logic

Tarek Sayed Ahmed

Department of Mathematics, Faculty of Science,
Cairo University, Giza, Egypt.

August 26, 2017

Outline

Summary of techniques and results

Atom canonicity and omitting types

Positive properties for L_n together with its guarded and clique–guarded fragments

Complete representations and neat embeddings

From now on, unless otherwise indicated, n is fixed to be a finite ordinal > 2.

- While the classical Orey-Henkin OTT holds for $L_{\omega,\omega}$, it is known [2] that the OTT fails for L_n in the following (strong) sense. For every $2 < n \leq l < \omega$, there is a countable and complete L_n atomic theory T, and a single type, namely, the type consisting of co-atoms of T, that is realizable in every model of T, but cannot be isolated by a formula using l variables.

- We prove stronger negative OTTs for L_n when types are required to be omitted with respect to certain (much wider) generalized semantics, called *m-flat* and *m–square* with $2 < n < m < \omega$.

- We use so-called *blow up and blur constructions*. Such subtle constructions may be applied to any two classes $\mathbf{L} \subseteq \mathbf{K}$ of completely additive BAOs. One takes an atomic $\mathfrak{A} \notin \mathbf{K}$ (usually but not always finite), blows it up, by splitting one or more of its atoms each to infinitely many subatoms, obtaining an (infinite) *countable* atomic $\mathfrak{Bb}(\mathfrak{A}) \in \mathbf{L}$, such that $\mathfrak{A}$ *is blurred* in $\mathfrak{Bb}(\mathfrak{A})$ meaning that $\mathfrak{A}$ *does not* embed in $\mathfrak{Bb}(\mathfrak{A})$, but $\mathfrak{A}$ embeds in the Dedekind-MacNeille completion of $\mathfrak{Bb}(\mathfrak{A})$, namely, $\mathfrak{Cm}\mathsf{At}\mathfrak{Bb}(\mathfrak{A})$.

- Then any class **M** say, between **L** and **K** that is closed under forming subalgebras will not be atom–canonical. We say, in this case, that **L** *is not atom–canonical with respect to* **K**.

This method is applied to $\mathbf{K} = \mathbf{S}\mathrm{RaCA}_l$, $l \geq 5$ and $\mathbf{L} = \mathrm{RRA}$ in [3] and to $\mathbf{K} = \mathrm{RRA}$ and $\mathbf{L} = \mathrm{RRA} \cap \mathrm{RaCA}_k$ for all $k \geq 3$ in [2], and will applied below to $\mathbf{K} = \mathbf{S}\mathrm{Nr}_n\mathrm{CA}_{n+k}$, $k \geq 3$ and $\mathbf{L} = \mathrm{RCA}_n$, where Ra denote the operator of forming relation algebra reducts, respectively, [4, Definition 5.2.7].

- Using variations on several blow up and blur constructions, we obtain negative results of the form: *There exists a countable, complete and atomic L_n theory T such that the type Γ consisting of co–atoms is realizable in every m–square model, but Γ cannot be isolated using $\leq l$ variables, where $n \leq l < m \leq \omega$.* Call it $\Psi(l, m)$, short for VT *fails at (the parameters) l and m.*

  Let $\mathsf{VT}(l, m)$ stand for VT *holds at l and m*, so that by definition $\Psi(l, m) \iff \neg\mathsf{VT}(l, m)$. We also include $l = \omega$ in the equation by defining $\mathsf{VT}(\omega, \omega)$ as VT holds for $L_{\omega,\omega}$: Atomic countable first order theories have atomic countable models. It is well known that $\mathsf{VT}(\omega, \omega)$ is a direct consequence of the Orey-Henkin OTT.

- We provide strong evidence that VT *fails everywhere* in the sense that for the permitted values $n \leq l, m \leq \omega$, namely, for $n \leq l < m \leq \omega$ and $l = m = \omega$, $\mathrm{VT}(l, m) \iff l = m = \omega$.

  From known algebraic results such as non-atom–canonicity of RCA_n and non-first order definability of the class of completely representable CA_ns, it can be easily inferred that $\mathrm{VT}(n, \omega)$ is false, that is to say, VT fails for L_n with respect to (usual) Tarskian semantics [5].

- From sharper algebraic results, we prove many other special cases for specific values of l and m, with $l < m$, that support the last equivalence. For example, from the non–atom canonicity of RCA_n with respect to the variety of CA_ns having $n+3$–square representations ($\supseteq \mathbf{S}\mathsf{Nr}_n\mathsf{CA}_{n+3}$), we prove $\Psi(n, n+k)$ for $k \geq 3$ and from the non–atom canonicity of $\mathsf{Nr}_n\mathsf{CA}_{n+k} \cap \mathsf{RCA}_n$ with respect to RCA_n for all $k \in \omega$, we prove $\Psi(l, \omega)$ for all finite $l \geq n$.

- Both results are obtained by blowing up and blurring finite algebras; a rainbow CA_n in the former case, and a finite RA (whose number of atoms depend on k) in the second case. In this case, we say (and prove) that VT fails *almost* everywhere.

- The non atom–canonicity of $\mathsf{Nr}_n\mathsf{CA}_{m-1} \cap \mathsf{RCA}_n$ with respect to the variety of CA_ns having m–square representations ($\supseteq \mathbf{S}\mathsf{Nr}_n\mathsf{CA}_m$) for all $2 < n < m < \omega$, implies that $\Psi(l, m)$ holds for all $2 < n \leq l < m \leq \omega$, in which case VT *fails everywhere.*

- This is reduced to (finding then) blowing up and blurring a finite relation algebra having a so-called strong $m-1$ blur and no m-dimensional relational basis for each $2 < n < m < \omega$.

Figuratively speaking, VT holds only at the limit when $l \to \infty$ and $m \to \infty$. So we can express the situation (using elementary Calculas terminology) as follows: For $2 < n \leq l < m < \omega$, $\mathrm{VT}(l, m)$ is false, but as l and m gets larger, $\mathrm{VT}(l, m)$ gets closer to VT, in symbols, $\lim_{l,m \to \infty} \mathrm{VT}(l, m) = \mathrm{VT}(\lim_{l \to \infty} l, \lim_{m \to \infty} m) = \mathrm{VT}(\omega, \omega)$.

From now on, unless otherwise indicated, n is fixed to be a finite ordinal > 2.

Definition 1.

Let $\mathfrak{A} \in \mathsf{CA}_n$ be atomic. Assume that $m, k \leq \omega$. The *atomic game* $G_k^m(\mathrm{At}\mathfrak{A})$*, or simply* G_k^m, is the game played on atomic networks of $\mathfrak{A}$ using m nodes and having k rounds. The ω–rounded game $\mathbf{G}^m(\mathrm{At}\mathfrak{A})$ or simply $\mathbf{G}^m$ is like the game $G_\omega^m(\mathrm{At}\mathfrak{A})$ except that $\forall$ has the advantage to reuse the m nodes in play.

Let A, B be two relational structures. Let $2 < n < \omega$. Then the colours used are:

- greens: g_i $(1 \leq i \leq n-2)$, g_0^i, $i \in A$,
- whites : $\mathrm{w}_i : i \leq n-2$,
- reds: r_{ij} $(i, j \in B)$,
- shades of yellow : y_S : S a finite subset of B or $S = B$.

A *coloured graph* is a graph such that each of its edges is labelled by the colours in the above first three items, greens, whites or reds, and some $n-1$ hyperedges are also labelled by the shades of yellow. Certain coloured graphs will deserve special attention.

Definition 2.

Let $i \in A$, and let M be a coloured graph consisting of n nodes $x_0, \ldots, x_{n-2}, z$. We call M *an* i *- cone* if $M(x_0, z) = \mathrm{g}_0^i$ and for every $1 \leq j \leq n-2$, $M(x_j, z) = \mathrm{g}_j$, and no other edge of M is coloured green. $(x_0, \ldots, x_{n-2})$ is called the *base of the cone*, z the *apex of the cone* and i the *tint of the cone*.

The rainbow algebra depending on A and B, from the class **K** consisting of all coloured graphs M such that:

The rainbow algebra depending on A and B, from the class **K** consisting of all coloured graphs M such that:

1. M is a complete graph and M contains no triangles (called forbidden triples) of the following types:

$$(\mathsf{g}, \mathsf{g}^{'}, \mathsf{g}^{*}), (\mathsf{g}_i, \mathsf{g}_i, \mathsf{w}_i) \qquad \text{any } 1 \leq i \leq n-2, \tag{1}$$

$$(\mathsf{g}_0^j, \mathsf{g}_0^k, \mathsf{w}_0) \qquad \text{any } j, k \in A, \tag{2}$$

$$(\mathsf{r}_{ij}, \mathsf{r}_{j'k'}, \mathsf{r}_{i^*k^*}) \qquad \text{unless } i = i^*,\ j = j' \ \&\ k' = k^* \tag{3}$$

   and no other triple of atoms is forbidden.

The rainbow algebra depending on A and B, from the class $\mathbf{K}$ consisting of all coloured graphs M such that:

2. If $a_0, \ldots, a_{n-2} \in M$ are distinct, and no edge (a_i, a_j) $i < j < n$ is coloured green, then the sequence $(a_0, \ldots, a_{n-2})$ is coloured a unique shade of yellow. No other $(n-1)$ tuples are coloured shades of yellow. Finally, if $D = \{d_0, \ldots, d_{n-2}, \delta\} \subseteq M$ and $M \upharpoonright D$ is an i cone with apex δ, inducing the order $d_0, \ldots, d_{n-2}$ on its base, and the tuple $(d_0, \ldots, d_{n-2})$ is coloured by a unique shade y_S then $i \in S$.

Let A and B be relational structures as above. Take the set J consisting of all surjective maps $a : n \to \Delta$, where $\Delta \in \mathbf{K}$ and define an equivalence relation on this set relating two such maps iff they essentially define the same graph; the nodes are possibly different but the graph structure is the same. Let At be the set of equivalences classes. We denote the equivalence class of a by $[a]$. Then define, for $i < j < n$, the accessibility relations corresponding to ijth–diagonal element, ith–cylindrifier, and substitution operator corresponding to the transposition $[i, j]$, as follows:

(1) $[a] \in E_{ij}$ iff $a(i) = a(j)$,

(2) $[a]T_i[b]$ iff $a \upharpoonright n \smallsetminus \{i\} = b \upharpoonright n \smallsetminus \{i\}$,

(3) $[a]S_{ij}[b]$ iff $a \circ [i, j] = b$.

Lemma 3.

If $\mathfrak{A} \in \mathbf{S}_c\mathrm{Nr}_n\mathrm{CA}_m$ *be atomic, then* $\exists$ *has a winning strategy in* $\mathbf{G}^m(\mathrm{At}\mathfrak{A})$.

Theorem 4.

1. *The variety* RRA *is not atom-canonical with respect to* $\mathbf{S}\mathrm{RaCA}_k$*, for any* $k \geq 6$*,*
2. *Let* $m \geq n+3$*. Then* RCA_n *is not-atom canonical with respect to* $\mathbf{S}\mathrm{Nr}_n\mathrm{CA}_m$*.*

Proof

Blowing up and blurring $\mathfrak{A}_{n+1,n}$ forming a weakly representable atom structure At:

Proof

Blowing up and blurring $\mathfrak{A}_{n+1,n}$ forming a weakly representable atom structure At:

Take the finite rainbow CA_n, $\mathfrak{A}_{n+1,n}$ where the reds R is the complete irreflexive graph n, and the greens are $\{g_i : 1 \leq i < n-1\} \cup \{g_0^i : 1 \leq i \leq n+1\}$, so that $G = n+1$. Denote the finite atom structure of $\mathfrak{A}_{n+1,n}$ by $\mathbf{At}_f$. One then replaces the red colours of the finite rainbow algebra of $\mathfrak{A}_{n+1,n}$ each by infinitely many countable reds (getting their superscripts from ω), obtaining this way a weakly representable atom structure **At**.

Proof

Blowing up and blurring $\mathfrak{A}_{n+1,n}$ forming a weakly representable atom structure At:

The atom structure **At** is like the weakly (but not strongly) representable atom structure of the atomic and countable and simple $\mathfrak{A} \in \mathsf{Cs}_n$ as defined in [2, Definition 4.1]; the sole difference is that we have $n+1$ greens and not ω–many as is the case in [2]. We denote the resulting term CA_n, $\mathfrak{Tm}\mathbf{At}$ by $\mathfrak{Bb}(\mathfrak{A}_{n+1,n}, \mathsf{r}, \omega)$ short hand for blowing up $\mathfrak{A}_{n+1,n}$ by splitting each *red graph (atom)* into ω many.

Proof

Blowing up and blurring $\mathfrak{A}_{n+1,n}$ forming a weakly representable atom structure At:

It can be shown exactly like in [2] that $\exists$ can win the rainbow ω–rounded game and build an n–homogeneous model M by using a shade of red ρ *outside* the rainbow signature, when she is forced a red; [2, Proposition 2.6, Lemma 2.7]. Using this, one proves like in *op.cit* that $\mathfrak{Bb}(\mathfrak{A}_{n+1,n}, \mathsf{r}, \omega)$ is representable as a set algebra having top element ${}^n\mathsf{M}$.

Proof (continuation)

Embedding $\mathfrak{A}_{n+1,n}$ **into** $\mathfrak{Cm}(\mathsf{At}(\mathfrak{Bb}(\mathfrak{A}_{n+1,n}, \mathsf{r}, \omega)))$**:**

Proof (continuation)

Embedding $\mathfrak{A}_{n+1,n}$ into $\mathfrak{Cm}(\mathrm{At}(\mathfrak{Bb}(\mathfrak{A}_{n+1,n}, \mathrm{r}, \omega)))$:

Let CRG_f be the class of coloured graphs on $\mathbf{At}_f$ and CRG be the class of coloured graph on $\mathbf{At}$. Write M_a for the atom that is the (equivalence class of the) surjection $a : n \to M$, $M \in \mathrm{CGR}$. We define the (equivalence) relation $\sim$ on $\mathbf{At}$ by $M_a \sim N_b$, $(M, N \in \mathrm{CGR})$ $\iff$ they are identical everywhere except at possibly at red edges: $M_a(a(i), a(j)) = \mathrm{r}^l \iff N_b(b(i), b(j)) = \mathrm{r}^k$, for some $l, k \in \omega$.

Proof (continuation)

Embedding $\mathfrak{A}_{n+1,n}$ into $\mathfrak{Cm}(\mathrm{At}(\mathfrak{Bb}(\mathfrak{A}_{n+1,n}, \mathrm{r}, \omega)))$:

We say that M_a is a *copy of* N_b if $M_a \sim N_b$. Define the map Θ from $\mathfrak{A}_{n+1,n} = \mathfrak{Cm}\mathbf{At}_f$ to $\mathfrak{Cm}\mathbf{At}$, by specifing first its values on $\mathbf{At}_f$, via $M_a \mapsto \bigvee_j M_a^{(j)}$ where $M_a^{(j)}$ is a copy of M_a. So each atom maps to the suprema of its copies. This map is well-defined because $\mathfrak{Cm}\mathbf{At}$ is complete. Furthermore, it can be checked that Θ is an injective a homomorphism.

Proof (continuation)

$\forall$ has a winning strategy in $\mathbf{G}^{n+3}\mathrm{At}(\mathfrak{A}_{n+1,n})$:

For him to win, $\forall$ lifts his winning strategy from the private Ehrenfeucht–Fraïssé forth game $\mathrm{EF}_{n+1}^{n+1}(n+1,n)$ (in $n+1$ rounds), to the graph game on $\mathbf{At}_f = \mathrm{At}(\mathfrak{A}_{n+1,n})$ orcing a win using $n+3$ nodes. He bombards $\exists$ with cones having common base and distinct green tints until $\exists$ is forced to play an inconsistent red triangle (where indicies of reds do not match). By Lemma 3, $\mathfrak{A}_{n+1,n} \notin \mathbf{S}_c\mathrm{Nr}_n\mathrm{CA}_{n+3}$. Since $\mathfrak{A}_{n+1,n}$ is finite, then $\mathfrak{A}_{n+1,n} \notin \mathbf{S}\mathrm{Nr}_n\mathrm{CA}_{n+3}$, for else $\mathfrak{A}^+_{n+1,n} = \mathfrak{A}_{n+1,n} \in \mathbf{S}_c\mathrm{Nr}_n\mathrm{CA}_{n+3}$. But $\mathfrak{A}_{n+1,n}$ embeds into $\mathfrak{Cm}\mathrm{At}\mathfrak{A}$, hence $\mathfrak{Cm}\mathbf{At} = \mathfrak{Cm}(\mathrm{At}\mathfrak{Bb}(\mathfrak{A}_{n+1,n}, \mathrm{r}, \omega))$ is outside the variety $\mathbf{S}\mathrm{Nr}_n\mathrm{CA}_{n+3}$, as well. We have proved that $\mathfrak{Tm}\mathbf{At} \in \mathrm{Cs}_n(\subseteq \mathrm{RCA}_n)$, while (its Dedekind-MacNeille completion) $\mathfrak{Cm}\mathbf{At} \notin \mathbf{S}\mathrm{Nr}_n\mathrm{CA}_{n+3}$, thereby proving the desired result.

- Fix $2 < n \leq l < m \leq \omega$. We turn to the statement $\Psi(l, m)$ as defined in the introduction. By an m–square model M of a theory T we understand an *m–square representation of the algebra $\mathfrak{Fm}_T$ with base* M*.*

  Let $\mathsf{VT}(l, m) = \neg\Psi(l, m)$, short for VT *holds 'at the parameters l and m'* where by definition, we stipulate that $\mathsf{VT}(\omega, \omega)$ is just VT for $L_{\omega,\omega}$. For $2 < n \leq l < m \leq \omega$ and $l = m = \omega$, we investigate the plausability of the following statement which we abbreviate by (**): $\mathsf{VT}(l, m) \iff l = m = \omega$.

  In other words: *Vaught's Theorem holds only in the limiting case when $l \to \infty$ and $m = \omega$ and not 'before'.*

In the next Theorem several conditions are given implying $\Psi(l, m)_f$ for various values of the parameters l and m where $\Psi(l, m)_f$ is the formula obtained from $\Psi(l, m)$ replacing square by flat.

Theorem 5.

Let $2 < n \leq l < m \leq \omega$. Then every item implies the immediately following one.

1. *There exists a finite relation algebra $\mathfrak{R}$ with a strong l–blur and no infinite m–dimensional hyperbasis,*
2. *There is a countable atomic $\mathfrak{A} \in \mathsf{Nr}_n\mathsf{CA}_l \cap \mathsf{RCA}_n$ such that $\mathfrak{Cm}\mathsf{At}\mathfrak{A}$ does not have an m–flat representation,*
3. *There is a countable atomic $\mathfrak{A} \in \mathsf{Nr}_n\mathsf{CA}_l \cap \mathsf{RCA}_n$ such that $\mathfrak{Cm}\mathsf{At}\mathfrak{A} \notin \mathbf{S}\mathsf{Nr}_n\mathsf{CA}_m$,*
4. *There is a countable atomic $\mathfrak{A} \in \mathsf{Nr}_n\mathsf{CA}_l \cap \mathsf{RCA}_n$ such that $\mathfrak{A}$ has no complete infinitary m–flat representation,*
5. *$\Psi(l', m')_f$ is true for any $l' \leq l$ and $m' \geq m$.*

The same implications hold upon replacing infinite m–dimensional hyperbasis by m–dimensional relational basis (not necessarily infinite), m–flat by m–square and $\mathbf{S}\mathsf{Nr}_n\mathsf{CA}_m$ by $\mathbf{S}\mathsf{Nr}_n\mathsf{D}_m$. Furthermore, in the new chain of implications every item implies the corresponding item in Theorem 5. In particular, $\Psi(l, m) \implies \Psi(l, m)_f$.

Proof

(1) $\implies$ (2):

Let $\mathfrak{R}$ be as in the hypothesis with strong l–blur (J, E). The idea is to 'blow up and blur' $\mathfrak{R}$ in place of the Maddux algebra $\mathfrak{E}_k(2,3)$ blown up and blurred in [2, Lemma 5.1], where $k < \omega$ is the number of non–identity atoms and k depends recursively on l, giving the desired strong l–blurness, cf. [2, Lemmata 4.2, 4.3]. Let $2 < n \leq l < \omega$. The relation algebra $\mathfrak{R}$ is blown up by splitting all of the atoms each to infinitely many giving a new infinite atom structure **At** denoted in [2, p.73] by At. One proves that the blown up and blurred atomic relation algebra $\mathfrak{Bb}(\mathfrak{R}, J, E)$ (as defined in [2]) with atom structure **At** is representable; in fact this representation is induced by a complete representation of its canonical extension, cf. [2, Item (1) of Theorem 3.2].

Proof

(1) $\implies$ (2):

Because (J, E) is a strong l–blur, then, by its definition, it is a strong j–blur for all $n \leq j \leq l$, so the atom structure $\mathbf{At}$ has a j–dimensional cylindric basis for all $n \leq j \leq l$, namely, $\mathsf{Mat}_j(\mathbf{At})$. For all such j, there is an RCA_j denoted on [2, Top of p. 9] by $\mathfrak{Bb}_j(\mathfrak{R}, J, E)$ such that $\mathfrak{Tm}\mathsf{Mat}_j(\mathbf{At}) \subseteq \mathfrak{Bb}_j(\mathfrak{R}, J, E) \subseteq \mathfrak{Cm}\mathsf{Mat}_j(\mathbf{At})$ and $\mathsf{At}\mathfrak{Bb}_j(\mathfrak{R}, J, E)$ is a weakly representable atom structure of dimension j, cf. [2, Lemma 4.3].

Proof

(1) $\implies$ (2):

Take $\mathfrak{A} = \mathfrak{Bb}_n(\mathfrak{R}, J, E)$. We claim that $\mathfrak{A}$ is as required. Since $\mathfrak{R}$ has a strong j–blur (J, E) for all $n \leq j \leq l$, then $\mathfrak{A} \cong \mathfrak{Nr}_n\mathfrak{Bb}_j(\mathfrak{R}, J, E)$ for all $n \leq j \leq l$ as proved in [2, item (3) p.80]. In particular, taking $j = l$, $\mathfrak{A} \in \mathsf{RCA}_n \cap \mathsf{Nr}_n\mathsf{CA}_l$. We show that $\mathfrak{Cm}\mathsf{At}\mathfrak{A}$ does not have an m–flat representation. Assume for contradicton that $\mathfrak{Cm}\mathsf{At}\mathfrak{A}$ does have an m–flat representation M. Then M is infinite of course. Since $\mathfrak{R}$ embeds into $\mathfrak{Bb}(\mathfrak{R}, J, E)$ which embeds into $\mathfrak{Ra}\mathfrak{Cm}\mathsf{At}\mathfrak{A}$, then $\mathfrak{R}$ has an m–flat representation with base M. But since $\mathfrak{R}$ is finite, $\mathfrak{R} = \mathfrak{R}^+$, so by [3, Theorem 13.46, (7) $\iff$ (11)] $\mathfrak{R}$ has an infinite m–dimensional hyperbasis, contradiction.

Proof (continuation)

(2) $\implies$ (3):

An algebra $\mathfrak{A}$ has an m-flat representation $\iff$ $\mathfrak{A} \in \mathbf{S}\mathrm{Nr}_n\mathrm{CA}_m$.

Proof (continuation)

(3) $\implies$ (4):

A complete m–flat representation of (any) $\mathfrak{B} \in \mathrm{CA}_n$ induces an m–flat representation of $\mathfrak{CmAt}\mathfrak{B}$ which implies that $\mathfrak{CmAt}\mathfrak{B} \in \mathbf{S}\mathrm{Nr}_n\mathrm{CA}_m$. To see why, assume that $\mathfrak{B}$ has an m–flat complete representation via $f : \mathfrak{B} \to \mathfrak{D}$, where $\mathfrak{D} = \wp(V)$ and the base of the representation $\mathrm{M} = \bigcup_{s \in V} \mathrm{rng}(s)$ is m–flat. Let $\mathfrak{C} = \mathfrak{CmAt}\mathfrak{B}$. For $c \in C$, let $c\downarrow = \{a \in \mathrm{At}\mathfrak{C} : a \leq c\} = \{a \in \mathrm{At}\mathfrak{B} : a \leq c\}$; the last equality holds because $\mathrm{At}\mathfrak{B} = \mathrm{At}\mathfrak{C}$. Define, representing $\mathfrak{C}$, $g : \mathfrak{C} \to \mathfrak{D}$ by $g(c) = \sum_{x \in c\downarrow} f(x)$, then g is a homomorphism into $\wp(V)$ having base M.

Proof (continuation)

(4) $\implies$ (5):

By [4, §4.3], we can (and will) assume that $\mathfrak{A} = \mathfrak{Fm}_T$ for a countable, simple and atomic theory L_n theory T. Let Γ be the n–type consisting of co–atoms of T. Then Γ is realizable in every m–flat model, for if M is an m–flat model omitting Γ, then M would be the base of a complete infinitary m–flat representation of $\mathfrak{A}$, and so $\mathfrak{A} \in \mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_m$ which is impossible. But $\mathfrak{A} \in \mathsf{Nr}_n\mathsf{CA}_l$, so using an argument similar to that used in [2, Theorem 3.1] we get that any witness isolating Γ needs more than l–variables. Spelling out more details, suppose for contradiction that ϕ is an l witness, so that $T \models \phi \to \alpha$, for all $\alpha \in \Gamma$, where (recall that) Γ is the set of coatoms. Then since $\mathfrak{A}$ is simple, we can assume without loss of generality, that it is set algebra with a countable base.

Proof (continuation)

(4) $\implies$ (5):

Let $\mathsf{M} = (M, R_i)_{i\in\omega}$ be the corresponding model (in a relational signature) to this set algebra in the sense of [4, section 4.3]. Let ϕ^{M} denote the set of all assignments satisfying ϕ in M. We have $\mathsf{M} \models T$ and $\phi^{\mathsf{M}} \in \mathfrak{A}$, because $\mathfrak{A} \in \mathsf{Nr}_n\mathsf{CA}_l$. But $T \models \exists x\phi$, hence $\phi^{\mathsf{M}} \neq 0$, from which it follows that ϕ^{M} must intersect an atom $\alpha \in \mathfrak{A}$ (recall that the latter is atomic). Let ψ be the formula, such that $\psi^{\mathsf{M}} = \alpha$. Then it cannot be the case that $T \models \phi \to \neg\psi$, hence ϕ is not a witness, contradiction and we are done.

Proof (continuation)

(4) $\implies$ (5):

For squareness the proofs are essentially the same undergoing the obvious modifications In the first implication 'infinite' in the hypothesis is not needed because any finite relation algebra having an infinite m–dimensional relational basis has a finite one, cf. [3, Theorem 19.18] which is not the case with hyperbasis, cf. [3, Prop. 19.19].

Summary of results on VT:

VT(n, ω)	no, [2] and Theorem 4
VT$(n, n+3)$	no, Theorem 4
VT$(n, n+2)_f$	no, if $\exists$ $\mathfrak{R}$ with n–blur and no $n+2$-hyp
VT(l, ω)	no, $\mathfrak{E}_k(2,3)$ has strong l-blur, and no ω-hyp
VT$(l, m)_f, l \leq m-1$	no, if $\exists$ $\mathfrak{R}$ with strong l-blur, and no m-hyp
VT$(l, m), l \leq m-1$	no, if $\exists$ $\mathfrak{R}$ with strong l-blur, and no m-bases
VT(ω, ω)	yes, VT for $L_{\omega,\omega}$.

Definition 6.

Let λ be a cardinal. Assume that $\mathfrak{A} \in \mathsf{RCA}_n$. If $\mathbf{X} = (X_i : i < \lambda)$ is a family of subsets of $\mathfrak{A}$, we say that $\mathbf{X}$ *is omitted in* $\mathfrak{C} \in \mathsf{Gs}_n$, if there exists an isomorphism $f : \mathfrak{A} \to \mathfrak{C}$ such that $\bigcap f(X_i) = \emptyset$ for all $i < \lambda$. If $X \subseteq \mathfrak{A}$ and $\prod X = 0$, then we refer to X as a *non-principal type* of $\mathfrak{A}$.

Theorem 7.

Let $\mathfrak{A} \in \mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_\omega$ *be countable. Let* $\lambda < 2^\omega$ *and let* $\mathbf{X} = (X_i : i < \lambda)$ *be a family of non-principal types of* $\mathfrak{A}$*. Then the following hold:*

1. *If* $\mathfrak{A} \in \mathsf{Nr}_n\mathsf{CA}_\omega$ *and the* X_i*s are non–principal ultrafilters, then* $\mathbf{X}$ *can be omitted in a* Gs_n*,*
2. *Every subfamily of* $\mathbf{X}$ *of cardinality* $< \mathfrak{p}$ *can be omitted in a* Gs_n*. Furthermore, if* $\mathfrak{A}$ *is simple, then every subfamily of* $\mathbf{X}$ *of cardinality* $< \mathsf{covK}$ *can be omitted in a* Cs_n*.*

Corollary 8.

Let n be any finite ordinal. Let T be a countable and consistent L_n theory and λ be a cardinal $< \mathfrak{p}$. Let $\mathbf{F} = (\Gamma_i : i < \lambda)$ be a family of non-principal types of T. Suppose that T admits elimination of quantifiers. Then the following hold:

1. *If ϕ is a formula consistent with T, then there is a model* M *of T that omits $\mathbf{F}$, and ϕ is satisfiable in* M. *If T is complete, then we can replace $\mathfrak{p}$ by* covK,
2. *If the non-principal types constituting $\mathbf{F}$ are maximal, then we can replace $\mathfrak{p}$ by 2^{ω}.*

Theorem 9.

For $2 < n < \omega$ the following hold:

1. $\mathrm{CRCA}_n \subseteq \mathbf{S}_c\mathrm{Nr}_n(\mathrm{CA}_\omega \cap \mathbf{At}) \cap \mathbf{At} \subseteq \mathbf{S}_c\mathrm{Nr}_n\mathrm{CA}_\omega \cap \mathbf{At}$. *At least two of these three classes are distinct,*
2. *All reverse inclusions and implications in the previous item hold, if algebras considered have countably many atoms,*
3. *All classes in the first item are closed under* $\mathbf{S}_c$ *(a fortiori under* $\mathbf{S}_d$*),* $\mathbf{P}$*, but are not closed under* $\mathbf{S}$*, nor* $\mathbf{H}$ *nor* $\mathbf{Ur}$*. Their elementary closure coincides with* LCA_n*,*
4. $\mathrm{Nr}_n\mathrm{CA}_\omega \subsetneq \mathbf{S}_d\mathrm{Nr}_n\mathrm{CA}_\omega \subseteq \mathbf{S}_c\mathrm{Nr}_n\mathrm{CA}_\omega \subsetneq \mathbf{ElS}_c\mathrm{Nr}_n\mathrm{CA}_\omega \subsetneq \mathrm{RCA}_n$. *Furthermore, the strictness of inclusions are witnessed by atomic algebras.*

Theorem 10.

Any class $\mathbf{K}$ *such that* $\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathsf{CRCA}_n \subseteq \mathbf{K} \subseteq \mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_{n+3}$, $\mathbf{K}$ *is not elementary.*

Proof

One takes a rainbow –like algebra based on the ordered structure $\mathbb{Z}$ and $\mathbb{N}$, that is similar but not identical to $\mathsf{CA}_{\mathbb{Z},\mathbb{N}}$; call this (complex) algebra $\mathfrak{C}$. The reds R is the set $\{\mathsf{r}_{ij} : i < j < \omega (= \mathbb{N})\}$ and the green colours used constitute the set $\{\mathsf{g}_i : 1 \leq i < n-1\} \cup \{\mathsf{g}_0^i : i \in \mathbb{Z}\}$. In complete coloured graphs the forbidden triples are like in usual rainbow constructions; more specifically the following are forbidden triangles in coloured graphs.

$$(\mathsf{g}, \mathsf{g}^{'}, \mathsf{g}^{*}), (\mathsf{g}_i, \mathsf{g}_i, \mathsf{w}_i), \qquad \text{any } 1 \leq i \leq n-2 \tag{1}$$

$$(\mathsf{g}_0^j, \mathsf{g}_0^k, \mathsf{w}_0) \qquad \text{any } j, k \in \mathsf{G} \tag{2}$$

$$(\mathsf{r}_{ij}, \mathsf{r}_{j'k'}, \mathsf{r}_{i^*k^*}) \qquad \text{unless } i = i^*,\ j = j' \text{ and } k' = k^*, \tag{3}$$

Proof

but now the triple $(\mathrm{g}_0^i, \mathrm{g}_0^j, \mathrm{r}_{kl})$ is also forbidden if $\{(i,k),(j,l)\}$ is not an order preserving partial function from $\mathbb{Z} \to \mathbb{N}$. It can be proved that $\exists$ has a winning strategy ρ_k in the k–rounded game $G_k(\mathrm{At}\mathfrak{C})$ for all $k \in \omega$. Hence, using ultrapowers and an elementary chain argument , one gets a countable (completely represenatble) algebra $\mathfrak{B}$ such that $\mathfrak{B} \equiv \mathfrak{A}$, and $\exists$ has a winning strategy in $G_\omega(\mathrm{At}\mathfrak{B})$.

Proof

On the other hand, one can show that $\forall$ has a winning strategy in $F^{n+3}(\mathsf{At}\mathfrak{C})$. The idea here, is that, as is the case with winning strategy's of $\forall$ in rainbow constructions, $\forall$ bombards $\exists$ with cones having distinct green tints demanding a red label from $\exists$ to appexes of succesive cones. The number of nodes are limited but $\forall$ has the option to re-use them, so this process will not end after finitely many rounds. The added order preserving condition relating two greens and a red, forces $\exists$ to choose red labels, one of whose indices form a decreasing sequence in $\mathbb{N}$. In ω many rounds $\forall$ forces a win, so by lemma 3 $\mathfrak{C} \notin \mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_{n+3}$.

Proof

Finally, we now construct two atomic algebras $\mathfrak{A}, \mathfrak{B} \in \mathsf{CA}_n$ such that, $\mathfrak{A} \in \mathsf{Nr}_n\mathsf{CA}_\omega$, $\mathfrak{B} \notin \mathbf{S}_d\mathsf{Nr}_n\mathsf{CA}_{n+1}$ and $\mathfrak{A} \equiv \mathfrak{B}$. Thus $\mathfrak{B} \in \mathbf{El}(\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathsf{CRCA}_n) \sim \mathbf{S}_d\mathsf{Nr}_n\mathsf{CA}_\omega$. Since $\mathbf{El}(\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathsf{CRCA}_n) \nsubseteq \mathbf{S}_d\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathsf{CRCA}_n$, there can be no elementary class between $\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathsf{CRCA}_n$ and $\mathbf{S}_d\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathsf{CRCA}_n$. Having already eliminated elementary classes between $\mathbf{S}_d\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathsf{CRCA}_n$ and $\mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_{n+3}$, we are done.

Algebras	At-can.	At. gen	El. gen.	Can.	Str is el.	CR is el.	VT
RCA_n, RRA	no	yes	yes	yes	no	no	no
$\mathbf{S}\mathsf{Nr}_n\mathsf{CA}_{n+1}$	yes	yes	yes	yes	yes	?	?
$\mathbf{S}\mathsf{RaCA}_3$, $\mathbf{S}\mathsf{RaCA}_4$	yes	yes	yes	yes	yes	yes	yes
$\mathbf{S}\mathsf{Nr}_n\mathsf{CA}_{n+2}$	?	yes	yes	yes	?	?	?
$\mathbf{S}\mathsf{RaCA}_5$	?	yes	yes	yes	?	yes	yes
$\mathbf{S}\mathsf{Nr}_n\mathsf{CA}_{n+k}$, $\mathbf{S}\mathsf{RaCA}_m$	no	yes	yes	yes	?	no	no
D_n, G_n	yes	yes	yes	yes	yes	yes	yes

In the next table we summarize the results obtained on non-first order definability:

Algebras	Elementary
$\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathsf{CRCA}_n \subseteq \mathbf{K} \subseteq \mathbf{S}_d\mathsf{Nr}_n\mathsf{CA}_{n+1}$	no
$\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathsf{CRCA}_n \subseteq \mathbf{K} \subseteq \mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_{n+3}$	no
$\mathsf{At}(\mathsf{Nr}_n\mathsf{CA}_\omega \cap \mathsf{CRCA}_n) \subseteq \mathbf{K} \subseteq \mathsf{At}\mathbf{S}_c\mathsf{Nr}_n\mathsf{CA}_{n+3}$	no
$\mathsf{Nr}_n\mathsf{CA}_\omega \subseteq \mathbf{K} \subseteq \mathsf{Nr}_n\mathsf{CA}_{n+1}$	no
$\mathbf{S}_c\mathsf{RaCA}_\omega \cap \mathsf{CRRA} \subseteq \mathbf{K} \subseteq \mathbf{S}_c\mathsf{RaCA}_6$	no
$\mathbf{S}_d\mathsf{RaCA}_\omega \cap \mathsf{CRRA} \subseteq \mathbf{K} \subseteq \mathbf{S}_c\mathsf{RaCA}_6$	no
$\mathsf{RaCA}_\omega \cap \mathsf{CRRA}_n \subseteq \mathbf{K} \subseteq \mathbf{S}_c\mathsf{RaCA}_6$	?
$\mathsf{At}(\mathsf{RaCA}_\omega \cap \mathsf{CRRA}) \subseteq \mathbf{K} \subseteq \mathsf{At}\mathbf{S}_c\mathsf{RaCA}_6$	no

- H. Andréka, M. Ferenczi and I. Németi (Editors), **Cylindric-like Algebras and Algebraic Logic.** Bolyai Society Mathematical Studies **22** (2013).
- H. Andréka, I. Németi and T. Sayed Ahmed, *Omitting types for finite variable fragments and complete representations.* Journal of Symbolic Logic. **73** (2008) pp. 65–89.
- H. Andréka and R. Thompson *A Stone type representation theorem for algebras of relations of higher rank.* Transactions of the American Mathematical Society, **309** (1988), p.671–682.
- L. Henkin, J.D. Monk and A. Tarski *Cylindric Algebras Part I,II*. North Holland, 1971, 1985.
- R. Hirsch, *Relation algebra reducts of cylindric algebras and complete representations*, Journal of Symbolic Logic, **72**(2) (2007), pp. 673–703.

- R. Hirsch *Corrigendum to 'Relation algebra reducts of cylindric algebras and complete representations'* Journal of Symbolic Logic, **78**(4) (2013), pp. 1345–1348.
- I. Hodkinson, *Atom structures of relation and cylindric algebras*. Annals of pure and applied logic, **89**(1997), p.117–148
- R. Hirsch and I. Hodkinson, *Relation Algebras by Games.* Studies In Logic. North Holland **147** (2002).
- B. Samir and T. Sayed Ahmed *A Neat Embedding Theorem for expansions of cylindric algebras.* Logic Journal of *IGPL* **15**(2007) pp.41–51.
- T. Sayed Ahmed *Completions, Complete representations and Omitting types*, in [1], pp. 186–205.

Thank you!