Understanding “Gauge”

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What does “**gauge**” mean?

First strand

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A **gauge theory** is a theory that posits strictly more structure than is necessary.

Second strand

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A **gauge theory** is a theory that bears a certain historical relationship to electromagnetism.

Second strand

For me, a gauge theory is any physical theory of a dynamic variable which [sic], at the classical level, may be identified with a connection on a principal bundle.

-Trautman 1980, p. 306

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- Newton-Cartan theory

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Preliminary Question: Can categorial methods help make a notion of "surplus structure" precise? (I say: **Yes.**)

Talk Overview

1. A motivating example
2. Comparing structure
3. Gauge theories and surplus structure

Classical electromagnetism

Consider electromagnetism in Minkowski spacetime.[1]

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EM_1

Dynamical variable: Faraday tensor, F_{ab}.

Equations of motion: $\nabla_{[a}F_{bc]} = \mathbf{0}$ and $\nabla_a F^{ab} = J^b$.

Models: (M, η_{ab}, F_{ab}).

EM_2

Dynamical variable: 4-vector potential, A_a.

Equations of motion: $\nabla_a \nabla^a A^b - \nabla^b \nabla_a A^a = J^b$.

Models: (M, η_{ab}, A_a).

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Given a Faraday tensor F_{ab}, there always exists a 4-vector potential A_a s.t. $F_{ab} = \nabla_{[a}A_{b]}$.

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An asymmetry in the relationship

A 4-vector potential determines a **unique** Faraday tensor.

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If $\nabla_{[a}A_{b]} = F_{ab}$, then $\nabla_{[a}\tilde{A}_{b]} = F_{ab}$, where $\tilde{A}_a = A_a + \nabla_a\psi$ for **any** smooth ψ. (**Gauge Transformation**)

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But there are **distinct** models of EM_2 that correspond to a **single** model of EM_1.

Thus, whatever structure distinguishes these distinct models of EM_2 is **surplus** structure.

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- Smooth manifolds < Lie groups
- Vector spaces < inner product spaces

More structure?

This idea can be made precise using the notion of a **forgetful functor**.

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- Preserves domain and codomain;
- Preserves identity; and
- Preserves composition.

Forgetful functors

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Example: There is a functor $F : \mathbf{Top} \to \mathbf{Set}$ that takes a topological space (X, τ) to the set X, and takes continuous functions $g : X \to X'$ to functions $g : X \to X'$.

Forgetful functors

How do we know a functor is forgetful?

Forgetful functors

A functor $F : \mathbf{C} \to \mathbf{D}$ is **full** if $(f : A \to B) \mapsto (F(f) : F(A) \to F(B))$ is surjective for all A and B.

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F is **faithful** if $(f : A \to B) \mapsto (F(f) : F(A) \to F(B))$ is injective for all A and B.

F is **essentially surjective** if for every object X of $\mathbf{D}$, there is an object A of $\mathbf{C}$ such that $F(A)$ is isomorphic to X.

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Forgetful functors

Baez-Dolan-Bartels-Barrett classification:

A functor forgets:

- **Nothing** if it is full, faithful, and essentially surjective. (**Equivalence of categories**)
- Only **structure** if it is faithful and essentially surjective.
- Only **properties** if it is full and faithful.
- Only **stuff** if it is full and essentially surjective.

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The functor from **Top** to **Set** that takes topological spaces to their underlying sets and continuous functions to their underlying functions forgets only **structure.**

The functor from **AbGrp** to **Grp** that takes Abelian groups and group homomorphisms to themselves forgets only **properties.**

The functor from **Set** to 1 that takes every set to the unique object $\bullet$ and every arrow to $1_\bullet$ forgets only **stuff.**

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Making “surplus structure” precise

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$\mathbf{EM}_1$: Objects are models (M, η_{ab}, F_{ab}), where F_{ab} satisfies Maxwell's Equations; Arrows are isometries that preserve F_{ab}.

Making “surplus structure” precise

We can think of EM_1 and EM_2 as categories.

$\mathbf{EM}_1$: Objects are models (M, η_{ab}, F_{ab}), where F_{ab} satisfies Maxwell’s Equations; Arrows are isometries that preserve F_{ab}.

$\mathbf{EM}_2$: Objects are models (M, η_{ab}, A_a), where A_a satisfies the required equation; Arrows are isometries that preserve A_a.

Making "surplus structure" precise

The map $A_a \mapsto \nabla_{[a}A_{b]} = F_{ab}$ determines a functor $F : \mathbf{EM}_2 \to \mathbf{EM}_1$. ($F$ acts trivially on arrows.)

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The map $A_a \mapsto \nabla_{[a}A_{b]} = F_{ab}$ determines a functor $F : \mathbf{EM}_2 \rightarrow \mathbf{EM}_1$. ($F$ acts trivially on arrows.)

This functor is **essentially surjective** and **faithful** but not **full**.

Making “surplus structure” precise

The map $A_a \mapsto \nabla_{[a}A_{b]} = F_{ab}$ determines a functor $F : \mathbf{EM}_2 \to \mathbf{EM}_1$. ($F$ acts trivially on arrows.)

This functor is **essentially surjective** and **faithful** but not **full.**

F forgets (only) **structure.**

Making “surplus structure” precise

Thus EM_2 has **surplus structure** in the sense that one can **forget** structure without affecting empirical adequacy.

Another example

Newtonian gravitation has surplus structure in the same sense.

Gauge transformations

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The functor $F : EM_2 \to EM_1$ is not full.

Thus, there is a sense in which EM_2 is “missing” arrows.

There are **non-isomorphic** models of EM_2 that map to the **same** model of EM_1.

A diagnostic tool

Rule of Thumb

A theory (or formulation of a theory) has "surplus structure" if and only if there are ***non-isomorphic*** *models that have the same representational capacities.*

A diagnostic tool

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Suppose you are given a theory and a collection of maps taking models to physically equivalent models.

(That is, suppose you are given a candidate “gauge theory” and class of “gauge transformations”.)

Ask: Are these maps **isomorphisms** of the models? **Yes** $\Rightarrow$ **no surplus structure**.

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Verdict: No surplus structure!

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The End

Thank you!

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A model of Newton-Cartan theory is a structure (M, t_a, h^{ab}, ∇), where ∇ satisfies the geometrized Poisson equation.

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Asymmetry! Again, the natural functor forgets **structure**.