Fragments of monadic second-order theories of the chronological accessibility relation

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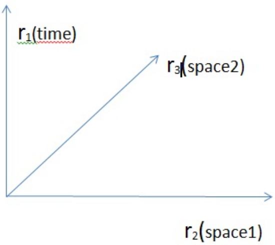
Outline

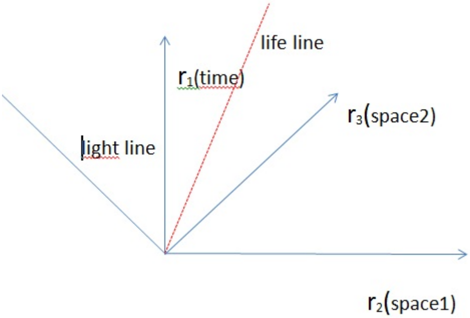
Spacetime $\mathbb{F}^n (n \geq 2)$

$\mathbb{F} \in \{\mathbb{R}, \mathbb{Q}\}$. Elements of $\mathbb{F}^n$ denoted by $(r_1, \ldots, r_n)$.



time dimension is 1
space dimension is $n - 1$

Life line, light line, speed of light

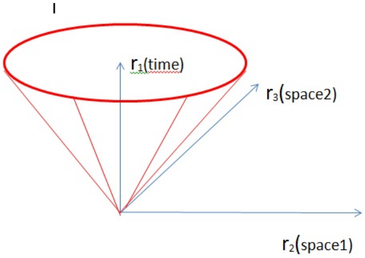


Life lines of pointlike bodies are straight lines of slope less then the speed of light(c). Assume $c = 1$.

case $\mathbb{Q}$: $c \in \mathbb{Q}$

Upper light cone

On $\mathbb{F}^n$ $(n > 1)$, $(r_1, \ldots, r_n) \blacktriangleleft (q_1, \ldots, q_n) :\Leftrightarrow$
$(r_1 - q_1)^2 > (r_2 - q_2)^2 + \ldots + (r_n - q_n)^2 \wedge r_1 < q_1$.



$(q_1, \ldots, q_n)$ is in the upper light-cone of $(r_1, \ldots, r_n)$
$\blacktriangleleft$ is called chronological accessibility relation

Other relations related to possible causality

$r << q$: Robb's after, q is in or on the upper light cone of r

$r =\blacktriangleleft q$, $r =<< q$: reflexive closure of $\blacktriangleleft$ and $<<$, resp.

First-order theories of causality relations

Robb(1914) $FOTH(\mathbb{R}^n, <<)$ is finitely axiomatizable
explicit axiomatization, far from transparent (source: Suppes (1973))

Brett Mundy (1986)

James P. Ax (1978)

Rob Goldblatt (1988): spacetime geometry can be built up by explicit first-order definitions starting from Robb's binary relation $<<$ (not like in Euclidean case)

Higher-order axiomatizations – selection

Walker (1959)

Schutz (2002)

logical expressive power to follow a classical physics book word-by-word (real numbers, quantification over mappings)

First-order axiomatization for the full playground

Andréka, Németi, Madarász ...

not only for the geometry: axioms talk about bodies, coordinates, observers, world view functions

conceptual analysis

The presented results support avoiding higher-order logic in the axiomatization process

Universal fragment of monadic second-order theory of $(\mathbb{F}^n, \blacktriangleleft)$ is rec. enumerable if and only if $\mathbb{F} = \mathbb{Q}$ and $n = 2$.

This case is too restricted to do relativity theory

Monadic second-order theories

standard version of the monadic second-order theory of a structure $(T, <)$

syntax:
variables $(x, y, \ldots)$ for individuals and for subsets $(X, Y, \ldots)$
$x < y$, $x \in X$

interpretation:
the *standard model* $\mathcal{M}_n$ of this language on $(\mathbb{Q}^n, \blacktriangleleft)$ $(n > 1)$ when the domain is $\mathbb{Q}^n$ and the interpretation of $<$ is $\blacktriangleleft$, further, the variables of the second sort range over *all* subsets of $\mathbb{Q}^n$ and the interpretation of $\in$ is the standard inclusion
variable valuation, satisfaction is defined in the expected way

Monadic second-order theories II.

The *monadic second-order theory* of $(\mathbb{Q}^n, \blacktriangleleft)$ is the set of true closed formulae of $\mathcal{M}_n$. Denoted by $MSOTH(\mathbb{Q}^n, \blacktriangleleft)$.

We define its $\forall\exists$-fragment as the set of the formulae in this theory of form $\forall V_1 \ldots \forall V_n \exists W_1 \ldots \exists W_m\ B$, where $n \geq 0$, $m \geq 0$, $V_1, \ldots V_n$ and $W_1, \ldots, W_m$ are subset variables and B itself is free from subset quantifiers, that is, one measure only the complexity of subset quantifications. $MSOTH_{\forall\exists}(T, <)$.

If $m = 0$ we obtain the definition of the $\forall$-fragment. $MSOTH_{\forall}(T, <)$.

Propositional temporal logic

model: time-dependent truth valuation of propositional variables

time-dependent truth value of formulae ($M, t \models a$)

logical operators to refer to truth values of formulae in another time points

For example, the rule for the evaluation of binary connective *Until*:
$M, t \models Until(A, B) :\Leftrightarrow$
$\exists x(t < x \wedge M, x \models A \wedge \forall u(t < u < x \rightarrow M, u \models B))$

Translation of temporal formulae into monadic second-order ones

$Until(A, B)^{mso} = \forall A \forall B \exists x(t < x \wedge x \in A \wedge$
$\forall u(t < u < x \rightarrow u \in B))$

well known: a temporal formula ϕ is temporal logical law over a time flow (T,<) if and only if ϕ^{mso} is in $MSOTH(T, <)$.

not need to formalize more exactly here, mentioned only for motivation

Burgess and Gurevich (1985) have decided linear temporal logic by this translation
$MSOTH(\mathbb{Q}, <)$ decidable
$MSOTH(\mathbb{R}, <)$ undec. but $MSOTH_{\forall}(\mathbb{R}, <)$ is dec.

Theorems

For any $n > 1$, $MSOTH_{\forall\exists}(\mathbb{F}^n, \blacktriangleleft)$ is not rec. enumerable.

$MSOTH_{\forall}(\mathbb{Q}^n, \blacktriangleleft)$ is rec. enumerable iff $n = 2$

$MSOTH_{\forall}(\mathbb{R}^n, \blacktriangleleft)$ is not rec. enumerable

will see what is the difference

Spatio-temporal logics

temporal logic with time flow $(\mathbb{F}^n, \blacktriangleleft)$ or other causality related relation

c

oined by A. Dragalin (source: V. Shehtman)
first axiomatizations: V. Shehtman, R.Goldblatt (S4.2) (1980)
Shapirovsky
Robin Hirsch and Mark Reynolds – at this conf.

Proof for the only positive rec. enumerability result

Rather routine.

Johan van Benthem (1983): $FOTH(\mathbb{Q}^2, \blacktriangleleft)$ is ω-categorical and finitely aximatized

My remark: if the first-order theory of a countably infinite structure $(T, <)$ is ω-categorical and rec. enumerable, then $MSOTH_{\forall}(T, <)$ is also rec. enumerable.

Idea: to take the first-order theory as an axiom set in a signature for $(T, \prec)$ extended by a finite number of unary predicate symbols.

Proof for not rec. enum., $MSOTH_{\forall\exists}(\mathbb{Q}^n, \blacktriangleleft)$

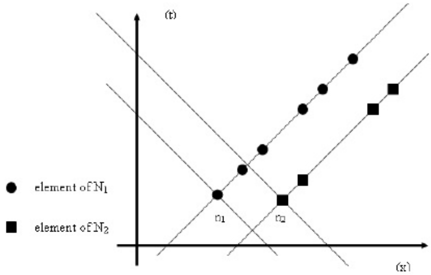
Abbreviations

$(x \triangleleft y) \rightleftharpoons \forall z(y \blacktriangleleft z \to x \blacktriangleleft z) \wedge \neg x \blacktriangleleft y \wedge \neg x = y$
y is on the boundary of the upper light-cone of x (directed optical accessibility)

$\cup(x, y; z) \rightleftharpoons (x \triangleleft z \wedge y \triangleleft z)$
existence and uniqueness of this upper bound z for x and y is not guaranteed, except for the case $n = 2$ – the intersection of light-cones of two rational points may not include rational points, if $n > 2$

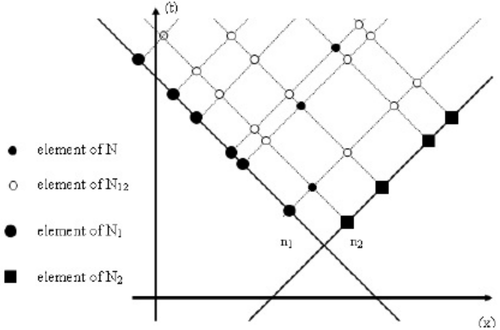
ν_1 is the monadic second-order formula expressing

For each $i \in \{1, 2\}$:
(i) N_i is discretely linear ordered by $\lhd$, minimum n_i, no maximum
(ii) Every lower Dedekind-cut Y (with respect to $\lhd$) of the light-like line M through N_i has the following property: there exist two consecutive points x, y in N_i such that $x \in Y \wedge y \notin Y$ holds.



ν_1 makes $(N_i, \lhd)$ isomorphic to $(\mathbb{N}, <)$.

ν describes a situation similar to



Subset N_{12} is like a grid and subset N pairs the elements of N_1 with the elements of N_2

this is expressible in universal monadic second-order logic

Universal fragment of dyadic second-order theory of $(\mathbb{N}, <)$

universally quantified variables for binary relations on $\mathbb{N}$

not recursively enumerable

a recursive translation m is given from this language into our monadic second-order language

Proved

for all closed dyadic formulae A and $n > 1$, A is in the dyadic second-order theory of $(\mathbb{N}, <)$ iff
$\forall N_1, N_2, N_{12}, N, n_1, n_2(\nu \rightarrow A^m) \in MSOTH(\mathbb{Q}^n, \blacktriangleleft)$

Case $n > 2$: why the $\forall$-fragment is also not rec. enumerable over $\mathbb{Q}$?

definable betweenness and equidistance also for rationals

Goldblatt's method did not work, new method was developed

the new definition working also for rationals

$$\beta_\sigma(x, z, y) \rightleftharpoons \sigma(x, y) \wedge \forall u(x \blacktriangleleft u \wedge y \blacktriangleleft u \rightarrow z \blacktriangleleft u) \wedge \forall u(u \blacktriangleleft x \wedge u \blacktriangleleft y \rightarrow u \blacktriangleleft z)$$

proof: elementary but lengthy

Dedekind-cut formula can be eliminated

Implications

in FOTH($\mathbb{Q}^2, \blacktriangleleft$) there is no definition for equidistance of betweenness

there exists an ω-categorical structure whose $MSOTH_{\forall\exists}$-fragment monadic second-oder theory is not rec. enumerable

$FOTH(\mathbb{Q}^n, \blacktriangleleft)$ is not ω-categorical or is not rec. enumerable – the wet can prove

Thank you for your attention.