Meaning, Truth, and the Diffeomorphism Invariance

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Three philosophical premises

Physicalism: everything is physical; all facts supervene on, or are necessitated by, the physical facts.

Empiricism: genuine information about the world can be acquired only by *a posteriori* means.

Formalism: logic and mathematics are thought of as statements about manipulations with meaningless symbols.

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Instead, I will discuss a few radical consequences of them, concerning the fundamental nature of logic and mathematics, and the structure of physical theories.

Program:

I. Outlines of the **physico-formalist philosophy of mathematics**

II. Physico-formalist account of **physical theory –** meaning and truth

III. Discussion of the problem of **diffeomorphism invariance** in GR

Physico-formalist philosophy of mathematics

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(L.E. Szabó, Formal System as Physical Objects: A Physicalist Account of Mathematical Truth, *International Studies in the Philosophy of Science* **17** (2003) 117–125.)

The problem:

If physicalism is true, then the logical/mathematical facts must be necessitated by the physical facts of the world.

> The aim of the project is to clarify how logical and mathematical facts can be accommodated in a purely physical ontology.

What are logical/mathematical facts? What is it that has to be accounted for within a physicalist ontology?

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This is the point where
PHYSICO-FORMALIST PHILOSOPHY OF MATHEMATICS
starts!

The question we are asking now is:

Where are the states of affairs located in the ontological picture of the (physical) world that make propositions like "$\Sigma \vdash A$" true or false?

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> **The PHYSICO-FORMALIST thesis:** A formal system should be regarded as a physical system which consists of signs and derivational mechanisms embodied in concrete physical objects and concrete physical processes.

Therefore, a "$\Sigma \vdash A$"-type mathematical proposition expresses an *objective fact* of the formal system as a particular portion *of the physical world*.

I argue in three steps.

(Step I) A formal systems can be represented in a physical system

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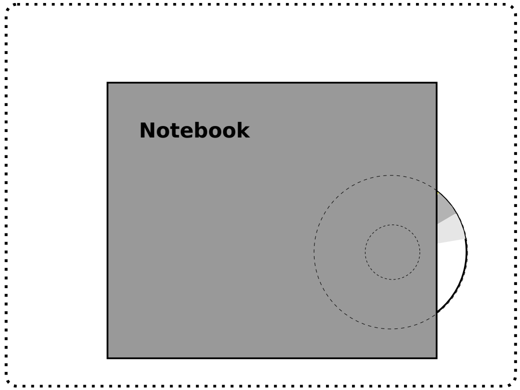

In some order, the computer lists the theorems and the proofs of a formal system. It is commonly accepted to say that in the "computer + CD" system we have "a physical representation of the formal system" in question.

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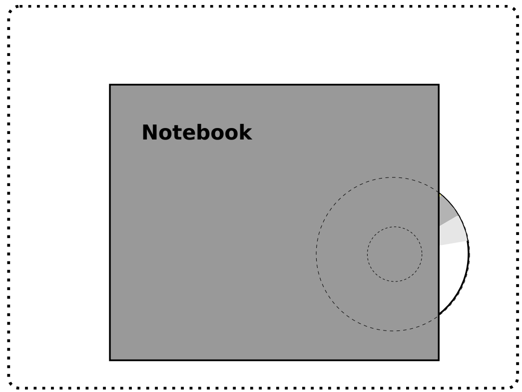

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In this represantation, "$\Sigma \vdash A$" (whether formula A is printed) is a fact of **the physical reality** inside the dotted line!

(Step II) We have access to a formal system only in some concrete physical representation

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> **[Mathematical truths] are revealed to us only through the physical world.** It is only physical objects, such as computers or human brains, that ever give us glimpses of the abstract world of mathematics. (Deutsch, Ekert, and Lupacchini 2000)

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> **[I]n order to think of a formal system at all we must think of it as represented somehow.** But when we think of it *as* formal system **we abstract from all properties peculiar to the representation**. (Curry 1951)

(Step III) Actually, there is nothing to be "represented"; there is nothing beyond the flesh and blood formal systems.

Abstraction is a motion from one physically existing formal system to another physically existing formal system.

Abstraction does not produce "abstract formal systems" over and above the physically existing "representations".

(For argumentation: L.E. Szabó, Mathematical facts in a physicalist ontology, *Parallel Processing Letters*, **22** (2012) 1240009)

Consequences

- Any fact about a formal system L – including a fact like "$\Sigma \vdash A$" – is a fact of the physical reality; which is
  - a posteriori
  - not necessary
  - not certain
  - independent from human mind
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- Deduction – as it is observation of a fact of a formal system as physical object – is a particular case of inductive generalization (from finite body of empirical observations), as in any other empirical sciences
- The reasoning cannot deliver to us more certainty than experience: *The certainty obtainable from experience is the best of all possible certainties*

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Semantics *S* is a kind of "correspondence" between (some of) the formulas of L and the states of affairs in U.

We distinguish between the following two facts:

(Fact I) *A is a theorem* in L: $\Sigma \vdash A$

(Fact II) *A is true*: according to the semantics S, A refers to a state of affairs in U, which is in fact the case.

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But, how can the physicalist account for the fact of meaning and truth?

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As we can learn from **Gödel's construction of representation of the meta-arithmetic statements in arithmetic**:

Formula A ***represents*** (means) a state of affairs a in U, if the following two conditions are met:

(a) There exist a family $\{A_\lambda\}_\lambda$ of formulas in L and a family $\{a_\lambda\}_\lambda$ of state of affairs in U, such that $A = A_{\lambda_0}$ and $a = a_{\lambda_0}$ for some λ_0.

(b) For all λ,

$$\begin{array}{rcl} \text{if } a_\lambda \text{ is the case in } U & \text{then} & \Sigma \vdash A_\lambda \\ \text{if } a_\lambda \text{ is not the case in } U & \text{then} & \Sigma \vdash \neg A_\lambda \end{array}$$

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Formula R ***represents*** (means) a state of affairs Pr in Meta-PA, if the following two conditions are met:

(a) There exist a family $\{R(x,y)\}_{(x,y)}$ of formulas in PA and a family $\{Pr(x,y)\}_{(x,y)}$ of state of affairs in Meta-PA, such that $R = R(x_0, y_0)$ and $Pr = Pr(x_0, y_0)$ for some (x_0, y_0).

(b) For all (x, y),

$$\text{if } Pr(x,y) \text{ is the case in Meta-PA} \quad \text{then} \quad \Sigma_{PA} \vdash R(x,y)$$
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The two conceptions, **meaning** and **truth**, are completely **inter-twined**.

Consequences

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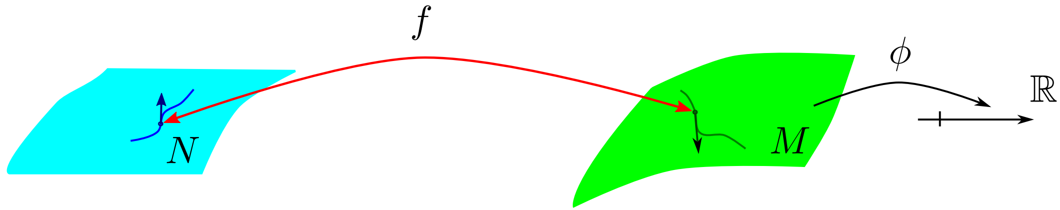
Condition (b) expresses *correlation between physical facts*. Combining this with the thesis of (1) *the causal closeness of the physical world*, and (2) *the principle of common cause*, one must conclude:

- ***Semantic relationship* must be brought about by the underlying causal processes of the physical world.**
- ***The truth of the physical theory* (consequently, our *knowledge*) must be brought about by the underlying causal processes of the physical world.** (Empiricism)

Diffeomorphism invariance



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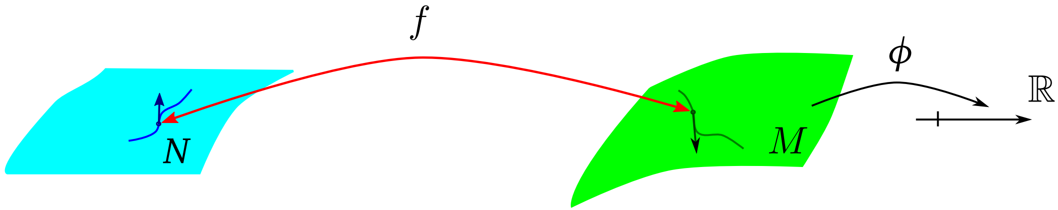

$$
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f^*\phi \quad &:= \quad \phi \circ f \\
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f_*\dot{\gamma} \quad &= \quad (\dot{f(\gamma)}) \\
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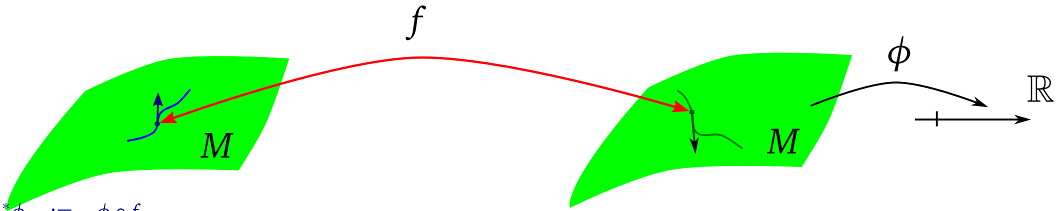

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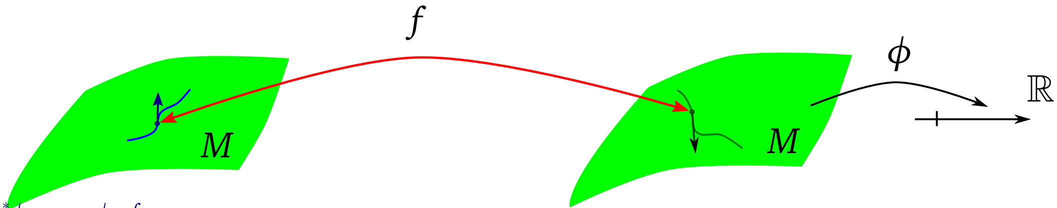
In GR, applying these for the case $N = M$, *the common view* is that:

(A) $\forall f \in Diff(M)$, (M, g) and $(M, f^* g)$ represent the same "physical space-time".

(B) $Diff(M)$ constitute the "group" of transformations against which the laws of physics (e.g. the Einstein eq.) must be covariant (General Covariance Principle)

(C) $Diff(M)$ is a "group" of gauge freedom; that is, the presence of surplus mathematical structure in GR that has no correlate in physical reality.



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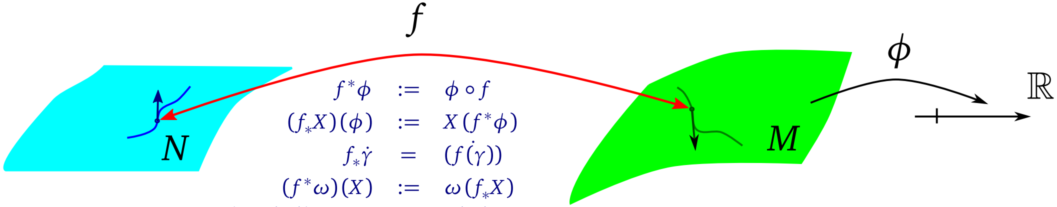

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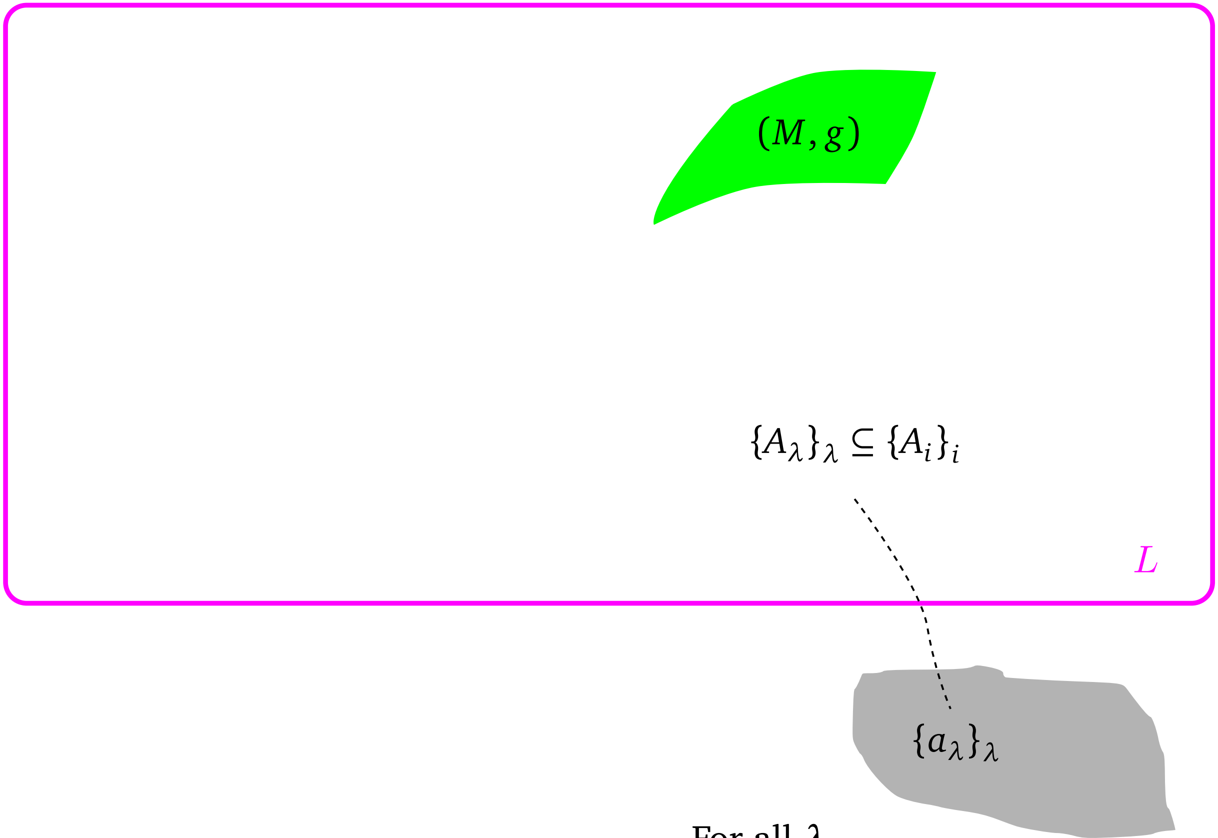
$\{B_i\}_i$ $\qquad\qquad$ $\{A_i\}_i$

such that for all i

$$B_i \longleftrightarrow A_i$$

L

With no meaning!



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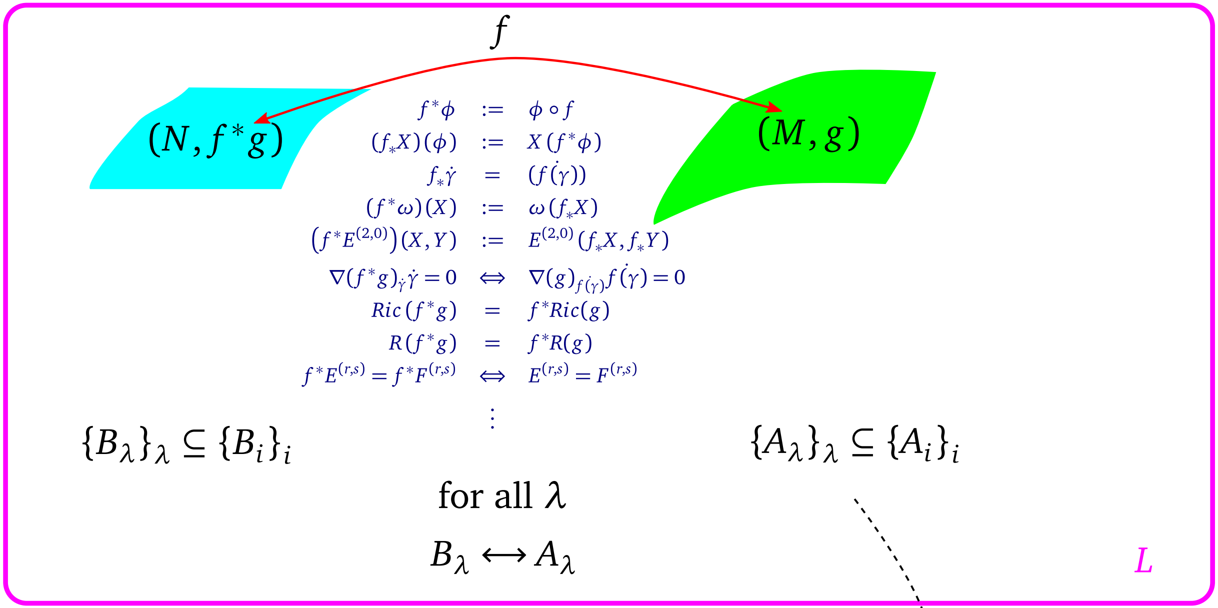

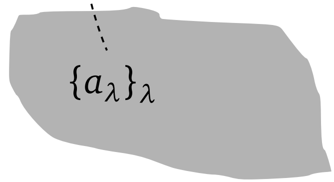

Therefore:

For all λ

if a_λ is the case then $\Sigma \vdash B_\lambda$

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That is, $\{B_\lambda\}_\lambda$ can carry exactly the same meaning as $\{A_\lambda\}_\lambda$.

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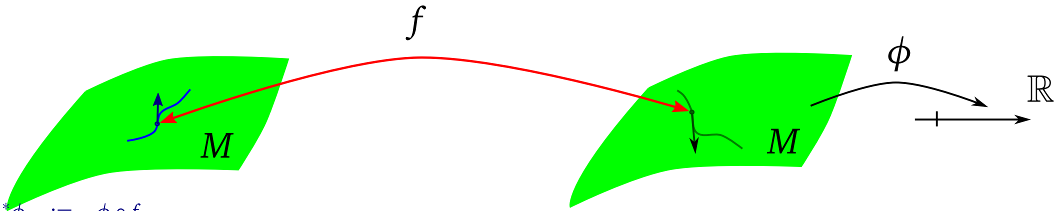

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The usual claim is based on a fundamental misunderstanding of the covariance principle!

The physical laws must be covariant against the (contingent) **transformation laws** of the physical quantities ascertained by the measuring devices moving in various ways.[1] (In the General Principle they may arbitrarily accelerate, etc.)

- Whether these laws have anything to do with $Diff(M)$, $Diff^*(M)$, $Diff_*(M), \ldots$ can be known **only by a posteriori means**;
- **prior** to which we should know how the physical quantities in question are **operationally defined**.[2]

None of them can be known from the trivial MATHEMATICAL fact (A)!

[1] M. Gömöri and L. E. Szabó: Formal statement of the special principle of relativity *Synthese* (2013), DOI: 10.1007/s11229-013-0374-1

[2] L. E. Szabó: Empirical Foundation of Space and Time, in *EPSA07*, Springer 2009.

$Diff(M) \neq$ **gauge group**

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- **Prior** to say that $Diff(M)$ generates surplus non-observable degrees of freedom, we should know what the **observable** quantities are;

- without which to say that the observable quantities are $Diff(M)$-invariant is a question begging.

- Why just $Diff(M)$?
  (A) is a trivial truth! For an arbitrary physical quantity X:

$$X = 2 \text{ iff } X^3 = 8$$

  Is that a gauge freedom? Does it mean that no physical quantity can express an observable physical fact?

Einstein equation

Einstein equation

- Einstein equation can be a true physical law even if it does not determine a unique solution by fixing the solution around the Hole or a Cauchy surface.

- Determinism can be true even if the Einstein equation does not provide a complete description of the world.