Squares of modal logics and relation algebras

Valentin Shehtman

Institute for Information Transmission Problems
& Higher School of Economics
& Moscow State University

Budapest, August 2015

Introduction

In this talk we present results on the meet of two fields — modal logic and algebraic logic. The third participant is game theory.

Applications of games to algebraic logic:

- R. Hirsch, I. Hodkinson. Relation algebras by games. Elsevier, 2002.

Games have also got closer to modal logic in recent years

- J. Van Benthem. Logic in games. MIT Press, 2013.

The presented research continues this trend

Relation algebras

Binary relations on a set constitute a boolean algebra with

extra operations: composition, inversion, and an extra constant, the diagonal relation.

A dream of describing properties of these operations in a nice way exists since the middle of 19^{th} century (De Morgan).

In 1941 Alfred Tarski proposed a list of extra axioms that should be added to axioms of boolean algebras.

Relation algebras

Binary relations on a set constitute a boolean algebra with

extra operations: composition, inversion, and an extra constant, the diagonal relation.

A dream of describing properties of these operations in a nice way exists since the middle of 19^{th} century (De Morgan).

In 1941 Alfred Tarski proposed a list of extra axioms that should be added to axioms of boolean algebras.

Tarski's axioms

$a \circ (b \cup c) = (a \circ b) \cup (a \circ c)$

$a \circ \delta = a$

$(a^{-1})^{-1} = a$

$(a \cup b)^{-1} = a^{-1} \cup b^{-1}$

$(a \circ b)^{-1} = b^{-1} \circ a^{-1}$

$a^{-1} \circ (-(a \circ b)) \leq - b$

Full relation algebras

$$\mathrm{Rel}(W) = (2^{W^2}, \circ, I_W, {}^{-1}),$$

where W is a set,

2^{W^2} is the full boolean algebra of binary relations on W,

$\circ$ is the composition of relations,

I_W is the diagonal,

${}^{-1}$ is the converse.

Relation algebras-2

RA is the variety of relation algebras.

Its subvariety of representable relation algebras **RRA** is generated by full relation algebras.

1. **RA** ≠ **RRA** (Lyndon, 1950).
2. Moreover, the equational theory Eq(**RRA**) is not finitely axiomatizable (Monk, 1964).

Unlike the boolean case, the equational theories are undecidable:

3. Eq(**RA**) is undecidable (Tarski, 1941).
4. Eq(**RRA**) is undecidable, and moreover, for finite relation algebras representability is undecidable (Hirsch&Hodkinson, 1999).

Relation algebras-3

5. Eq($\mathbf{RA}_{\text{fin}}$), Eq($\mathbf{RRA}_{\text{fin}}$) are also undecidable.

(Tarski's problem solved by Andréka & Givant & Németi, 1997)

How about fragments of these theories?

6. In the signature ($\circ$, $\cap$) the variety generated by representable algebras is finitely axiomatizable (Andréka & Mikulász, 2011)

Relation algebras-4

7. In the signature ($\circ$, $\cap$, $^{-1}$) the variety generated by representable algebras is not f.a. (Hodkinson & Mikulász, 2011)

Question What happens in the signature (Boolean, $\circ$)?

Our approach: we do not restrict the signature, but

the term formation rules. This means that we extract other operations from relation algebras.

Modal algebras from relation algebras

A (normal) n-modal algebra is a boolean algebra with extra n unary operations $\Diamond_1$,..., $\Diamond_n$ distributing over $\cup$ and preserving 0.

In relation algebras we have

$$a \circ (b \cup c) = (a \circ b) \cup (a \circ c),$$

so every fixed a gives us a modal algebra.

But we also have

$$(b \cup c) \circ a = (b \circ a) \cup (c \circ a).$$

Different left and right multiplications generate a polymodal algebra with nontrivial identities. This leads us to products of modal logics.

Modal propositional language

N-modal formulas are built from a countable set of proposition letters PL=$\{p_1,p_2,...\}$ using boolean connectives and unary modal connectives $\Box_1, ..., \Box_N$;as usual $\Diamond_i = \neg \Box_i \neg$

If N=1 we denote the modalities just by $\Box$ and $\Diamond$.

The modal depth md(A) is defined by induction:

$md(p_i)=0$, $md(\neg A)=md(A)$,

$md(A \vee B)= md(A \wedge B) = max(md(A),md(B))$,

$md(\Box_i A)=md(A)+1$

Kripke frames and models-1

An N-modal Kripke frame is a nonempty set with N binary relations $F = (W, R_1, \ldots, R_N)$.

A valuation in F is a function $\theta : PL \rightarrow 2^W$ (so $\theta(p_i) \subseteq W$).

(F, θ) is a *Kripke model* over F.

In *k-weak Kripke models* only the letters $p_1, \ldots, p_k$ are evaluated.

Kripke frames and models-2

The inductive truth definition ($M,x \vDash A$) is standard.

- $M,x \vDash p_i$ iff $x \in \theta(p_i)$
- $M,x \vDash \Box_i A$ iff $\forall y(xR_i y \Rightarrow M,y \vDash A)$
- $M,x \vDash \Diamond_i A$ iff $\exists y(xR_i y \;\&\; M,y \vDash A)$

A formula A is valid in a frame F (in symbols, $F \vDash A$) if A is true at all points in every Kripke model over F.

Logics-1

An *N-modal logic* is a set of N-modal formulas L with the following properties:

- L contains all boolean tautologies
- L is closed under Modus Ponens: if A, $A \rightarrow B \in L$, then $B \in L$.
- L is closed under Substitution:

  if $A(p_1,\ldots,p_n) \in L$, then $A(B_1,\ldots,B_n)$ (for any formulas $B_1,\ldots,B_n$)

- if $A \in L$, then $\Box_i A \in L$
- $\Box_i(A \rightarrow B) \rightarrow (\Box_i A \rightarrow \Box_i B) \in L$

The *minimal logic* $\mathbf{K}_N$ is the smallest such set; $\mathbf{K}$ denotes $\mathbf{K}_1$.

Logics-2

For a set Γ of formulas in the language of L and a logic L

L+Γ denotes the smallest modal (intermediate) logic containing (L∪Γ).

Logics of the form **K**+Γ (or **H**+Γ), where Γ is finite, are called *finitely axiomatizable.*

Logics-4

<u>Proposition</u> (Soundness theorem) For a Kripke frame F

L(F) := { A | F ⊨ A} is a modal logic (the *logic of* F).

- **L**($\mathcal{C}$) := $\bigcap$ {**L**(F)|F ∈ $\mathcal{C}$} (the *logic of a class of frames* $\mathcal{C}$).

  Logics of the form **L**($\mathcal{C}$) are called *Kripke complete*.

- If F is finite, **L**(F) is called *tabular* (or *finite*)
- If $\mathcal{C}$ consists of finite frames, **L**($\mathcal{C}$) has the finite model property (FMP). Or:

  L has the FMP iff L is an intersection of tabular logics.

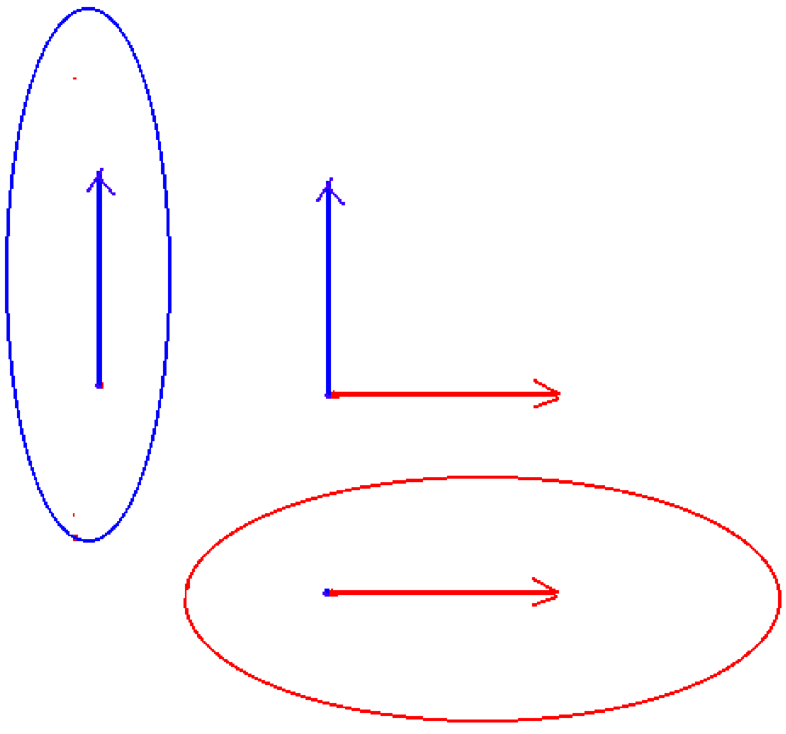
<u>Proposition</u> (Harrop's theorem) If L is finitely axiomatizable and has the FMP, then L is decidable.

Products of frames

Def. The product of two Kripke frames $(W,R_1,\ldots,R_n)\times(V,S_1,\ldots,S_m):=(W\times V,R_{11},\ldots,R_{n1},S_{12},\ldots,S_{m2})$, where

$$(x_1,y_1)R_{i1}(x_2,y_2) \text{ iff } x_1R_ix_2 \;\&\; y_1=y_2$$

$$(x_1,y_1)S_{j2}(x_2,y_2) \text{ iff } x_1=x_2 \;\&\; y_1S_jy_2$$



Products of modal logics

Def. The product of two modal logics

$$L_1 \times L_2 := \mathbf{L}(\{F_1 \times F_2 \mid F_1 \vDash L_1, F_2 \vDash L_2\})$$

AXIOMATIZATION PROBLEM: *to find axioms of* $L_1 \times L_2$ *given the axioms of* L_1, L_2

Def. The fusion of two modal logics with disjoint modalities

$$L_1 * L_2 := \text{the smallest logic containing } L_1 \text{ and } L_2$$

Def. The commutative join of two modal logics with disjoint modalities

$\Box_i$ $(1 \leq i \leq n)$, $\blacksquare_j$ $(1 \leq j \leq m)$

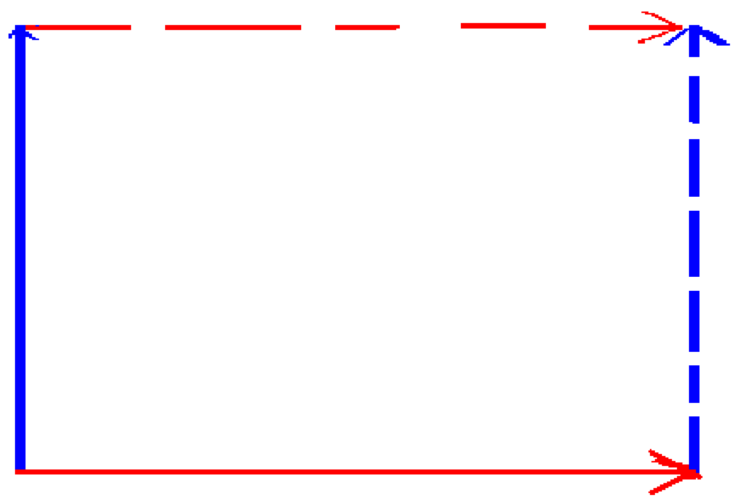
$[L_1, L_2] := L_1 * L_2 + \Box_i \blacksquare_j p \leftrightarrow \blacksquare_j \Box_i p + \Diamond_i \blacksquare_j p \to \blacksquare_j \Diamond_i p$ (for any i, j)

Remark. If the modalities are not disjoint, we can change them.

Product of modal logics-2

These are Sahlqvist formulas expressing the following properties of the relations in the product frame

$$\Diamond_i \blacksquare_j p \to \blacksquare_j \Diamond_i p:$$



Products of modal logics-3

$$\Box_i \blacksquare_j p \leftrightarrow \blacksquare_j \Box_i p$$

$R_{i1} \circ S_{k2} = S_{k2} \circ R_{i1}$ (commutativity)

Def. Logics L_1, L_2 are product-matching if $L_1 \times L_2 = [L_1, L_2]$.

Squares

For a class of frames C put

$$C^2 := \{F \times F \mid F \models C\}.$$

For a modal logic Λ put

$$\Lambda^2 := \Lambda \times \Lambda$$

Proposition 1 [Gabbay,Sh 2000]

(1) $\Lambda^2 = L(\{F \times F \mid F \models \Lambda\})$

(Squares of logics are determined by squares of frames)

(2) $L_1 \times L_2$ is embeddable in $(L_1 * L_2)^2$.

(Products are reducible to squares)

Segerberg squares

These are square frames with additional functions. Krister Segerberg (1973) studied a special type - squares of frames with the universal relation.

He considered the following functions on squares.

σ_0: $(x,y) \mapsto (y,x)$ (the diagonal symmetry)

σ_1: $(x,y) \mapsto (y,y)$ (the first diagonal projection)

σ_2: $(x,y) \mapsto (x,x)$ (the second diagonal projection)

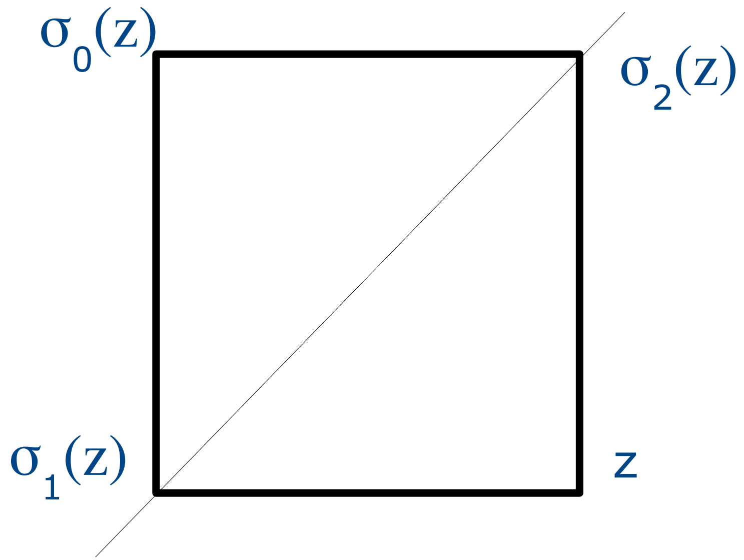
These functions can be associated with extra modal operators $\bigcirc$, $\bigcirc_1$, $\bigcirc_2$. So in square frames they are interpreted as follows:

$(x,y) \vDash \bigcirc A$ iff $(y,x) \vDash A$

$(x,y) \vDash \bigcirc_1 A$ iff $(y,y) \vDash A$

$(x,y) \vDash \bigcirc_2 A$ iff $(x,x) \vDash A$

Segerberg squares-2



Formally we can define the Segerberg square of a frame

$F=(W, R_1,..., R_n)$ as the (2n+3)-frame $F^2_{Sg} := (F^2, \sigma_0, \sigma_1, \sigma_2)$

(where σ_i are the functions on W^2 described above).

Respectively, the Segerberg square of an n-modal logic Λ is the logic of the Segerberg squares of its frames

$$\Lambda^2_{Sg} := L(\{F^2_{Sg} \mid F \vDash \Lambda\}).$$

TOMORROW (OR SUCCESSOR) LOGIC

$$\mathbf{SL} := \mathbf{K} + \Diamond p \leftrightarrow \Box p$$

This well-known logic is also due to Segerberg (1967). It is complete w.r.t. the frame

(the successor relation on natural numbers).

Every logic of a frame with a functional accessibility relation is an extension of **SL**.

Axiomatizing Segerberg squares

Soundness Here are some formulas valid in Segerberg squares. The corresponding semantic conditions for an arbitrary (2n+3)-frame $(V, X_1, \ldots, X_n, Y_1, \ldots, Y_n, f_0, f_1, f_2)$

are in the right column; here fg denotes the composition of functions: $(fg)(x)=f(g(x))$

- The **SL**-axioms for the circles $\bigcirc$, $\bigcirc_1$, $\bigcirc_2$.

(Sg1) $\bigcirc\bigcirc p \leftrightarrow p$ $\quad f_0 f_0 = 1_V$ (the identity function on V)

i.e., f_0 is an involution

(Sg2) $\bigcirc_1\bigcirc_1 p \leftrightarrow \bigcirc_1 p$ $\quad f_1 f_1 = f_1$

i.e., the image of f_1 consists of fixed points

(In Segerberg squares this image is the diagonal)

Axiomatizing Segerberg squares-2

(Sg3) $\bigcirc\bigcirc_1 p \leftrightarrow \bigcirc_2 p$ $\quad f_1 f_0 = f_2$

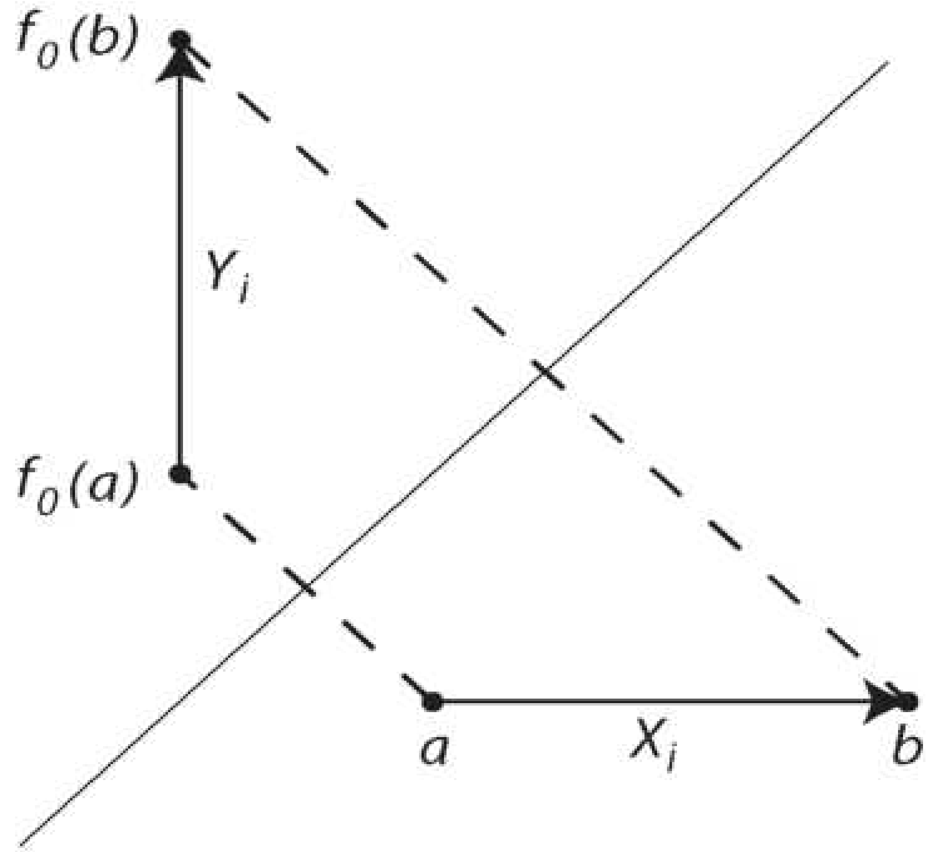
This axiom allows us to eliminate $\bigcirc_2$

(Sg4) $\bigcirc_1\bigcirc p \leftrightarrow \bigcirc_1 p$ $\quad f_0 f_1 = f_1$

The image of f_1 consists of fixed points of f_0

(in Segerberg squares: every diagonal point is self-symmetric).

Axiomatizing Segerberg squares-3



(Sg5)

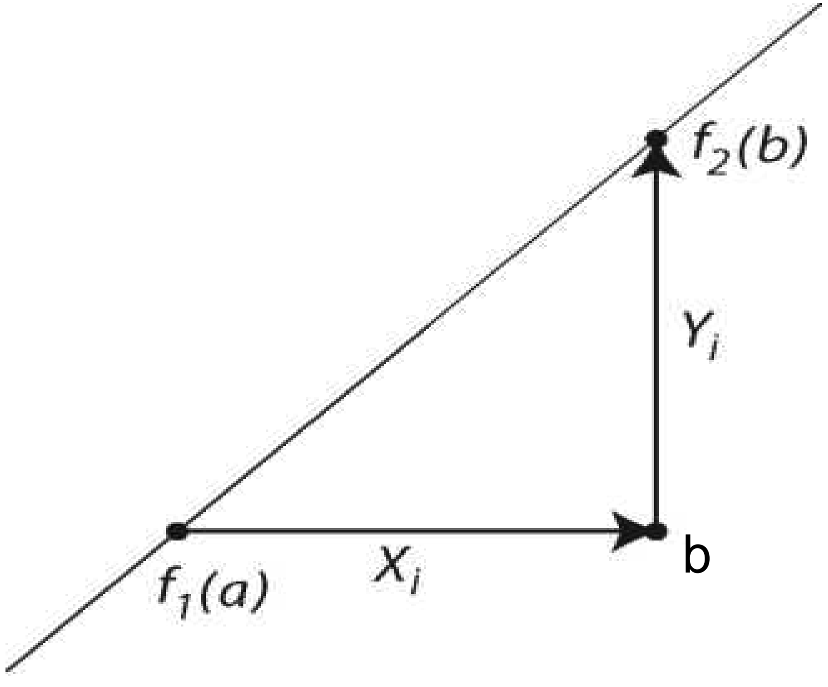
(Sg5) $\bigcirc\Box_i\bigcirc p \leftrightarrow \blacksquare_i p$ $\quad aX_i b \Rightarrow f_0(a)Y_i f_0(b)$

In Segerberg squares: the diagonal symmetry is an isomorphism between R_{i1} and R_{i2}. This axiom allows us to eliminate $\blacksquare_i$

Axiomatizing Segerberg squares-4

(Sg6) $\bigcirc_1 \Box_i (\blacksquare_i p \rightarrow \bigcirc_2 p)$ $f_1(a) R X_i b \Rightarrow b R Y_i f_2(b)$

In Segerberg squares: If $(y,y) R_{i1} (x,y)$ (i.e. $y R_i x$), then $(x,y) R_{i2} (x,x)$.

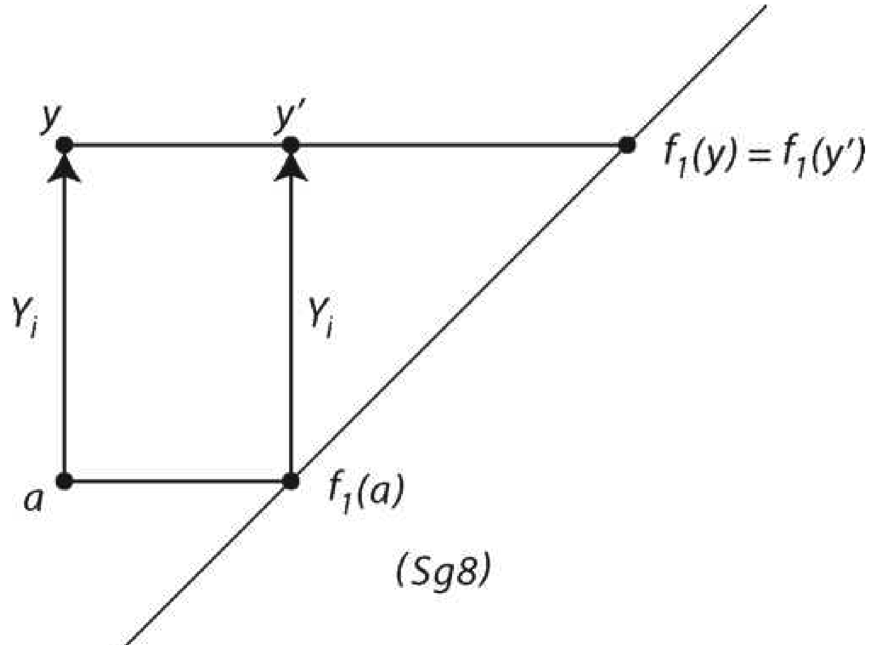


(Sg6)

Axiomatizing Segerberg squares-4

(Sg7) $\bigcirc_1 p \rightarrow \Box_i \bigcirc_1 p$ $aX_i b \Rightarrow f_1(a) = f_1(b)$

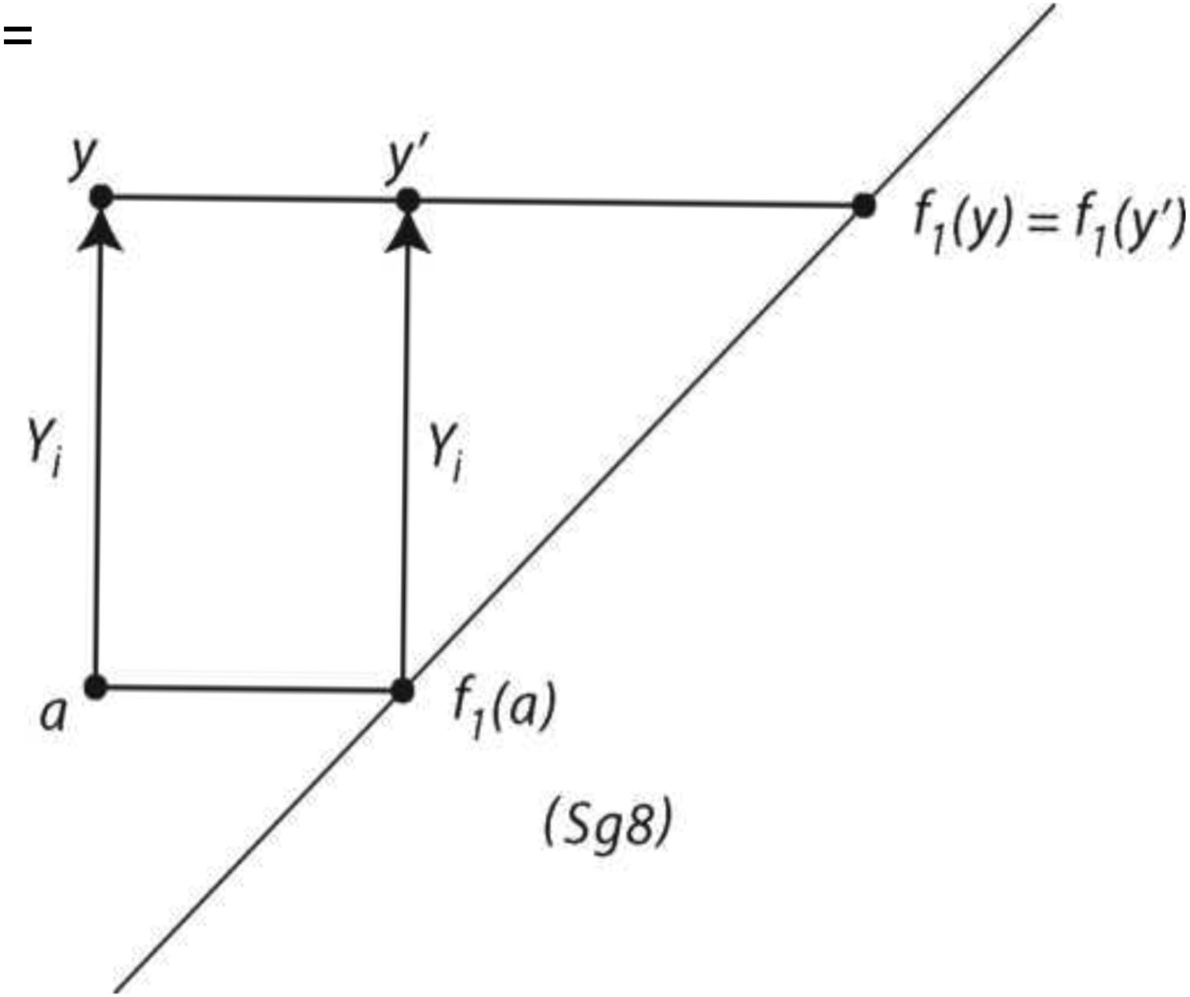
In Segerberg squares: horizontally accessible points are in the same horizontal row.



Axiomatizing Segerberg squares-5

(Sg8) $\blacksquare_i \bigcirc_1 p \leftrightarrow \bigcirc_1 \blacksquare_i \bigcirc_1 p$ $f_1[Y_i(a)] = f_1[Y_i(f_1(a))]$

In Segerberg squares: the rows vertically accessible from a=(x,y) and $f_1(a)$=



Axiomatizing Segerberg squares-6

Further on we regard a Segerberg square F^2_{Sg} of a frame

$F=(W, R_1,\dots, R_n)$ as the (2n+3)-frame

$$(W^2, R_{11},\dots, R_{n1}, \sigma_0, \sigma_1)$$

Respectively the Segerberg square Λ^2_{Sg} of an n-modal logic Λ is an (n+2)-modal logic in the language $\Box_1, \dots, \Box_n, \bigcirc, \bigcirc_1$

Def. For a modal logic Λ, put

$[\Lambda, \Lambda]⦿ := [\Lambda, \Lambda]$ + **SL*SL** (for $\bigcirc, \bigcirc_1$) +

{(Sg1),(Sg2),(Sg4), (Sg6), (Sg7), (Sg8)}.

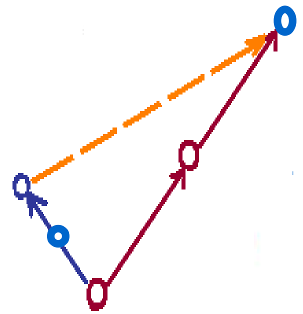
Now (Sg3), (Sg5) become definitions

Examples of product-matching logics

Def. A Horn sentence is a classical first-order sentence of the form $\forall x \forall y \forall \mathbf{z}\, (\varphi(x,y,\mathbf{z}) \to R(x,y))$,

where φ is positive, $R(x,y)$ is atomic. A modal formula A is Horn if the class of its frames V(A) is axiomatizable by a Horn sentence.

Examples: $(\Diamond \ldots \Diamond)\, \Box p \to (\Box \ldots \Box)\, p$



Completeness theorems

Def. A modal logic is Horn axiomatizable if if it is axiomatizable by formulas that are either variable-free or correspond to Horn sentences.

Completeness theorem for products

[BOOK03>>Gabbay,Sh 1998]

If two modal logics are Horn axiomatizable and Kripke complete, then they are product-matching.

Completeness theorem for Segerberg squares

[Sh2011, 2012]

If a logic Λ is Horn axiomatizable, then $\Lambda^2{}_{Sg}=[\Lambda,\Lambda]^{\odot}$

Remark on Segerberg's logic

Segerberg (1973) axiomatized

the logic of Segerberg squares of universal frames $(W, W \times W)$.
In this case (Sg8) becomes trivial and (Sg6) should be replaced
with a stronger axiom: $\Box p \rightarrow \bigcirc_1 p$

This logic is not a Segerberg square in our sense; it is a proper extension of $\mathbf{S5}^2_{Sg}$

The finite model property

Def. QT-formulas:

$\Box_i p \rightarrow \Box_i^k p$ (generalized transitivity)

$\Diamond_i \Box_i p \rightarrow p$ (symmetry)

QTC-logic is axiomatizable by formulas that are either variable-free or QT-formulas.

$K.t_n$ is the minimal n-temporal logic

(axiomatized by $\Diamond_i \Box_{-i} p \rightarrow p$ for i = ±1,..., ±n).

The finite model property-2

Theorem on the fmp for products [Sh 2005]

If L_2 is a QTC-logic, then $\mathbf{K.t}_n \times L_2 = [\mathbf{K.t}_n, L_2]$ has the fmp.

Theorem on the fmp for Segerberg squares [Sh 2014]

$(\mathbf{K.t}_n)^2_{Sg}$ has the fmp.

Product and square fmp

Def. A product logic $L_1 \times L_2$ has the product fmp if it is complete w.r.t product of finite frames.

A Segerberg square L^2_{Sg} has the square fmp if it is complete w.r.t Segerberg squares of finite frames.

Theorems on the product fmp

1. $\mathbf{K}_n^2$, $\mathbf{D}_n^2$, $\mathbf{T}_n^2$ have the product fmp [Gabbay&Sh 2000]
2. $\mathbf{K.t}_n \times \mathbf{K}_n$ has the product fmp [Gabbay&Sh 2002]

Conjecture (very probable) $(\mathbf{K.t}_n)^2$ has the product fmp.

Square fmp

Theorems on the square fmp

1. $(\mathbf{K}_n{}^2)_{Sg}$ has the square fmp [Sh 2011,2012]

2. $(\mathbf{T}_n{}^2)_{Sg}$, $(\mathbf{D}_n{}^2)_{Sg}$ have the square fmp [Sh 2015]

Problems Does $(\mathbf{K.t}_n)^2{}_{Sg}$ have the product fmp?

Does $(\mathbf{KB})^2{}_{Sg}$ have the product fmp?

Note that **KB** is embeddable in **K.t** by interpreting

$\Box p$ as $(\Box_1 p \wedge \Box_{-1} p)$

We may suppose that both answers are negative.

From modal formulas to relation algebra terms

We define the translation of a modal formula A in the language of $(\mathbf{K.t}_n{}^2)_{Sg}$ into a relational term A^∇ by induction (for k=1,2,...;i=1,...,n)

$p_k{}^\nabla = p_k$ for a proposition letter p_k , $\bot^\nabla = 0$,

$(A \to B)^\nabla = A^\nabla \to B^\nabla$

$(\Diamond_i A)^\nabla = r_i \circ A^\nabla$, $(\Diamond_{-i} A)^\nabla = (r_i)^{-1} \circ A^\nabla$,

$(\blacklozenge_i A)^\nabla = A^\nabla \circ (r_i)^{-1}$, $(\blacklozenge_{-i} A)^\nabla = A^\nabla \circ r_i$,

$(\bigcirc A)^\nabla = (A^\nabla)^{-1}$, $(\bigcirc_1 A) = 1 \circ (A^\nabla \cap \delta)$

From modal logics to relation algebras

Embedding theorem 1 [Sh 2015]

The following conditions are equivalent

1. $(\mathbf{K.t}_n{}^2)_{Sg} \vdash A$
2. $RA \models A^\nabla = 1$
3. $RRA \models A^\nabla = 1$

Embedding theorem 2 [Sh 2015]

The following conditions are equivalent

1. $(\mathbf{K}_n{}^2)_{Sg} \vdash A$
2. $RA \models A^\nabla = 1$
3. $RRA \models A^\nabla = 1$
4. $RA_{fin} \models A^\nabla = 1$
5. $RRA_{fin} \models A^\nabla = 1$

From modal logics to relation algebras

Embedding theorem 2 [Sh 2015]

The following conditions are equivalent

1. $(\mathbf{T}_n{}^2)_{Sg} \vdash A$
2. $RA \models 1 \leq r_1 \cap \ldots \cap r_n \rightarrow A^{\nabla} = 1$
3. $RRA \models 1 \leq r_1 \cap \ldots \cap r_n \rightarrow A^{\nabla} = 1$
4. $RA_{fin} \models 1 \leq r_1 \cap \ldots \cap r_n \rightarrow A^{\nabla} = 1$
5. $RRA_{fin} \models 1 \leq r_1 \cap \ldots \cap r_n \rightarrow A^{\nabla} = 1$

Bisimulation games-1

Def For a k-weak Kripke model $M=(W,R_1,\ldots,R_N,\theta)$ consider the *0-equivalence* relation between points

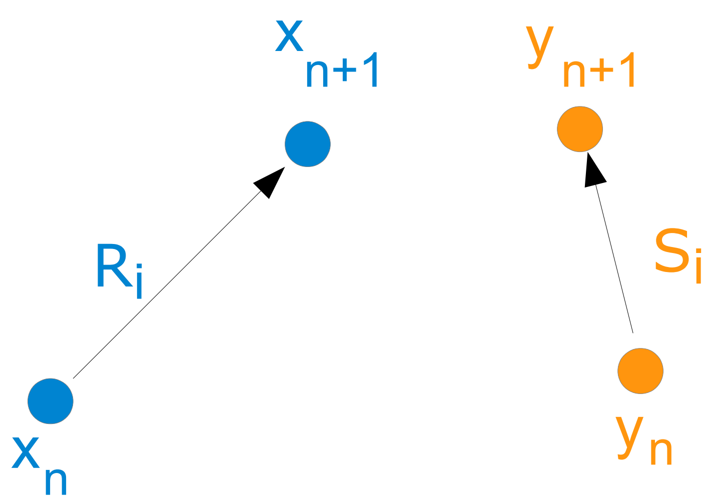
$$x \equiv_0 y := \forall j \leq k\ (M,x \vDash p_j \Leftrightarrow M,y \vDash p_j)$$

Given M and two points $x_0 \equiv_0 y_0$ we can play the *r-round bisimulation game* $BG_r(M,x_0,y_0)$.

Players: Spoiler (Abelard) vs Duplicator (Eloïse).

Bisimulation games-2

The initial position in $BG_r(M,x_0,M',y_0)$ is (x_0,y_0).



<u>Round (n+1)</u>

- Spoiler chooses i, x_{n+1} [or y_{n+1}] such that $x_n R_i x_{n+1}$ [$y_n R_i y_{n+1}$]
- Duplicator chooses y_{n+1} [x_{n+1}] such that $y_n R_i y_{n+1}$ [$x_n R_i x_{n+1}$] and $x_{n+1} \equiv_0 y_{n+1}$

- A player loses if he/she cannot move.
- Duplicator wins after r rounds.

Bisimulation games-3

Def Formula and game *n-equivalence* relations (on M)

- $x \equiv_n y$:= for any $A(p_1,\ldots,p_k)$ of modal depth $\leq n$

$$M,x \vDash A \Leftrightarrow M',y \vDash A$$

- $x \sim_n y$:= Duplicator has a winning strategy in $BG_n(M,x,y)$

Main Theorem on finite bisimulation games

$$\equiv_n \; = \; \sim_n$$

Formula depth-1

The *modal depth of a formula A in a modal logic L*

$md_L(A) := \min\{md(B) \mid L \vdash A \leftrightarrow B\}$

The *modal depth of a logic L*

$md(L) := \min\{md_L(A) \mid \text{A is in the language of L}\}$

Formula depth-2

Canonical model theorem For any modal logic L (weak or not) one can construct the *canonical model* M_L such that for any A in the language of L

$$M_L \vDash A \text{ iff } L \vdash A$$

In every model we have a decreasing sequence $\equiv_0 \supseteq \equiv_1 \ldots$

$$\equiv_\infty := \bigcap_n \equiv_n$$

Formula depth-3

Lemma 1 Every set $W/\equiv_n$ $(= W/\sim_n)$ is finite.

Lemma 2 $x \equiv_\infty y$ iff for any $A(p_1,...,p_k)$ $(M,x \vDash A \Leftrightarrow M,y \vDash A)$

Lemma 3 In canonical models: $x \equiv_\infty y$ iff $x=y$.

Stabilization theorem If $\equiv_n = \equiv_{n+1}$ in every $M_{L\lceil k}$ (bisimulation games *stabilize at n*), then $md(L) \leq n$.

Local tabularity-1

$L\lceil k$ denotes the restriction of a logic L to formulas in variables $p_1,...,p_k$. The sets $L\lceil k$ are called *weak modal logics*

Def A modal logic L is *locally tabular (or locally finite)* if for any k there are finitely many formulas in $p_1,...,p_k$ up to equivalence in L.

Equivalently: A modal logic L is locally tabular if all its weak fragments $L\lceil k$ are tabular.

Local tabularity-2

Equivalent definitions of local tabularity for a modal logic L:

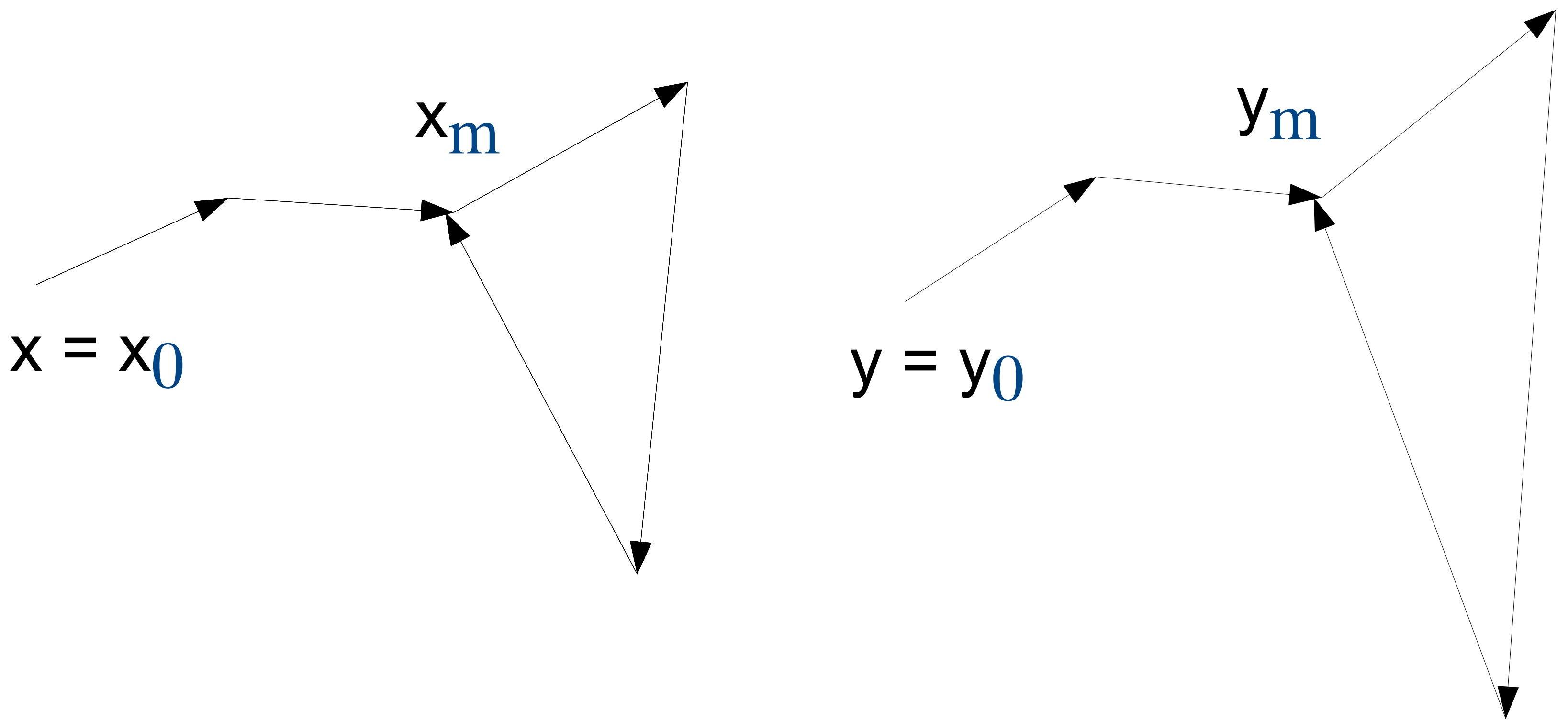
- The variety of L-algebras is *locally finite* : every finitely generated L-algebra is finite
- For every finite k, the free k-generated L-algebra (the *Lindenbaum algebra* of $L\lceil k$) is finite
- Every weak canonical model $M_{L\lceil k}$ is finite.

<u>Proposition</u> Every modal logic of finite modal depth is locally tabular.

Lemma on repeating positions

Let M be a Kripke model, x, y ∈ M. Suppose x $\equiv_n$ y and

moreover, the Duplicator has a winning strategy s in $BG_n(x; y)$ such that every play controlled by s has at least two repeating positions. Then x $\equiv_{n+1}$ y.



Correlation between properties of logics

TABULARITY ⇒ FMD ⇒ LOCAL TABULARITY ⇒ FMP

1. Theorem If F is finite, then $md(L(F)) \leq |F|^2+1$.

Proof: The Pigeonhole principle gives repeating positions.

3. Well-known

2. Easy: there are finitely many k-formulas of bounded modal depth up to equivalence in the basic modal logic.

PROBLEM 1 Does every locally tabular logic have the finite modal depth? (Conjecture:no)

PROBLEM 2 Is there a better upper bound for modal depth of tabular logics? (Conjecture:yes)

Examples of FMD-logics-1

$$md(\mathbf{K} + \Box^n\bot) = n\text{-}1$$

and more generally,

$$md(\mathbf{K}_N + \Box^n\bot) = n\text{-}1$$

where

$$\Box A := \Box_1 A \wedge \ldots \wedge \Box_N A.$$

The axiom $\Box^n\bot$ forbids paths of length n in Kripke frames:

$x_1Rx_2\ldots Rx_n$, where $R = R_1 \cup \ldots \cup R_N$

Proof for the upper bound: every play of a bisimulation game contains at most (n-1) rounds.

An earlier result: $\mathbf{K}_N + \Box^n\bot$ is locally tabular (Gabbay & Sh, 1998; a routine proof by induction).

Modal depth of Segerberg squares

Theorem $md((K_n + \Box^n\bot)^2)_{Sg} \leq m(m+1)+1$.

Corollary $(K_n^2)_{Sg}$ is the intersection of all these logics, so it has the fmp.

Some references

D. Gabbay, A. Kurucz, F. Wolter, M. Zakharyaschev. Many-dimensional Modal Logics: Theory and Applications. Elsevier, 2003.

D. Gabbay, V. Shehtman. Products of modal logics, part 1. Logic Journal of the IGPL, v. 6, pp. 73-146, 1998.

V. Shehtman. Filtration via bisimulation. In: Advances in Modal Logic, Volume 5. King's College Publications, 2005, pp. 289-308