On Einstein Algebras and Relativistic Spacetimes

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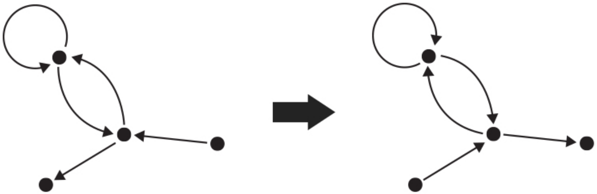
Criterion: Theories T_1 and T_2 are **equivalent** if the category $\mathbf{C}_1$ of models of T_1 is either equivalent *or dual* to the category $\mathbf{C}_2$ of models of T_2.



Category of Relativistic Spacetimes (**GR**)

Objects: (M, g)

Arrows: Isometric embeddings: smooth maps $\varphi : (M, g) \to (M', g')$ s.t. $\varphi^*(g') = g$

- An *algebra* A is a real vector space with a commutative, associative product and a multiplicative identity.
- An *(algebra) homomorphism* is a map between algebras that preserves the vector space operations, product, and multiplicative identity.
- $|A|$ denotes the collection of homomorphisms from A to $\mathbb{R}$, called the *points* of A.
- A is *geometric* if $\bigcap_{p \in |A|} \ker(p) = \{\mathbf{0}\}$.

- $\tilde{A} = \{\tilde{f} : |A| \to \mathbb{R} \ \mid \ \exists f \in A \text{ s.t. } \tilde{f}(x) = x(f) \forall x \in |A|\}$
- Operations on $\tilde{A}$:

$$(\tilde{f} + \alpha\tilde{g})(x) = \tilde{f}(x) + \alpha\tilde{g}(x) = x(f) + \alpha x(g)$$
$$(\tilde{f} \cdot \tilde{g})(x) = \tilde{f}(x) \cdot \tilde{g}(x) = x(f) \cdot g(f)$$

- A geometric $\Rightarrow \tau : A \to \tilde{A}$ as $f \mapsto \tilde{f}$ is an isomorphism.

- The *weak topology* on $|A|$ is the coarsest topology on $|A|$ s.t. every $\tilde{f} \in \tilde{A} \cong A$ is continuous.
- Given alg. homomorphism $\psi : A \to B$, the map $|\psi| : |B| \to |A|$ as $x \mapsto x \circ \psi$ is continuous in the weak topology.

- A is *complete* if for every function $f : |A| \to \mathbb{R}$ which is s.t. for every $p \in |A|$ there is a neighborhood O of p in $|A|$ and an element $\bar{f} \in A$ such that $f\restriction_O \equiv \bar{f}\restriction_O$, $f \in \tilde{A} \cong A$.
- A *smooth algebra* is a complete, geometric algebra with a countable open covering $\{U_k\}$ s.t. each U_k is isomorphic to a subset of $C^\infty(\mathbb{R}^n)$.
- n is the *dimension* of A.

F : **SmoothMan** $\rightarrow$ **SmoothAlg**

- Given a smooth manifold M, $F(M) = C^\infty(M)$.
- Given a smooth map $\varphi : M \rightarrow N$, $F(\varphi)$ is the map $\hat{\varphi} : C^\infty(N) \rightarrow C^\infty(M)$ as $\hat{\varphi}(f) = f \circ \varphi$ for any $f \in C^\infty(N)$.

G : **SmoothAlg** $\to$ **SmoothMan**

- Given a smooth algebra A, $G(A) = |A|$, with charts given by smoothness structure of A.
- Given an algebra homomorphism $\psi : A \to B$, $G(\psi)$ is the map $|\psi| : |B| \to |A|$ between the manifolds $G(B)$ and $G(A)$.

- Correspondence $\theta : M \to G \circ F(M) = |C^\infty(M)|$ as

$$\theta(p)(f) = f(p)$$

  for all $p \in M$, $f \in C^\infty(M)$.
- Correspondence $\eta : A \to F \circ G(A) = C^\infty(|A|)$ as

$$\eta(f)(p) = p(f)$$

  for all $f \in A$, $p \in |A|$.

Theorem

SmoothMan *and* **SmoothAlg** *are dual.*

Idea: F and G are contravariant functors and "up to isomorphism" inverses of one another.

Upshot:

$$\begin{aligned}
A &\quad\leftrightarrow\quad G(A) = |A| \\
M &\quad\leftrightarrow\quad F(M) = C^\infty(M) \\
\psi : A \to B &\quad\leftrightarrow\quad G(\psi) = |\psi| : |B| \to |A| \\
\varphi : M \to N &\quad\leftrightarrow\quad F(\varphi) = |\hat{\varphi}| : C^\infty(N) \to C^\infty(M)
\end{aligned}$$

Derivations on Smooth Algebras

- A *derivation on A* is an $\mathbb{R}$-linear map $\hat{X} : A \to A$ that satisfies the Leibniz rule,

$$\hat{X}(fg) = f\hat{X}(g) + g\hat{X}(f) \qquad \forall f, g \in A.$$

- Derivations at a point:

$$\hat{X}_p : A \to \mathbb{R} \text{ as } \hat{X}_p(f) = \hat{X}(f)(p)$$

- Correspond to vector fields X on $G(A) = |A|$ as

$$\hat{X}_p(f) = X_p(f)$$

for $f \in \tilde{A} \cong A$ and $p \in |A|$.

Metrics on Smooth Algebras

- The space of derivations on A is a module $\Gamma(A)$ over A, with dual module $\Gamma^*(A)$.
- A *metric* on A is a module isomorphism $\hat{g} : \Gamma(A) \to \Gamma^*(A)$ s.t. $\hat{g}(\hat{X})(\hat{Y}) = \hat{g}(\hat{Y})(\hat{X})$ for all $\hat{X}, \hat{Y} \in \Gamma(A)$.
- The *signature* of $\hat{g}$ is the pair $(m, n-m)$, where m is unique s.t. there exists a basis $\xi_1, \ldots, \xi_n$ for the tangent space T_pA such that

$$\begin{array}{rcrcl} \hat{g}(\xi_i, \xi_i) & = & +1 & \text{if} & 1 \leq i \leq m \\ \hat{g}(\xi_j, \xi_j) & = & -1 & \text{if} & m < j \leq n \\ \hat{g}(\xi_i, \xi_j) & = & 0 & \text{if} & i \neq j \end{array}$$

Definition

An *Einstein algebra* is a pair $(A, \hat{g})$ where A is a smooth algebra and $\hat{g}$ is a $(1, n-1)$ metric on A.

Category of Einstein Algebras (**EA**)

Objects: Einstein algebras $(A, \hat{g})$

Arrows: Smooth algebra homomorphisms that preserve the metric.

Lemma

(1) g *a* $(1, n-1)$ *metric on* $M \Rightarrow \hat{g}$ *a* $(1, n-1)$ *metric on* $F(M) = C^\infty(M)$, *where* $\hat{g}(\hat{X})(\hat{Y}) := g(X, Y)$;

(2) $\hat{h}$ *a* $(1, n-1)$ *metric on* $A \Rightarrow |\hat{h}|$ *a* $(1, n-1)$ *metric on* $G(A) = |A|$, *where* $|\hat{h}|(X, Y) := \hat{h}(\hat{X})(\hat{Y})$;

(3) $|\hat{g}| = g$;

(4) $\widehat{|\hat{h}|} = \hat{h}$.

$J : \mathbf{GR} \to \mathbf{EA}$

- $J(M,g) = (C^\infty(M), \hat{g})$, where $\hat{g}$ is given by Lemma.
- Given an isometric embedding $\varphi : (M,g) \to (M', g')$,

$$J(\varphi) = F(\varphi) = \hat{\varphi} : C^\infty(M') \to C^\infty(M).$$

$K : \mathbf{EA} \to \mathbf{GR}$

- $K(A, \hat{g}) = (|A|, |\hat{g}|)$, where $|\hat{g}|$ is given by Lemma.
- Given an EA homomorphism $\psi : (A, \hat{g}) \to (A', \hat{g}')$,

$$K(\psi) = G(\psi) = |\psi| : |A'| \to |A|.$$

Theorem

The categories **EA** *and* **GR** *are dual*

Idea: J and K are contravariant functors and "up to isomorphism" inverses of one another.

Upshot:

$$\begin{array}{rcl} (A, \hat{g}) & \leftrightarrow & K(A, \hat{g}) = (|A|, \hat{g}) \\ (M, g) & \leftrightarrow & J(M, g) = (C^\infty(M), \hat{g}) \\ \psi : (A, \hat{g}) \to (B, \hat{g}') & \leftrightarrow & K(\psi) = |\psi| : (|B|, |\hat{g}'|) \to (|A|, |\hat{g}|) \\ \varphi : (M, g) \to (N, g') & \leftrightarrow & J(\varphi) = |\hat{\varphi}| : (C^\infty(N), \hat{g}') \to (C^\infty(M), \hat{g}) \end{array}$$

Overview

- Criterion of theoretical equivalence that allows for categorical duality.
- Case Study: Einstein algebras and relativistic spacetimes.
- Claim: Duality of **EA** and **GR** shows how they are empirically equivalent.

Thank you.