Relativistic spacetime from events

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Our examination of the continuum problem contributes to critical epistemology's investigation into the relations between what is immediately (intuitively) given and the formal (mathematical) concepts through which we seek to construct the given in geometry and physics. (H. Weyl, Das Kontinuum)

But you have correctly grasped the drawback that the continuum brings. If the molecular view of matter is the correct (appropriate) one, i.e., if a part of the universe is to be represented by a finite number of moving points, then the continuum of the present theory contains too great a manifold of possibilities. I also believe that this too great is responsible for the fact that our present means of description miscarry with the quantum theory. The problem seems to me how one can formulate statements about a discontinuum without calling upon a continuum (space-time) as an aid; the latter should be banned from the theory as a supplementary construction not justified by the essence of the problem, which corresponds to nothing real. But we still lack the mathematical structure unfortunately. How much have I already plagued myself in this way!
[A. Einstein, Letter to Dällebach, Nov. 1916.]

The property of magnitudes on account of which no part of them is the smallest (no part is simple) is called their continuity. Space and time are quanta continua because no part of them can be given except as enclosed between boundaries (points and instants), thus only in such a way that this part is again a space or a time. Space therefore consists only of spaces, time of times. Points and instants are only boundaries, i.e., mere places of their limitation; but places always presuppose those intuitions that limit or determine them, and from mere places, as components that could be given prior to space or time, neither space nor time can be composed. (A169-70/B211-2)

[...] since indivisibles have no parts, they must be in contact with one another as whole with whole. And if they are in contact with one another as whole with whole, they will not be continuous: for that which is continuous has distinct parts [...] as we saw, no continuous thing is divisible into things without parts [...] Moreover, it is plain that everything continuous is divisible into divisibles that are infinitely divisible. [Physics 231b2-18]

It may be argued that we are not in agreement with experience in taking our undefined element of time to be an instant, and that this element should be a duration, to be pictured as an interval. This is certainly true, and we hope later to replace the instants, temporal relations and the temporal axiom of the present paper by still more fundamental ideas in closer agreement with experience. These will give rise to instants as defined elements, and, except for signal correspondences which will then refer to durations, the remainder of the present paper will be valid. (A.G. Walker, Foundations of Relativity I, p.321)

Definition

An event structure $\mathcal{W}$ is a tuple $(W, O, \preceq, B, E, \mathbb{B}, \mathbb{E}, \otimes_f, \otimes_p)$ where $O, \preceq, B, E, \mathbb{B}, \mathbb{E}$ are binary relations and $\otimes_f, \otimes_p$ are partial binary operations on W, satisfying the axioms of GLT:

- $E(a,b) \wedge \mathbb{E}(a,b) \to \bot$
- $B(a,b) \wedge \mathbb{E}(a,b) \to \bot$
- $E(a,b) \vee \mathbb{E}(a,b)$
- $B(a,b) \vee \mathbb{B}(a,b)$
- $\mathbb{E}(a,b) \vee \mathbb{E}(b,a)$
- $\mathbb{B}(a,b) \vee \mathbb{B}(b,a)$
- $\mathbb{E}(a,b) \wedge \mathbb{E}(b,c) \to \mathbb{E}(a,c)$
- $\mathbb{B}(b,a) \wedge \mathbb{B}(c,b) \to \mathbb{B}(c,a)$
- $a \preceq b \leftrightarrow \mathbb{B}(b,a) \wedge \mathbb{E}(b,a)$
- $O(a,b) \leftrightarrow \exists c(c \preceq a \wedge c \preceq b)$
- $\exists c(a \preceq c \wedge b \preceq c)$

For $\otimes_f, \otimes_p$ we have the following:

- $\mathbb{B}(a, a \otimes_f b)$
- $a \otimes_f b \preceq b$
- $\mathbb{B}(a, b) \rightarrow a \otimes_f b = b$
- $\mathbb{B}(b, a) \wedge O(a, b) \rightarrow \mathbb{B}(a \otimes_f b, a) \wedge \mathbb{E}(a \otimes_f b, b)$
- $\mathbb{B}(a, b) \rightarrow \mathbb{B}(a \otimes_f c, b \otimes_f c)$
- $(a \otimes_f b) \otimes_f c = a \otimes_f (b \otimes_f c)$
- Dually for $\otimes_p$ exchanging $\mathbb{B}$ with $\mathbb{E}$

- We also assume the axiom $a \preceq b \wedge b \preceq a \rightarrow a = b$, which is really not needed (and is philosophically unwarranted) but simplifies the treatment

Lemma

Let A be a finite multiset of events with enumeration $a_1, \ldots, a_n$ such that for all k, $\mathbb{B}(a_1, a_k)$ and $\mathbb{E}(a_n, a_k)$. Then there exists c such that for all k, $a_k \preceq c$, $\mathbb{B}(a_1, c)$, $\mathbb{E}(a_n, c)$

- O can be taken as a primitive to allow for its interpretation as a contact relation, i.e. $O(a,b)$ if a,b are "infinitesimally close", highlighting the constructive role of the operations $\otimes_f, \otimes_p$

- $\otimes_f, \otimes_p$ can be made total by introducing an "empty" event 0 satisfying $B(0,a) \wedge E(0,a)$, and modifying the other axioms accordingly

- The intended intepretation of $a \otimes_f b$ is "that part of b which is in the causal future of a", i.e., "which can be causally influenced by a" (similarly for $a \otimes_p b$)

- Since $\mathbb{E}, \mathbb{B}$ are reflexive and transitive they each generate an Alexandroff topology on events, called respectively the past and future topology.

Definition

A formula is positive primitive if it is constructed from atomic formulas using only $\vee, \bigvee, \wedge, \exists, \bot$. A formula is geometric if it is of the form:

$$\forall \bar{x}(\theta(\bar{x}) \rightarrow \psi(\bar{x}))$$

For $\theta(\bar{x}), \psi(\bar{x})$ positive primitive

- The geometric fragment of first order logic plays an important role in category theory
- It constitutes a Glivenko class
- Geometric theories are decidable
- It has been used to provide a formalization of Kant's theory of judgments

Theorem

Let ϕ be a geometric formula in the signature of GT. Then $GT \models \phi$ iff ϕ holds on all finite models of GT

The above result *does not* extend beyond geometric formulas.

Definition

A boundary in an event structure $\mathcal{W}$ is a triple $(Past, Pres, Fut)$ such that:

1. $Past \cup Pres \cup Fut = W$
2. $Past$ is $\mathbb{E}$-open, Fut is $\mathbb{B}$-open
3. $\forall a \in Past \forall b \in Fut : \neg O(a, b)$
4. $Pres$ is not empty and for all $a, b \in Pres : O(a, b)$
5. If $Past$ is not empty then $\forall a \in Pres \exists b \in Past : O(a, b)$
6. If Fut is not empty then $\forall a \in Pres \exists b \in Fut : O(a, b)$

Theorem

The set of boundaries $\mathcal{B}(\mathcal{W})$ of an event structure determines a linear order by letting, for $i, j \in \mathcal{B}(\mathcal{W})$, $i \leq j$ iff $Past(i) \subseteq Past(j)$. In ZF the linear order is a complete (linearly ordered) lattice, and hence the order topology on $\mathcal{B}(\mathcal{W})$ is compact Hausdorff.

An alternative way of conceiving boundaries in an event structure is by means of bi-continuous maps. Let $\mathbb{M}$ be the event structure with domain $\{p, c, f\}$ and relations $\neg O(p, f), \mathbb{B}(p, c), \mathbb{E}(f, c), \{p, f\} \preceq c$. Then:

Lemma

There is a 1-1 correspondence between boundaries in an event structure $\mathcal{W}$ and bi-continuous maps $f : \mathcal{W} \to \mathbb{M}$ satisfying:

1. *f is surjective on $\{p, c\}$ or $\{c, f\}$*
2. *there is no map f' satisfying 1 such that $f'^{-1}(c) \subset f^{-1}(c)$*

Definition

Let $\mathcal{W}, \mathcal{W}'$ be event structures. A morphism $f : \mathcal{W} \to \mathcal{W}'$ is a map preserving both $\mathbb{B}, \mathbb{E}$

Lemma

A morphism $f : \mathcal{W} \to \mathcal{W}'$ is such that if $a \otimes_f b$ is defined in $\mathcal{W}$, then $f(a) \otimes_f f(b)$ is defined in $\mathcal{W}'$ and $f(a) \otimes_f f(b) = f(a \otimes_f b)$ (similarly for $\otimes_p$)

Definition

Let $\mathcal{W}, \mathcal{W}'$ be such that $\mathcal{W}'$ is a substructure of $\mathcal{W}$. A retraction of $\mathcal{W}$ onto $\mathcal{W}'$ is a morphism which is the identity on $\mathcal{W}'$.

Definition

The event structure $\mathcal{E}[0,1]$ on $([0,1], \tau, \leq)$ is defined by letting:

- $W = \{U \in \tau \mid U \text{ is order convex}\}$
- $\mathbb{B}(a, b)$ if $b \subseteq {\uparrow}a$
- $\mathbb{E}(a, b)$ if $b \subseteq {\downarrow}a$
- $E(a, b)$ if $\mathbb{E}(b, a)$ and not $\mathbb{E}(a, b)$ (similarly for B)
- $O(a, b)$ if $a \cap b \neq \emptyset$
- $a \otimes_f b$ is $\uparrow a \cap b$ if this intersection is non empty, undefined otherwise (similarly for $\otimes_p$)

Lemma

- $\mathcal{BE}[0,1]$ *is isomorphic to* $[0,1]$
- $\mathcal{BE}\mathbb{R}$ *is isomorphic to* $[0,1]$
- $\mathcal{BE}\mathbb{Q}$ *is isomorphic to* $[0,1]$
- *In general assuming classical logic, for any linear order* $(X, \leq)$*,* $\mathcal{BE}(X, \leq)$ *is a completion of* $(X, \leq)$

The isomorphism relies on classical logic and breaks down in intuitionistic logic. Furthermore, the isomorphism abstracts away the internal structure of "Kantian boundaries":

- $[0, 1]$ possesses only two non-cut points, 0 and 1
- For $x \in \mathcal{W}$ let F_x be the associated present and construct a graph (V, R) where $V = \mathcal{B}\mathcal{W}$ and $R(v, v')$ if $\exists a \in v \exists b \in v' : O(v, v')$.
- The graph (V, R) is complete, hence if $|V|$ is finite it a simplex of dimension $|V| - 1$ and the order of time is represented by an hamiltonian path.
- Removing a point means removing a vertex of this simplex, which yeilds a simplex one dimension lower - and the flow of time is not disrupted

1. An individual point in [a continuum] is non-independent, i.e., is pure nothingness when taken by itself, and exists only as a point of transition (which, of course, can in no way be understood mathematically); 2. it is due to the essence of time (and not to contingent imperfections in our medium) that a fixed temporal point cannot be exhibited in any way, that always only an approximate, never an exact determination is possible. [Hermann Weyl, Das Kontinuum]

Definition

Let $\mathcal{W}, \mathcal{W}'$ be event structures, $\mathcal{W}' \hookrightarrow \mathcal{W}$, $\mathcal{W}'$ finite. Then we define the map $h : \mathcal{W} \to \mathcal{W}'$ as follows. For $a \in W$, let $U(a) \subseteq W'$ be the smallest $\mathbb{E}$-open set U s.t. in $\mathcal{W}$, $a \in \downarrow_{\mathbb{E}} U$; if U does not exist, let $U(a) = W'$. Define $V(a)$ analogously with respect to $\mathbb{B}$. Then let

$$h(a) = c_f \otimes_f c_p$$

Where c_f is the exact cover of $V(a)$ and c_p is the exact cover of $U(a)$ in W'.

Lemma

h is a retraction such that generally $a \preceq h(a)$

Lemma

Consider the set $\mathcal{W}_{<\omega}\Omega$ of finite substructures of $\Omega = \mathcal{E}([0,1] \cap \mathbb{Q})$ ordered according to the substructure relation, i.e., $W' \leq W$ if W' is a substructure of W. Then $(\mathcal{W}_{<_\omega}\Omega, \leq)$ is a directed poset

Lemma

The tuple $(\mathcal{W}_{<\omega}\Omega, \leq, h_{st})$ is an inverse system where $h_{st} : \mathcal{W}_t \to \mathcal{W}_s$ for $t \geq s$ is defined as h was defined before. The inverse limit $\mathcal{V}$ of this inverse system is non-empty and such that $\mathcal{V} \models GT$

Theorem

Let ϕ be a geometric sentence in the vocabulary $(\mathbb{B}, \mathbb{E}, \otimes_f, \otimes_p)$. Then $\mathcal{V} \models \phi$ iff $\Omega \models \phi$

Theorem

Let ϕ be a geometric sentence in the vocabulary $(\mathbb{B}, \mathbb{E}, \otimes_f, \otimes_p)$. Then $GT \vdash \phi$ iff $\Omega \models \phi$

Lemma

$\mathcal{B}(\mathcal{V})$ is lattice isomorphic to $\mathcal{B}(\Omega)$

Theorem

$\mathcal{B}(\mathcal{V})$ is lattice isomorphic to $[0,1]$

- These results offer a way of approximating the unit interval which is alternative (more logical) to that pursued in digital topology by Kopperman et. Al.
- One can on this basis already obtain foundations for physics which are more adherent to experience by adopting an axiomatization of relativity theory based on world-lines of particles, along the lines of, e.g., A.G. Walker or the Andréka-Neméti group
- However, to be in closer agreement with experience, one would need a reconstruction of relativistic spacetimes starting from events with the same topological dimension as the space to be recovered

Definition

The Alexandroff topology on a spacetime $\mathcal{M}$ has as a basis: $\{I^+(p) \cap I^-(p) \mid p, q \in \mathcal{M}\}$

Theorem

A spacetime $\mathcal{M}$ is strongly causal iff its Alexandroff topology is Hausdorff iff its Alexandroff topology is the manifold topology

Definition

Let M^d be the d-dimensional Minkowski spacetime with $d \geq 2$ and define the event structure $\mathcal{E}M^d$ as follows:

- $W = \{I^+(p) \cap I^-(p) \mid p, q \in \mathcal{M} \cap \mathbb{Q}\}$
- $\mathbb{E}(a, b)$ if $b \subseteq\downarrow a$
- $\mathbb{B}(a, b)$ if $b \subseteq\uparrow a$
- $O(a, b)$ if $a \cap b \neq \emptyset$
- $a \otimes_f b$ is defined as $\uparrow a \cap b$ if this is not empty, undefined otherwise
- $a \otimes_p b$ is defined as $\downarrow a \cap b$ if this is not empty, undefined otherwise

The axiom system GLT can then be weakened to an axiom system GCT by giving up the linearity axioms for $\mathbb{E}, \mathbb{B}$ and add further axioms in light of the above interpretation

- one then goes on to generalize the inverse limit construction from the linear case by:
  (i) defining a retraction $h : \mathcal{W} \to \mathcal{W}'$ for any $\mathcal{W}, \mathcal{W}'$ satisfying GCT with $\mathcal{W}' \hookrightarrow \mathcal{W}$ (very straightforward)
  (ii) defining a generalized notion of Walker boundary for the non-linear case (much trickier)
- More abstractly it must also be possible to carry out the present construction for a variety of partially ordered topological spaces satisfying suitable conditions, which might not be necessarily Haudorff, such as the Khalimsky line
- finally one could try to study causal sets dynamics "bottom up"