Guarded fragment of FOL without equality

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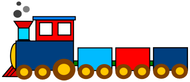
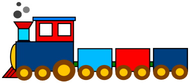
Algebraization of logics



Algebraic Logic

Algebraization of logics

Algebra

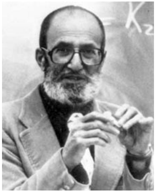


Logic

Algebraic Logic








Fragments of Logics

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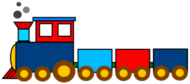
Variants of algebras  Fragments of logic

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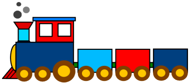
Variants of algebras  Fragments of logic

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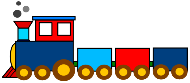
Variants of algebras  Fragments of logic

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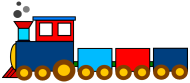
Variants of algebras  Fragments of logic

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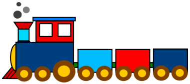

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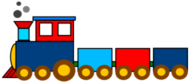
Variants of algebras  Fragments of logic

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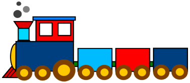
Variants of algebras  Fragments of logic

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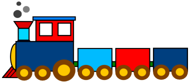
Variants of algebras  Fragments of logic

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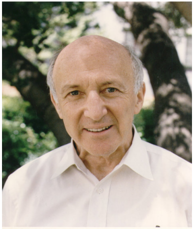
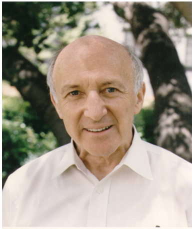
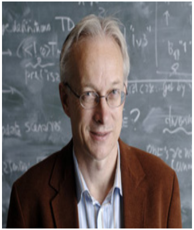
   - Expressive power!







Definition

The guarded fragment $GF(n, \neq)$ is a fragment of $FOL(n, \neq)$ with only guarded (bounded) quantification,

$$\exists \bar{v}(G(\bar{v}) \wedge \phi(\bar{v})) \qquad \exists \bar{v}(G(\bar{v}) \rightarrow \phi(\bar{v}))$$

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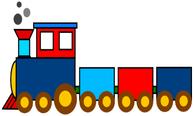
Definition

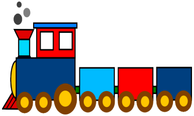
The class Crs_n^{df}, of diagonal free relativized set algebra of dimension n:

$$\mathfrak{A} \cong \mathfrak{B} \subseteq \langle \mathcal{P}(V), \cup, \cap, \setminus, \emptyset, V, C_i^{[V]} \rangle,$$

where $V \subseteq {}^n W$ and $C_i^{[V]}(X) = \{y \in V : \exists x \in X(x \equiv_i y)\}$.



Theorem (Andreka, Nemeti, van Benthem)



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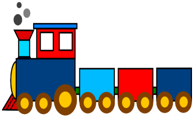


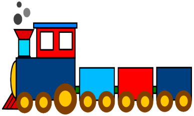
Theorem (Andreka, Nemeti, van Benthem)

Crs_n^{df}

- *Finitely axiomatizable.*

$GF(n, \neq)$

- *St. sound and complete.*



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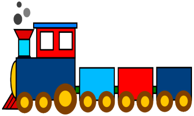

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- *Finite base property.*

$GF(n, \neq)$

- *St. sound and complete.*
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- *Finite model property.*

Important:

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Normal Disjunctive Forms

- $Crs_n^{df} \models F_q^{n,m}$ forms a partition of the unit.
- There is an algorithm to determine, for every term $\tau \in \mathfrak{Tm}_{m,df_n}$, a finite set of normal forms of the same degree such that Crs_n^{df} thinks that τ is zero or equals the disjunction of these normal forms.

Proof

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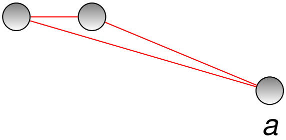

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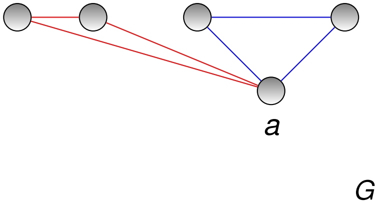

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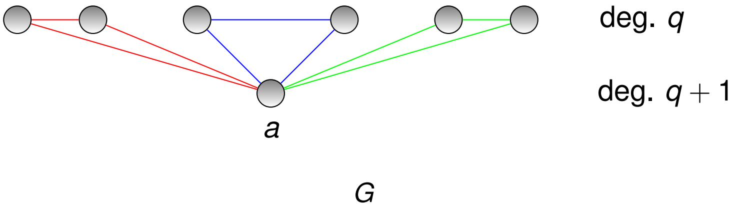

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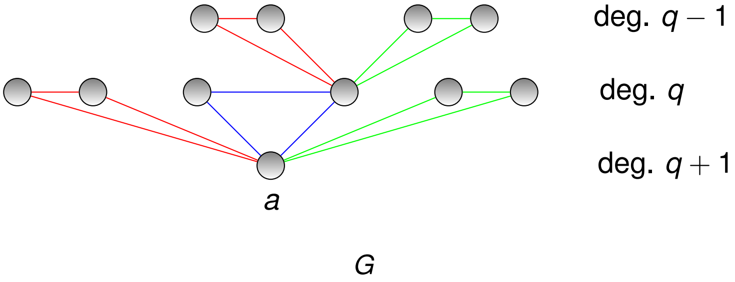

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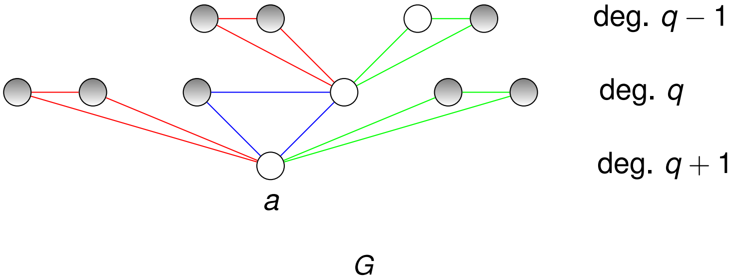

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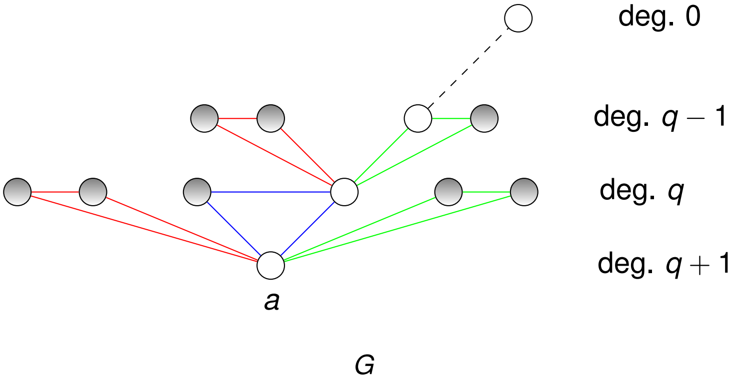

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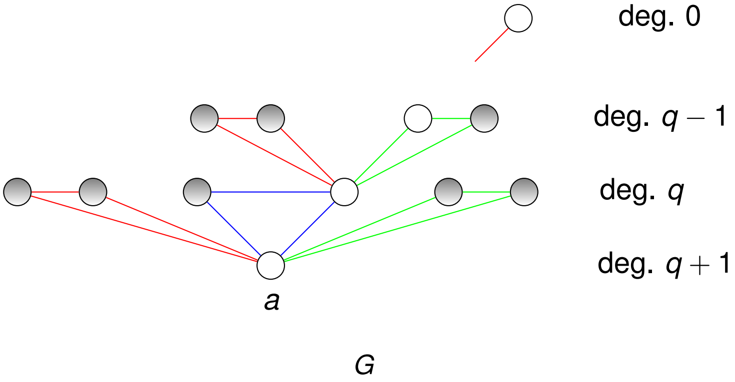

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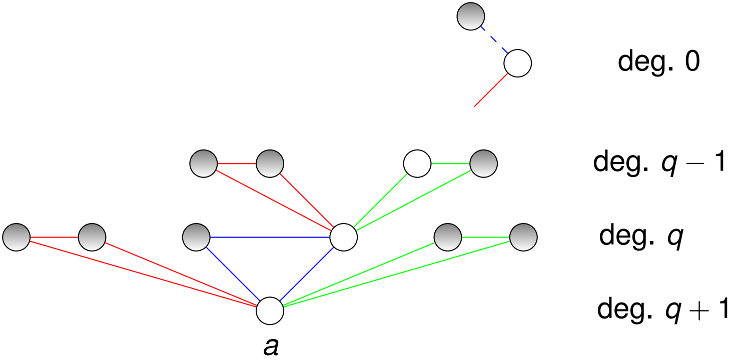

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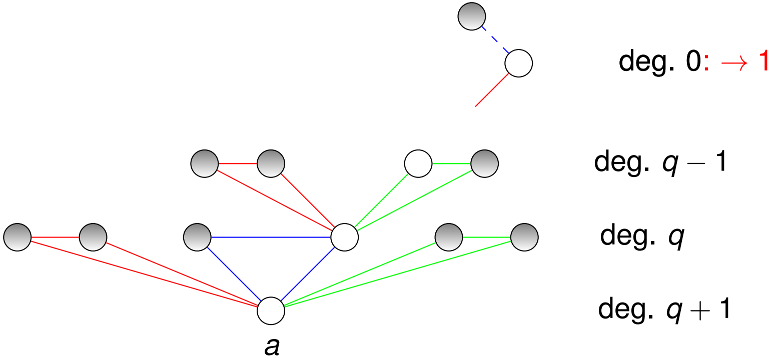

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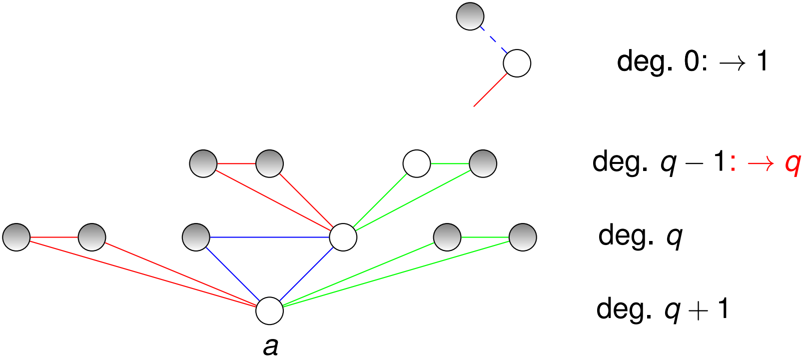

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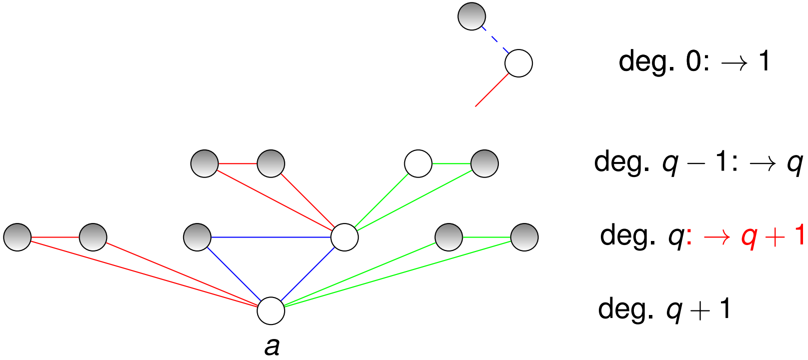

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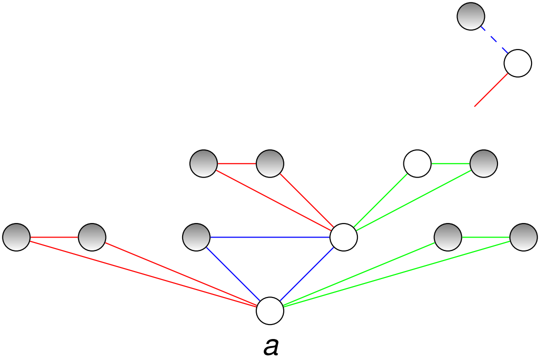

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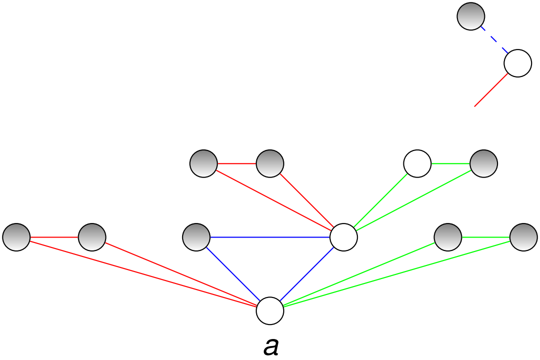

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Main results

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- Finite base property.
- Decidability of the equational theory.
- Decidability of the universal theory.
- Super amalgamation property.
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