Classification of absorbent-continuous, densely ordered and complete, group-like FL_e-chains

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Residuated semigroups

- *Ideal theory of commutative rings with unit*
  [Ward, M. and R. P. Dilworth, Residuated lattices, Transactions of the American Mathematical Society 45: 335-354, 1939]
- *[L. Fuchs, Partially Ordered Algebraic Systems, Pergamon Press, Oxford-London-New York-Paris (1963)]*
- *[G. Birkhoff, Lattice Theory, Amer. Math. Soc. Colloquium Publications, third edition (Amer. Math. Soc., RI), 1973.]*

- *Boolean algebras*
- *Heyting algebras*
  [Johnstone, P. T. (1982). Stone spaces. Cambridge: Cambridge University Press.]
- *complemented semigroups*
  [Bosbach, B. (1969). Komplementäre Halbgruppen. Axiomatik und Arithmetik. Fundamenta Mathematicae, 64, 257–287.]
- *bricks*
  [Bosbach, B. (1981a). Concerning bricks. Acta Mathematica Hungarica, 38, 89–104.]
- *residuation groupoids*
  [Bosbach, B. (1978). Residuation groupoids and lattices. Studia Scientiarum Mathematicarum Hungarica, 13, 433–451.]
- *semiclans*
  [Bosbach, B. (1981b). Concerning semiclans. Archiv der Mathematik, 37, 316–324.]
- *Bezout monoids*
  [Ánh, P. N., Márki, L., & Vámos, P. (2012). Divisibility theory in commutative rings: Bezout monoids. Transactions of the American Mathematical Society, 364, 3967–3992.]
- *MV-algebras*
  [Chang, C. C. (1958). Algebraic analysis of many-valued logic. Transactions of American Mathematical Society, 88, 456–490.]
  [Cignoli, R., D'Ottaviano, I. M. L., & Mundici, D. (2000a). Algebraic foundations of many-valued reasoning. Dordrecht: Kluwer.]
- *BL-algebras*
  [Hájek, P. (1998). Metamathematics of fuzzy logic. Dordrecht: Kluwer.]
- *lattice-ordered groups*

Residuated lattices are algebraic counterparts of Substructural Logics

- *Substructural Logics*
  [Galatos, N., Jipsen, P., Kowalski, T., Ono, H. Residuated Lattices: An Algebraic Glimpse at Substructural Logics, Volume 151. (2007). Studies in Logic and the Foundations of Mathematics, 532.]
- *Examples*
  classical logic, intuitionistic logic, relevance logics, many-valued logics, mathematical fuzzy logics, linear logic, along with their non-commutative versions

Classifications of residuated lattices

- *Every cancellative, Archimedean, naturally and totally ordered semigroup can be embedded into the additive semigroup of the real numbers.*
  [O. Hölder, Die Axiome der Quantität und die Lehre vom Mass, Berichte über die Verhandlungen der Königlich Sachsischen Gesellschaft der Wissenschaften zu Leipzig, Mathematisch-Physische Classe, 53 (1901), 1–64]

- *A continuous semigroup operations over intervals of real numbers is order isomorphic to a subsemigroup of the additive semigroup of the real numbers iff its multiplication is cancellative*
  [J. Aczél, Lectures on Functional Equations and Their Applications, Academic Press, New York-London, 1966.]

Classifications of residuated lattices

- *Every Archimedean, naturally and totally ordered semigroup in which the cancellation law does not hold can be embedded into either the real numbers in the interval [0, 1] with the usual ordering and $ab = \max(a + b, 1)$ or the real numbers in the interval [0, 1] and the symbol ∞ with the usual ordering and $ab=a+b$ if $a+b\leq 1$ and $ab=\infty$ if $a+b>1$.*
  Every naturally totally ordered, commutative semigroup is uniquely expressible as the ordinal sum of a totally ordered set of ordinally irreducible such semigroups
  [A. H. Clifford: Naturally totally ordered commutative semigroups, Amer. J. Math., 76 vol. 3 (1954), 631–646.]

Classifications of residuated lattices

- *Topological semigroups over compact manifolds with connected, regular boundary B such that B is a subsemigroup:*
  1. a subclass of compact connected Lie groups and via classifying (I)-semigroups
  2. (I)-semigroups are ordinal sums of three basic multiplications which an arc may possess.
  [P.S. Mostert, A.L. Shields, On the structure of semigroups on a compact manifold with boundary, Ann. Math., 65 (1957), 117–143]

(I)-Semigroups

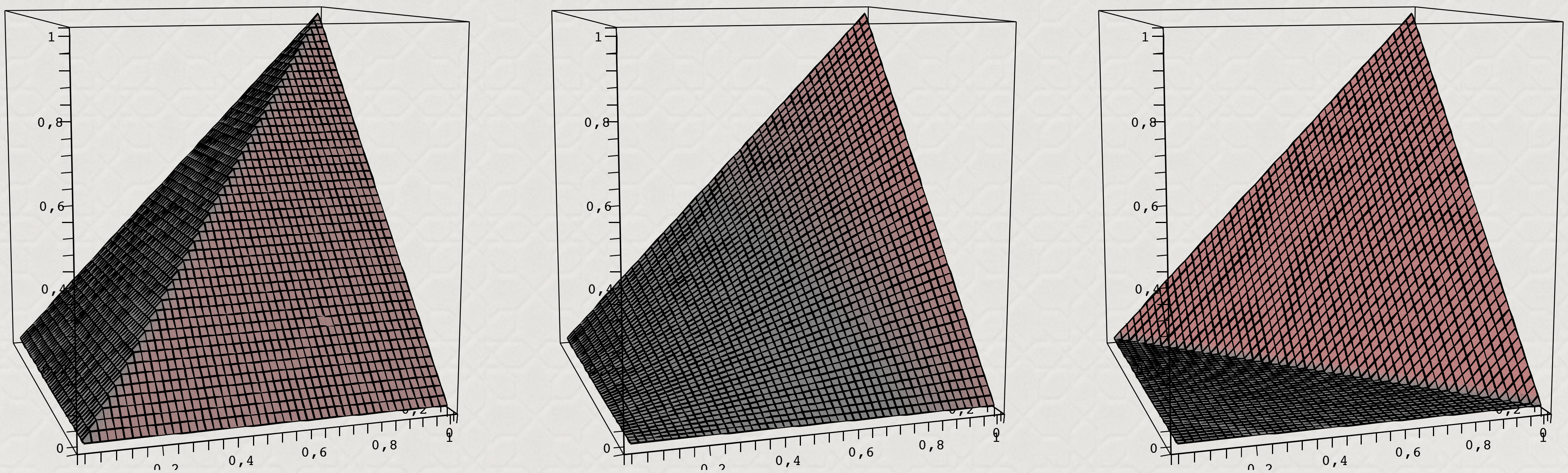


Figure 1: Minimum (left), product (center) and Łukasiewicz t-norms (right)

Classifications of residuated lattices

- *BL-algebras are subdirect poset products of MV-chains and product chains.*
  P. Jipsen, F. Montagna, Embedding theorems for classes of GBL-algebras, Journal of Pure and Applied Algebra, 214 vol. 9 (2010), 1559–1575
- *Our main theorem, for a more specific class of chains, and under the condition that the positive and the negative cones of the algebra are dually isomorphic*
  S. Jenei, F. Montagna, Strongly Involutive Uninorm Algebras, Journal of Logic and Computation 23:(3) pp. 707-726. (2013)
- *Our main theorem, for a more specific class of chains*
  S. Jenei, F. Montagna: A classification of certain group-like FL_e-chains, Synthese, (2014). doi:10.1007/s11229-014-0409-2 (papers from Logic and Relativity 2012, honoring István Németi's 70th birthday)

Residuated lattices

An algebra $\mathbf{A} = (A, \wedge, \vee, \cdot, \backslash, /, 1, 0)$ is called a *full Lambek algebra* or an *FL-algebra*, if

- $(A, \wedge, \vee)$ is a lattice (i.e., $\wedge$, $\vee$ are commutative, associative and mutually absorptive),
- $(A, \cdot, 1)$ is a monoid (i.e., $\cdot$ is associative, with unit element 1),
- $x \cdot y \leq z$ iff $y \leq x\backslash z$ iff $x \leq z/y$, for all $x, y, z \in A$,
- 0 is an arbitrary element of A.

- *Residuated lattices are exactly the 0-free reducts of FL-algebras. (FL-algebras are residuated lattices with a constant f.)*
- *FL_e-algebra: FL-algebra such that · is commutative. Notation $y/x = x \rightarrow y$*
- *t (truth) for 1, f (false) for 0*

FL_e-algebras

- $x' = x \to f$
- *involutive:* $x'' = x$ ($f' = t$ follows)
- *group-like:* involutive and $t = f$
- *All lattice-ordered groups are group-like FL_e-algebras.*
- *Absorbent continuity :* For $x \in X^-$, $a(x) \otimes x = x$ holds, where $a(x) = \inf\{ u \in X^- : u \otimes x = x \}$

BL-algebras

A **hoop** is an algebra $\mathbf{A} = \langle A, \rightarrow, \cdot, 1\rangle$ such that $\langle A, \cdot, 1\rangle$ is a commutative monoid and for all $x, y, z \in A$

(1) $x \rightarrow x = 1$.
(2) $x \cdot (x \rightarrow y) = y \cdot (y \rightarrow x)$.
(3) $x \rightarrow (y \rightarrow z) = (x \cdot y) \rightarrow z$.

A **Wajsberg hoop** is a hoop satisfying the equation

$$(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x.$$

A **bounded** hoop is an algebra $\mathbf{A} = \langle A, \rightarrow, \cdot, 0, 1\rangle$ such that $\langle A, \rightarrow, \cdot, 1\rangle$ is a hoop and $0 \leq a$ for all $a \in A$; a **Wajsberg algebra** is a bounded Wajsberg hoop.

A **basic hoop** is a subdirect product of totally ordered hoops;

A **BL-algebra** is a bounded basic hoop. A **Product algebra** is a BL-algebra satifying the equations $x \wedge \neg x = 0$ and $\neg\neg x \leq (yx \rightarrow zx) \rightarrow (y \rightarrow z)$. A **Gödel algebra** is a BL-algebra satisfying te equation $xx = x$

Ordinal Sums

Let $\langle I, \leq \rangle$ be a totally ordered set. For all $i \in I$ let $\mathbf{A}_i$ be a hoop such that for $i \neq j$, $A_i \cap A_j = \{1\}$. Then $\bigoplus_{i \in I} \mathbf{A}_i$ (the **ordinal sum** of the family $(\mathbf{A}_i)_{i \in I}$) is the structure whose base set is $\bigcup_{i \in I} A_i$ and the operations are

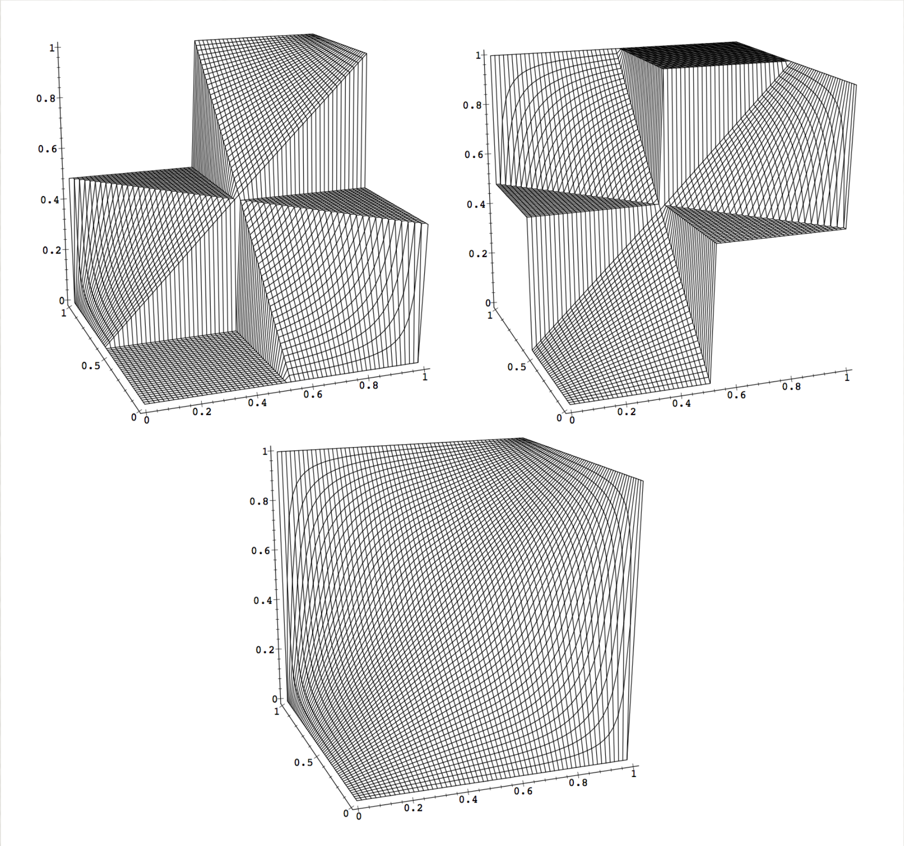
$$x \to y = \begin{cases} x \to^{\mathbf{A}_i} y & \text{if } x, y \in A_i \\ y & \text{if } x \in A_i \text{ and } y \in A_j \text{ with } i > j \\ 1 & \text{if } x \in A_i \setminus \{1\} \text{ and } y \in A_j \text{ with } i < j \end{cases}$$

$$x \cdot y = \begin{cases} x \cdot^{\mathbf{A}_i} y & \text{if } x, y \in A_i \\ y & \text{if } x \in A_i \text{ and } y \in A_j \setminus \{1\} \text{ with } i > j \\ x & \text{if } x \in A_i \setminus \{1\} \text{ and } y \in A_j \text{ with } i < j \end{cases}$$

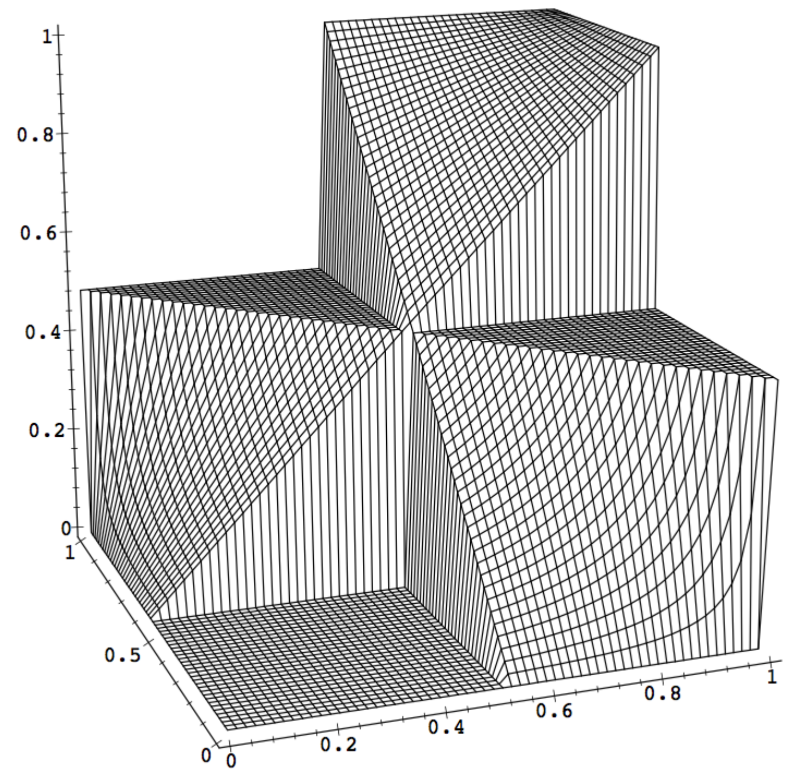
Theorem *Every totally ordered hoop (BL-algebra) is the ordinal sum of a family of Wajsberg hoops (whose first component is a Wajsberg algebra).*

- *P. Aglianò, F. Montagna, Varieties of BL-algebras I: general properties, Journal of Pure and Applied Algebra, 181 (2–3), 2003, 105–129*

Twin rotation



[S. Jenei, H. Ono, On involutive FL_e-monoids, Archive for Mathematical Logic, 51:(7-8) pp. 719-738. (2012)]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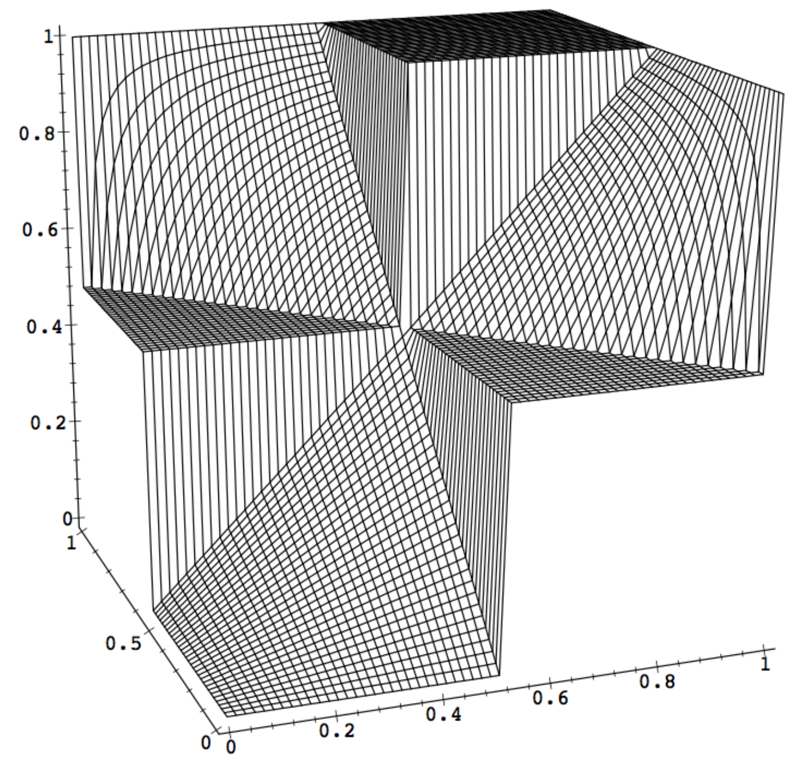
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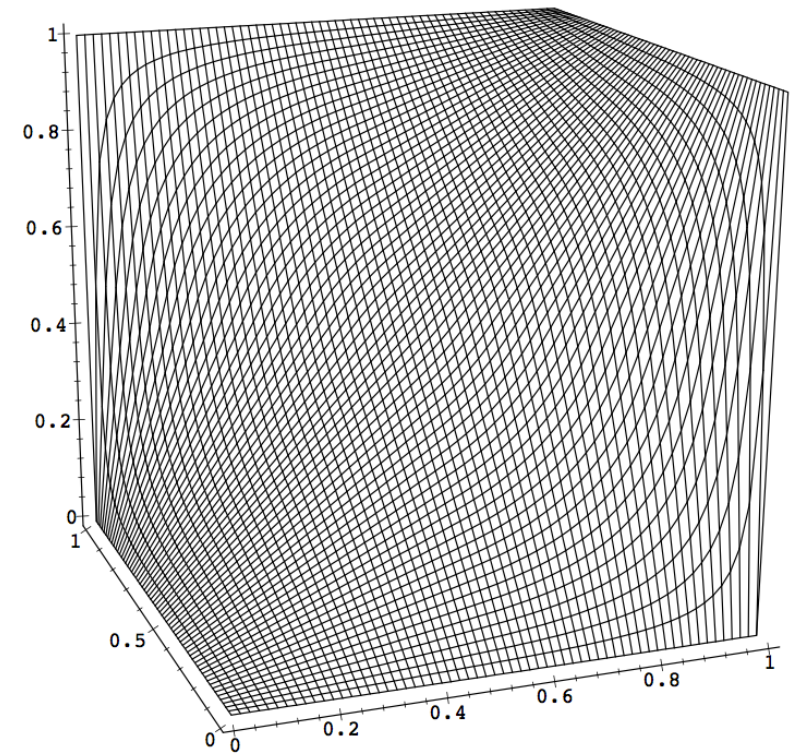
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Definition 7 (*Twin-rotation construction*) Let $(X_1, \leq)$ be a partially ordered set with top element t, and and $(X_2, \leq)$ be a partially ordered set with bottom element t such that the connected ordinal sum $os_c\langle X_1, X_2\rangle$ of X_1 and X_2 (that is putting X_1 under X_2, and identifying the top of X_1 with the bottom of X_2) has an order reversing involution $'$. Denote the partial order of $os_c\langle X_1, X_2\rangle$ also by $\leq$. Let $(X_1, \otimes)$ and $(X_2, \oplus)$ be commutative semigroups, both with neutral element t. Assume that $(X_1, \otimes)$ is residuated and assume that all residua $x \to_\oplus y$ exist if $x, y \in X_2, x \leq y$.[4] Assume, in addition, that

(a) in case $t' \in X_1$ we have $x \to_\otimes t' = x'$ for all $x \in X_1, x \geq t'$, and
(b) in case $t' \in X_2$ we have $x \to_\oplus t' = x'$ for all $x \in X_2, x \leq t'$.

Let

$$\mathcal{U}_\otimes^\oplus = \langle os_c\langle X_1, X_2\rangle, \circledast, \leq, t, f\rangle$$

where $f = t'$ and $\circledast$ is defined as follows:

$$x \circledast y = \begin{cases} x \otimes y & \text{if } x, y \in X_1 \\ x \oplus y & \text{if } x, y \in X_2 \\ (x \to_\oplus y')' & \text{if } x \in X_2, y \in X_1, \text{ and } x \leq y' \\ (y \to_\oplus x')' & \text{if } x \in X_1, y \in X_2, \text{ and } x \leq y' \\ (y \to_\otimes (x' \wedge t))' & \text{if } x \in X_2, y \in X_1, \text{ and } x \not\leq y' \\ (x \to_\otimes (y' \wedge t))' & \text{if } x \in X_1, y \in X_2, \text{ and } x \not\leq y' \end{cases}. \quad (14)$$

Call $\circledast$ (resp. $\mathcal{U}_\otimes^\oplus$) the twin-rotation of $\otimes$ and $\oplus$ (resp. of the first and the second partially ordered monoid).

Main Theorem of the talk

- *If U is an absorbent-continuous, group-like FL_e-algebra on a complete, order dense chain (with involution ') then U is the twin-rotation of a BL-algebra and its de Morgan dual $x+y=(x'\cdot y')'$, where the BL-algebra has components, which are either cancellative or Boole-algebras over two elements, and the BL-algebra cannot have two consecutive cancellative components.*

[S. Jenei, Classification of absorbent-continuous, densely ordered, complete, group-like FL_e-chains (submitted)]

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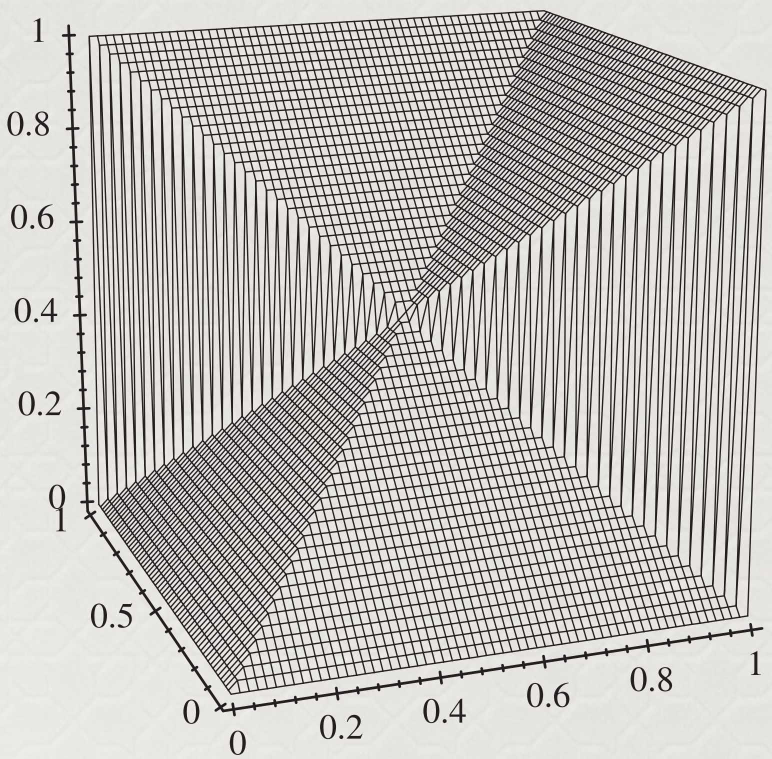
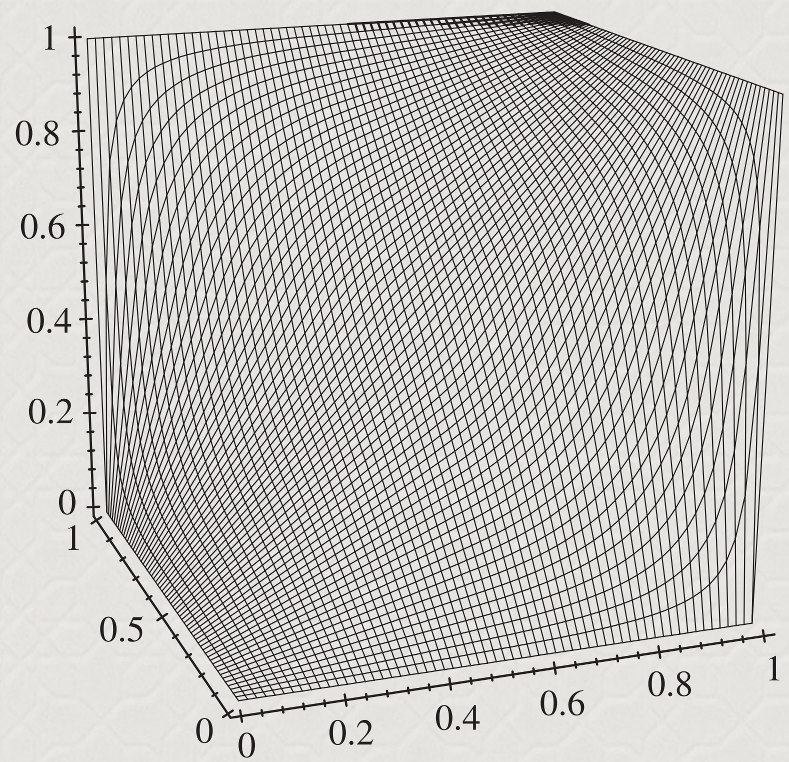
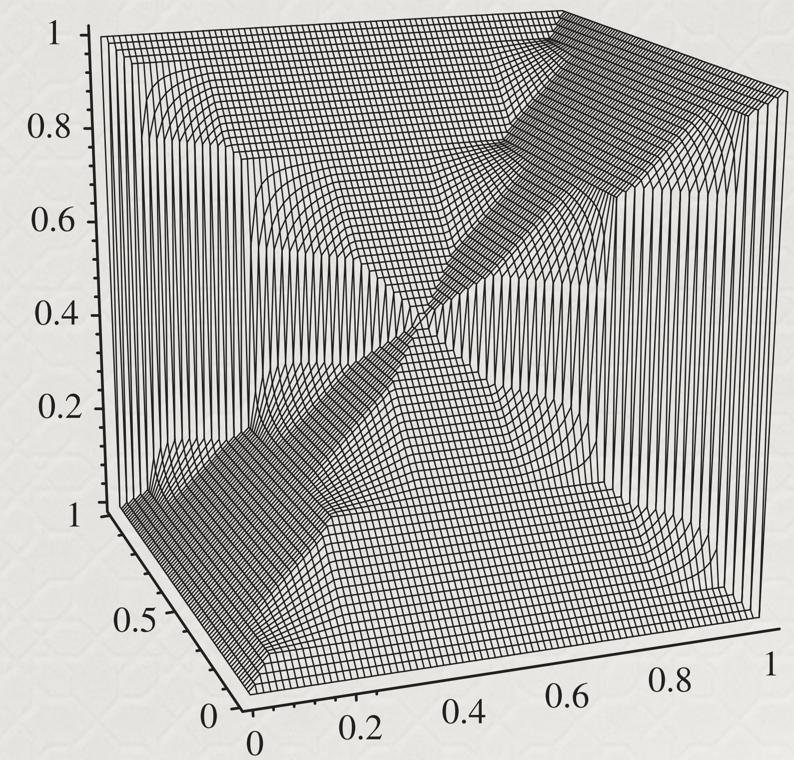
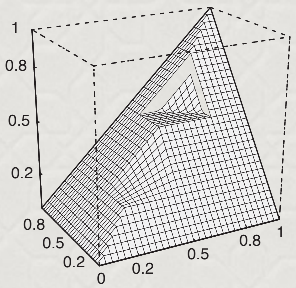
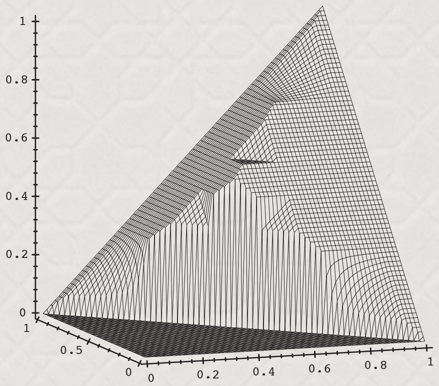
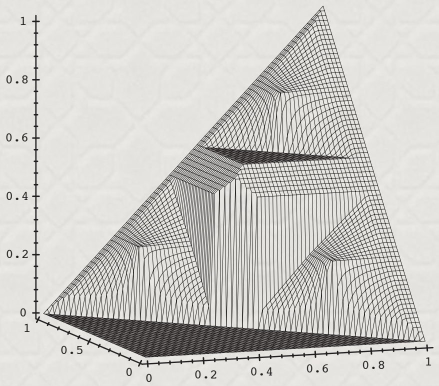
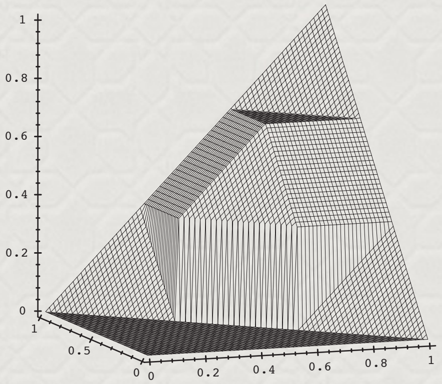
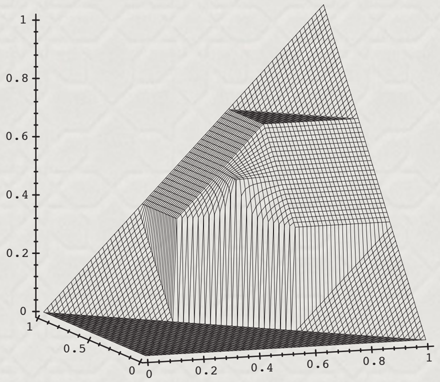
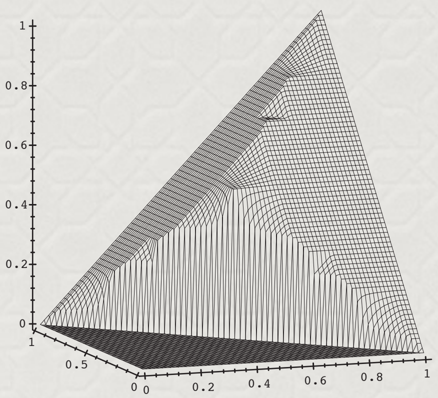
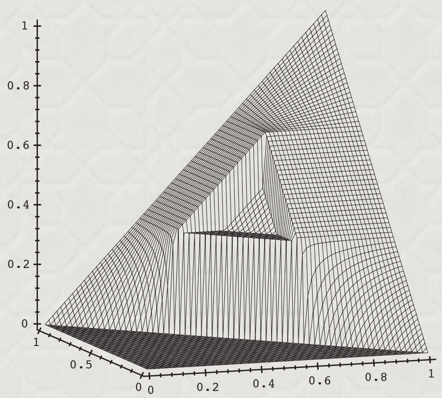
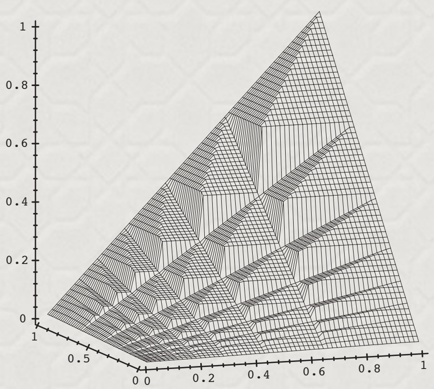
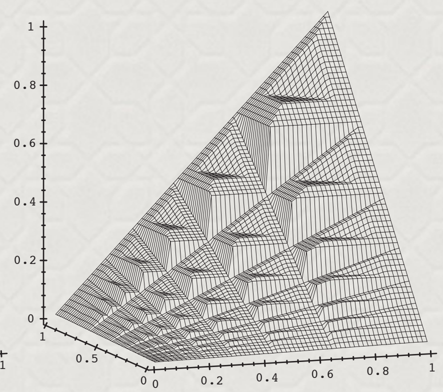
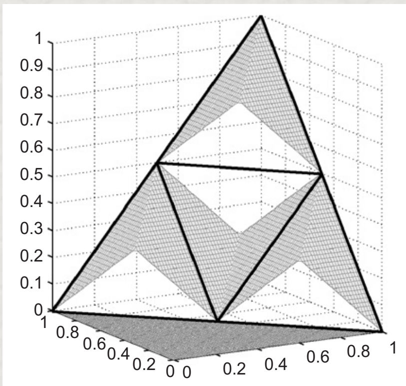
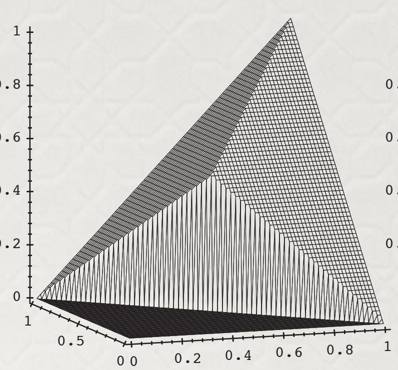
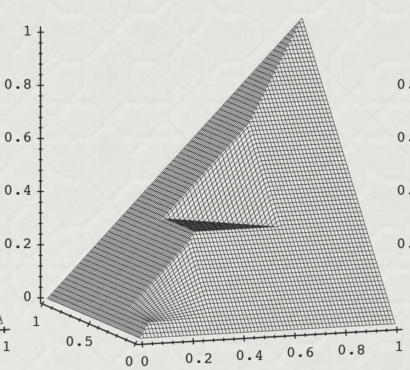
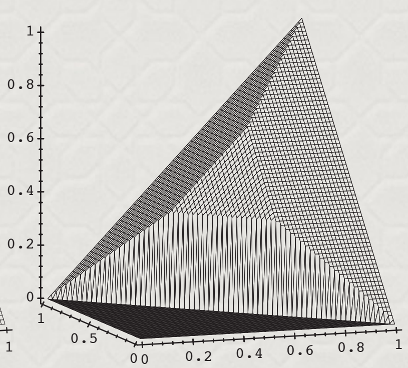
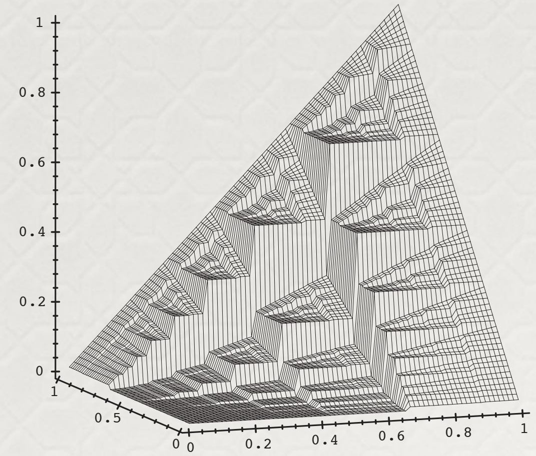
[S. Jenei, Classification of absorbent-continuous, densely ordered, complete, group-like FL_e-chains (submitted)]

Absorbent-continuous, complete, order-dense, group-like FL_e-chains over [0,1]







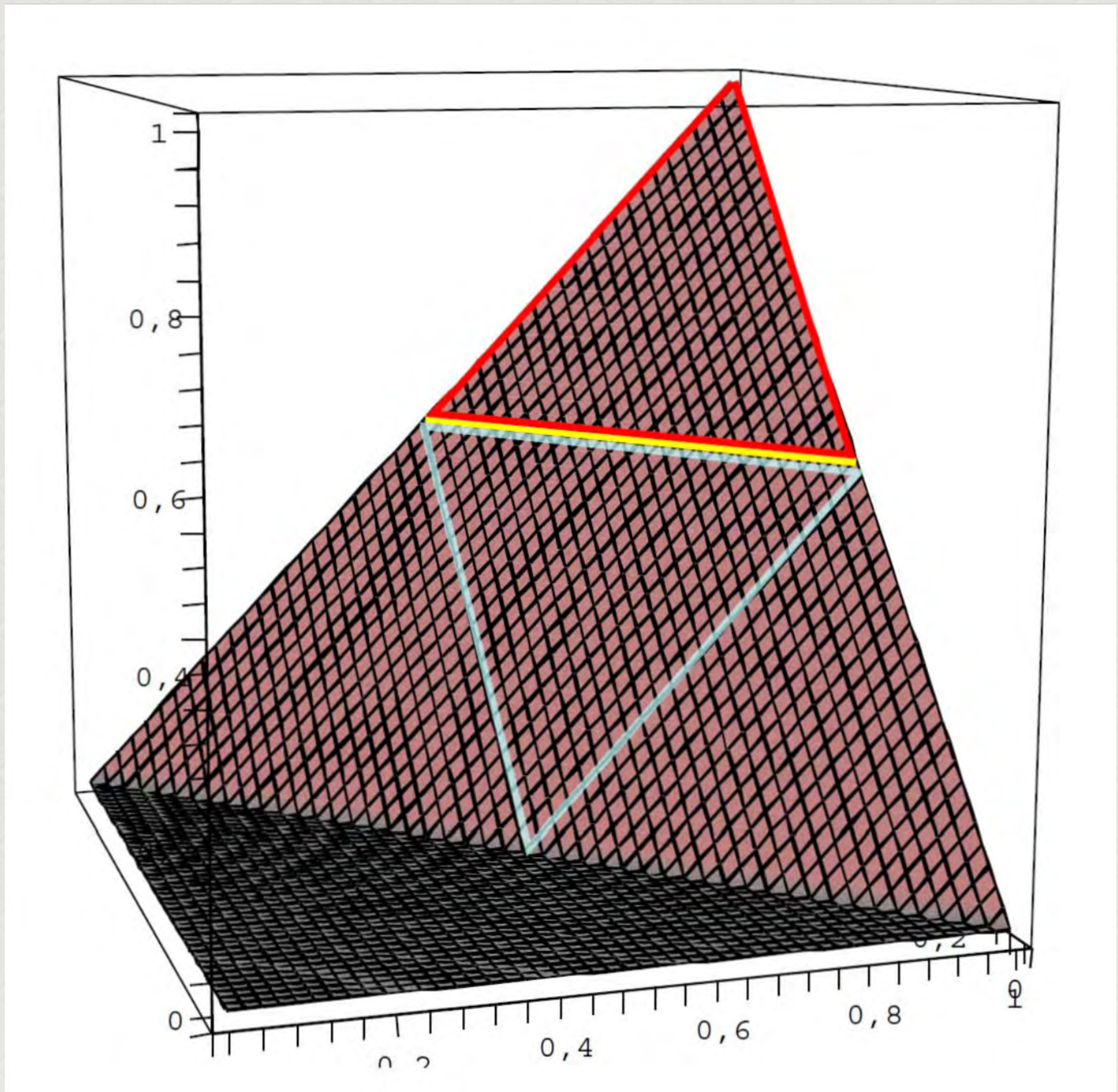















The proof uses geometric aspects of associativity

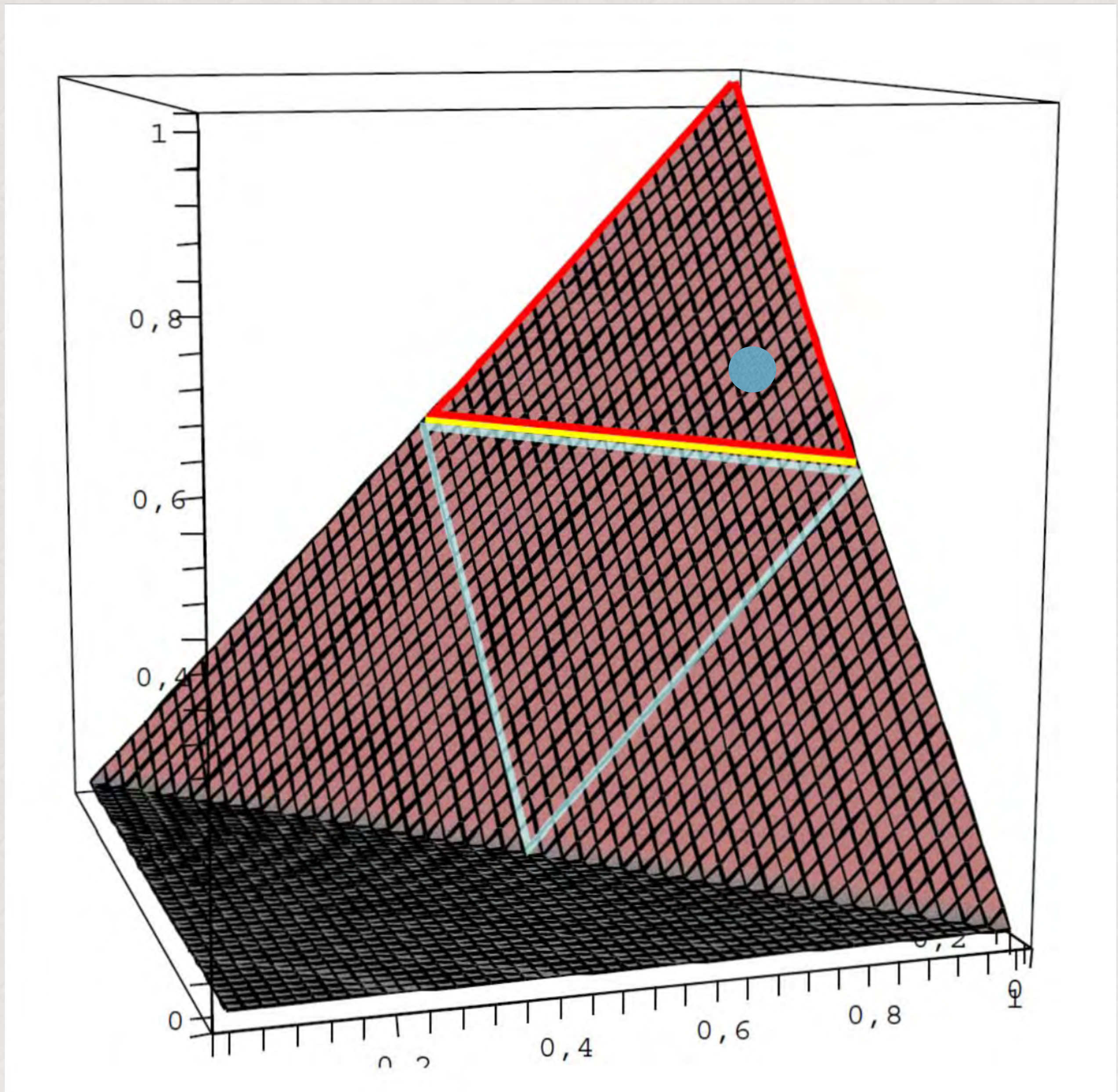
- *S. Jenei, On the geometry of associativity, Semigroup Forum 74:(3) pp. 439-466. (2007)*
- *S. Jenei, On the reflection invariance of residuated chains, Annals of Pure and Applied Logic 161:(2) pp. 220-227. (2009)*

  S. Jenei, Erratum to "On the reflection invariance of residuated chains" [Ann. Pure Appl. Logic 161 (2009) 220-227], Annals of Pure and Applied Logic 161:(12) pp. 1603-1604. (2010)

Main tool 1: Reflection lemma



Main tool 1: Reflection-invariance



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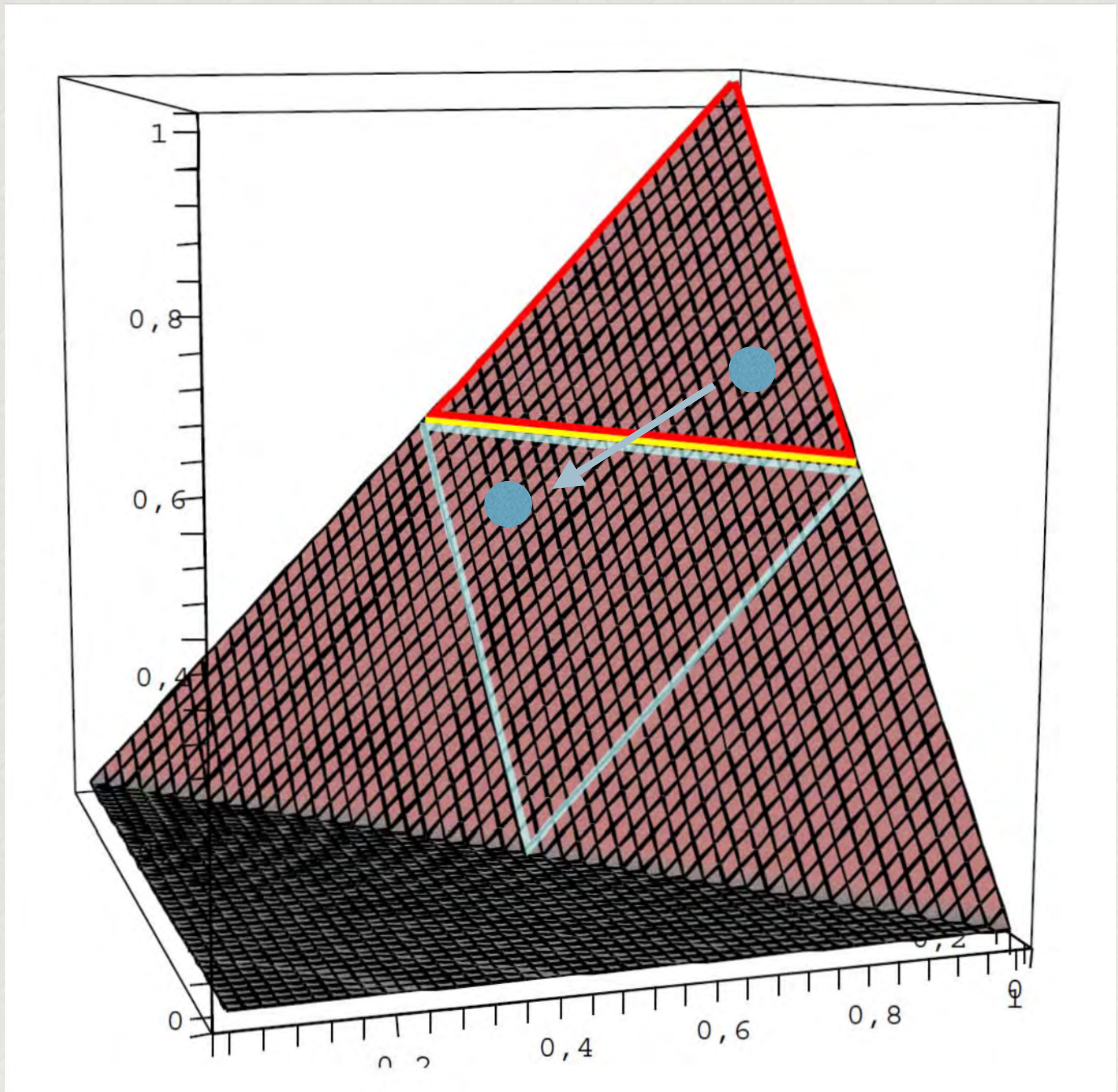
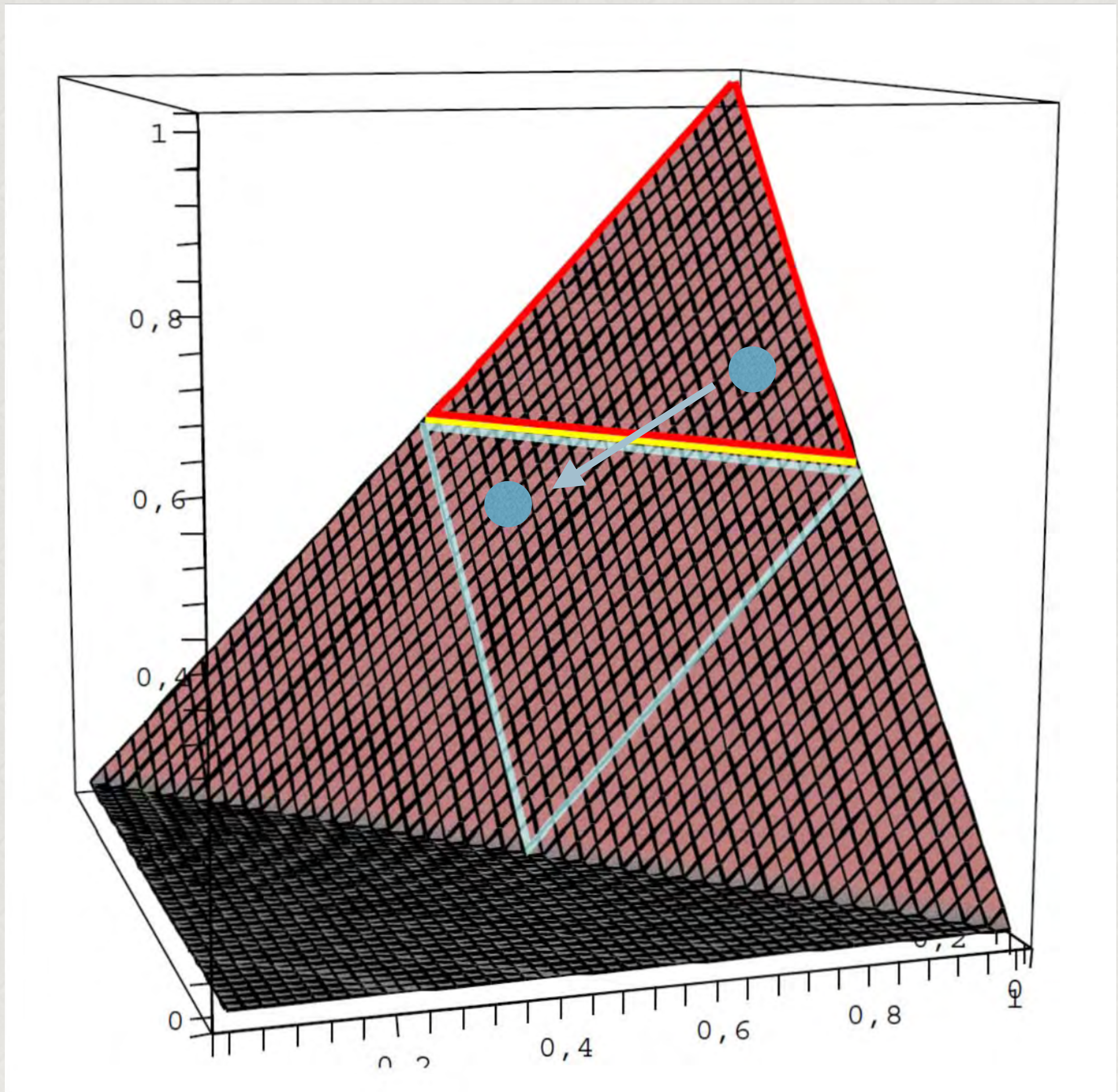


Main tool 1: Reflection-invariance

$x' * y' = (x * y)'$



Main tool 1: Reflection-invariance

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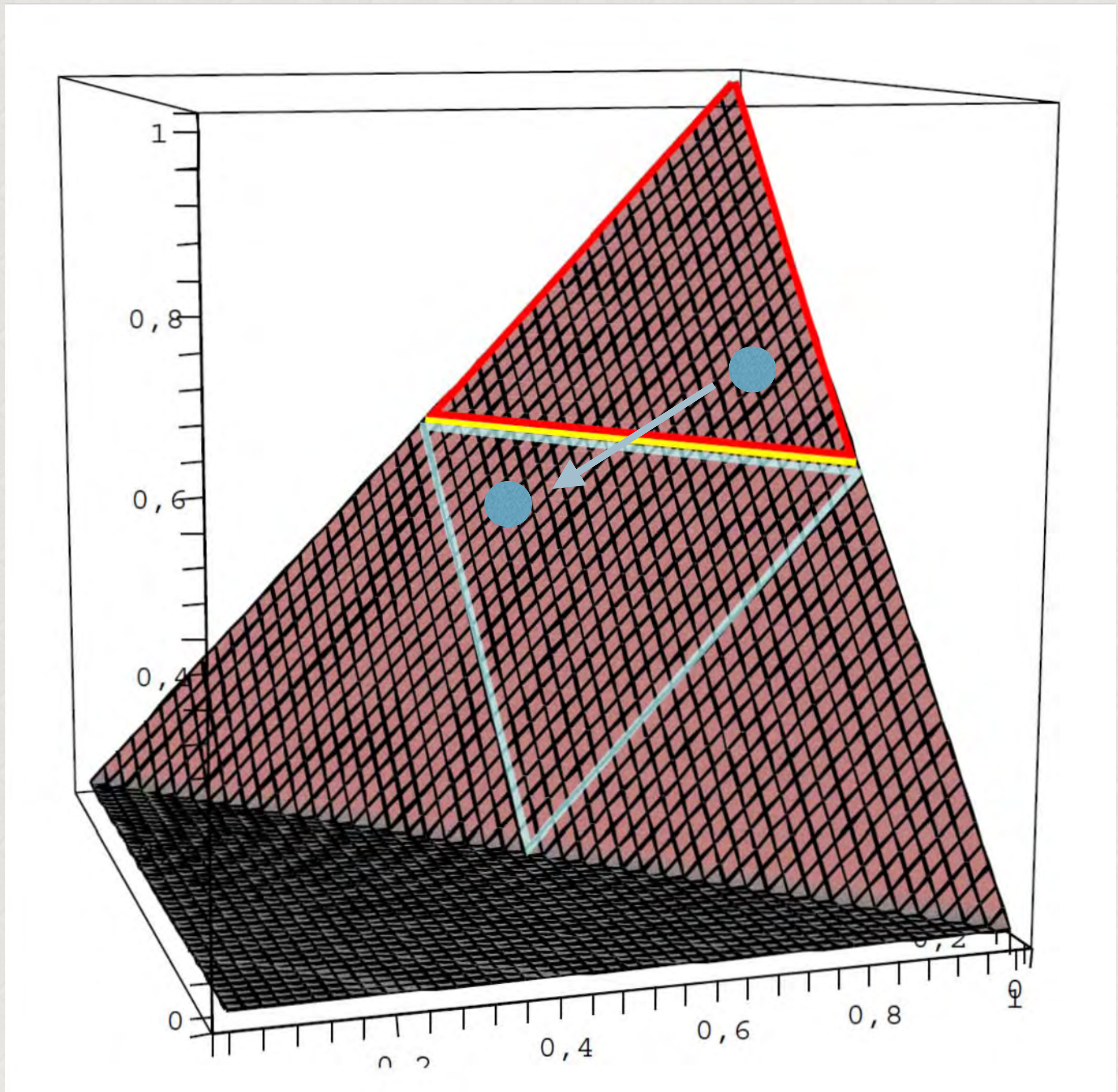
$(x' * y')' = x * y$



S. Jenei
Classification of absorbent-continuous, densely ordered, complete, group-like FL_e-chains (submitted)

- ***Main Tool*** $(x' * y')' = x * y$

Lemma 3 (Reflection Lemma) *Let* $(X, \wedge, \vee, \circledast, \rightarrow_{\circledast}, t, f)$ *be a group-like* FL_e*-algebra over a complete, order-dense chain. For* $\top \neq x, y \in X$,

$$(x' \circledast y')' = x \circledast_{co} y = x \circledast_Q y.$$

Definition 2 For a partially-ordered groupoid $(X, \leq, \circledast)$ over a complete lattice and for $x, y \in X \setminus \{\top\}$ define

$$\begin{aligned} x \circledast_{co} y &= \inf\{x_1 \circledast y_1 \mid x_1 > x, y_1 > y\}, \\ x \circledast_Q y &= \inf\{x \circledast y_1 \mid y_1 > y\}. \end{aligned}$$

Ongoing work

- ***Main Theorem of the talk***

  If U is an absorbent-continuous, group-like FL_e-algebra on a complete, order dense chain, with involution ′ then U is the twin-rotation of a BL-algebra and its de Morgan dual with respect to ′, where the BL-algebra has components, which are either cancellative or MV-algebras over two elements, and the BL-algebra cannot have two consecutive cancellative components.

Ongoing work

- Uninorms can be viewed (as in Girard's linear logic) as fusion operators suitable for interpreting combinations of premises or resources

- ***Uninorm logic UL is an extension of Multiplicative additive intuitionistic linear logic MAILL with the axiom $((A \to B) \wedge t) \vee ((B \to A) \wedge t)$.***
  - Proving or disproving the standard completeness of IUL

  [G. Metcalfe, F. Montagna. Substructural fuzzy logics. Journal of Symbolic Logic, 7, 834–864, 2007.]

Thank you for your attention.